LECTURE 24: Neutrinos

Overview:

1- 2-Family Oscillations

2- 3-Family Oscillations

3- Experimental Results

→ Main text is Thomson

I used Burgess, and Akhmedov (mainly), C. Giunti as references

↳ MSW Effect

NEUTRINO MIXING

IF NEUTRINOS HAVE MASS, WE CAN HAVE WEAK EIGENSTATES THAT ARE DIFFERENT THAN MASS EIGENSTATES:

WEAK EIGENSTATES	e^-	μ^-	τ^-
	ν_e	ν_μ	ν_τ
MASS EIGENSTATES	e^-	μ^-	τ^-
	ν_1	ν_2	ν_3

MIXING MATRIX
"PMNS"
PONTECORVO, MAKI
NAKAGAWA, SAKATA

$$(\nu_e, \nu_\mu, \nu_\tau) = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

WE'LL START WITH 2-FAMILY MIXING FIRST

2-Family Mixing

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

m_1, m_2 will be masses of two steps

θ mixing angle

- Note that different masses for ν_1 and ν_2 imply different velocities

$$\nu_e = \cos\theta\,\nu_1 + \sin\theta\,\nu_2$$
$$\nu_\mu = -\sin\theta\,\nu_1 + \cos\theta\,\nu_2$$

For state of mass, energy, momentum given by m_i, E_i, p_i:

$$\nu_i(T,x) = \nu_i(0,0)\, e^{i\varphi_i(T,x)}$$
$$\rightarrow \varphi_i = E_i T - p_i x \quad , \quad i = 1,2$$

with initial state given by: $T = x = 0$ $\quad \nu_e(0) = 1$, $\nu_\mu(0) = 0$

$\nu_1(0) = \nu_e(0)\cos\theta$

$\nu_2(0) = \nu_e(0)\sin\theta$

2-FAMILY MIXING (CONT.)

As a function of T and x:

$$\nu_e(T,x) = \cos\theta\, \nu_1(T,x) + \sin\theta\, \nu_2(T,x)$$

$$P_{e\to e} = \left|\frac{\nu_e(T,x)}{\nu_e(0,0)}\right|^2 = \left|\cos^2\theta\, e^{i\varphi_1(T,x)} + \sin^2\theta\, e^{i\varphi_2(T,x)}\right|^2$$

$$= 1 - \sin^2 2\theta \sin^2\left(\frac{\varphi_1-\varphi_2}{2}\right)$$

$$\varphi_1 - \varphi_2 = (E_1 - E_2)T - (p_1 - p_2)x$$

with $T = x\left(\frac{E_1+E_2}{p_1+p_2}\right)$ we get:

$$\varphi_1 - \varphi_2 = \frac{(E_1^2 - E_2^2) - (p_1^2 - p_2^2)}{p_1+p_2} \cdot x = \frac{m_1^2 - m_2^2}{p_1+p_2} x$$

$$= \frac{\Delta m^2}{2p} x \approx \frac{\Delta m^2}{2E} x$$

2-FAMILY MIXING (CONT.)

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\theta \, \sin^2\left(\frac{\Delta m^2 x}{4E_\nu}\right)$$

$$P(\nu_e \to \nu_\mu) = \sin^2 2\theta \, \sin^2 \underbrace{\left(\frac{\Delta m^2 x}{4E_\nu}\right)} \to \left(\frac{1.27\, \Delta m^2 L}{E_\nu}\right)$$

L in km
E_ν in GeV
m in eV

Max mixing at $\theta = \frac{\pi}{4}$
and $L = \frac{L_{osc}}{2}$

3-FAMILY MIXING

with 3 generations we will have 3 Δm^2 values but 2 are independent

$$\Delta m_{12}^2 = m_1^2 - m_2^2 \,, \quad \Delta m_{23}^2 = m_2^2 - m_3^2 \,, \quad \Delta m_{31}^2 = m_3^2 - m_1^2$$

PMNS will have 4 indep. parameters, like CKM

use $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$: $\theta_{12}, \theta_{23}, \theta_{13}, \underline{\varphi}$

★ **Hence the two-flavour oscillation probability is:**

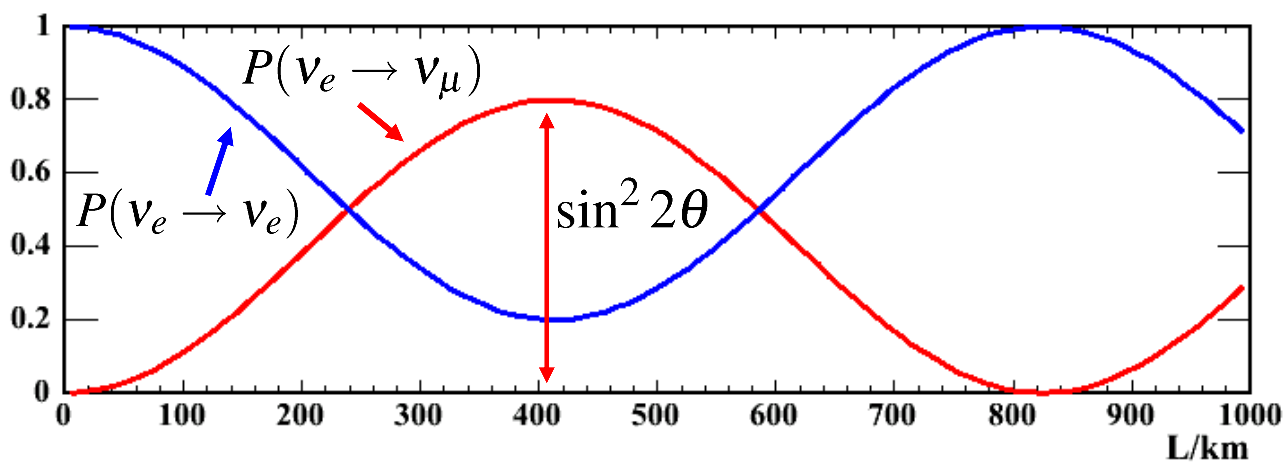
$$\boxed{P(\nu_e \to \nu_\mu) = \sin^2 2\theta \sin^2\left(\frac{\Delta m_{21}^2 L}{4E}\right)}$$

with $\boxed{\Delta m_{21}^2 = m_2^2 - m_1^2}$

★ **The corresponding two-flavour survival probability is:**

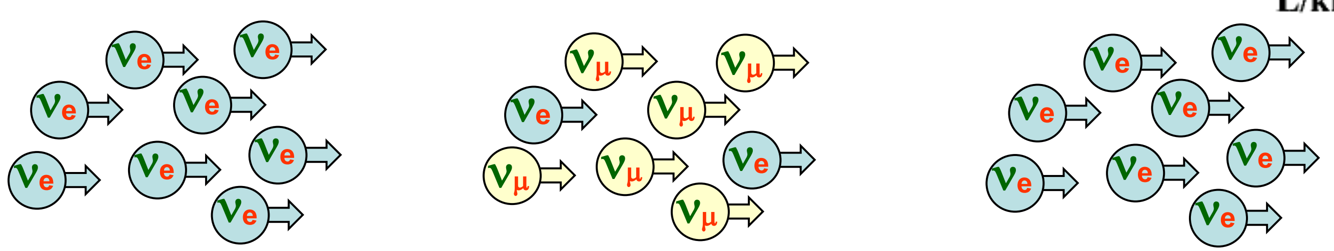
$$\boxed{P(\nu_e \to \nu_e) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m_{21}^2 L}{4E}\right)}$$

• **e.g.** $\Delta m^2 = 0.003\,\mathrm{eV}^2, \quad \sin^2 2\theta = 0.8, \quad E_\nu = 1\,\mathrm{GeV}$



• **wavelength**

$$\boxed{\lambda_{\mathrm{osc}} = \frac{4\pi E}{\Delta m^2}}$$



3-Family Mixing (cont.)

We will need a 3x3 matrix, the PMNS matrix

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

Unitary $\rightarrow$ $$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U_{e1}^* & U_{\mu 1}^* & U_{\tau 1}^* \\ U_{e2}^* & U_{\mu 2}^* & U_{\tau 2}^* \\ U_{e3}^* & U_{\mu 3}^* & U_{\tau 3}^* \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

At Time $= 0$, let's say we start with ν_e

$$|\psi(0)\rangle = |\nu_e\rangle \equiv U_{e1}^* |\nu_1\rangle + U_{e2}^* |\nu_2\rangle + U_{e3}^* |\nu_3\rangle$$

Time evolution:

$$|\psi(x,t)\rangle = U_{e1}^* |\nu_1\rangle e^{-i\varphi_1} + U_{e2}^* |\nu_2\rangle e^{-i\varphi_2} + U_{e3}^* |\nu_3\rangle e^{-i\varphi_3}$$

$$\begin{aligned} = & \; U_{e1}^* \left(U_{e1} |\nu_e\rangle + U_{\mu 1} |\nu_\mu\rangle + U_{\tau 1} |\nu_\tau\rangle \right) e^{-i\varphi_1} \\ + & \; U_{e2}^* \left(U_{e2} |\nu_e\rangle + U_{\mu 2} |\nu_\mu\rangle + U_{\tau 2} |\nu_\tau\rangle \right) e^{-i\varphi_2} \\ + & \; U_{e3}^* \left(U_{e3} |\nu_e\rangle + U_{\mu 3} |\nu_\mu\rangle + U_{\tau 3} |\nu_\tau\rangle \right) e^{-i\varphi_3} \end{aligned}$$

Grouping Terms for all eigenstates:

$$|\Psi(x,t)\rangle = \left(U_{e1}^* U_{e1}\, e^{-i\varphi_1} + U_{e2}^* U_{e2}\, e^{-i\varphi_2} + U_{e3}^* U_{e3}\, e^{-i\varphi_3}\right)|\nu_e\rangle$$
$$\left(U_{e1}^* U_{\mu 1}\, e^{-i\varphi_1} + U_{e2}^* U_{\mu 2}\, e^{-i\varphi_2} + U_{e3}^* U_{\mu 3}\, e^{-i\varphi_3}\right)|\nu_\mu\rangle$$
$$\left(U_{e1}^* U_{\tau 1}\, e^{-i\varphi_1} + U_{e2}^* U_{\tau 2}\, e^{-i\varphi_2} + U_{e3}^* U_{\tau 3}\, e^{-i\varphi_3}\right)|\nu_\tau\rangle$$

Oscillation probability of $\nu_e \to \nu_\mu$: $P(\nu_e \to \nu_\mu) = |\langle \nu_\mu | \Psi(x,t)\rangle|^2$

$$= |U_{e1}^* U_{\mu 1} e^{-i\varphi_1} + U_{e2}^* U_{\mu 2}\, e^{-i\varphi_2} + U_{e3}^* U_{\mu 3}\, e^{-i\varphi_3}|^2$$

note: if phases the same, $P(\nu_e \to \nu_\mu) = 0$ because of Unitarity of PMNS matrix

$$P(\nu_e \to \nu_\mu) = |U_{e1}^* U_{\mu 1}|^2 + |U_{e2}^* U_{\mu 2}|^2 + |U_{e3}^* U_{\mu 3}|^2 +$$
$$2\,\mathrm{Re}\left(U_{e1}^* U_{\mu 1} U_{e2} U_{\mu 2}^* e^{-i(\varphi_1 - \varphi_2)}\right) +$$
$$2\,\mathrm{Re}\left(U_{e1}^* U_{\mu 1} U_{e3} U_{\mu 3}^* e^{-i(\varphi_1 - \varphi_3)}\right) +$$
$$2\,\mathrm{Re}\left(U_{e2}^* U_{\mu 2} U_{e3} U_{\mu 3}^* e^{-i(\varphi_2 - \varphi_3)}\right)$$

We use: $|U_{e1}^* U_{\mu 1}|^2 + |U_{e2}^* U_{\mu 2}|^2 + |U_{e3}^* U_{\mu 3}|^2 +$
$$2\,\mathrm{Re}\left(U_{e1}^* U_{\mu 1} U_{e2} U_{\mu 2}^* + U_{e1}^* U_{\mu 1} U_{e3} U_{\mu 3}^* + U_{e2}^* U_{\mu 2} U_{e3} U_{\mu 3}^*\right) = 0$$

And obtain:

$$P(\nu_e \to \nu_\mu) = 2\,\mathrm{Re}\left(U_{e_1}^* U_{\mu_1} U_{e_2} U_{\mu_2}^* \left[e^{i(\varphi_2-\varphi_1)}-1\right]\right)$$
$$2\,\mathrm{Re}\left(U_{e_1}^* U_{\mu_1} U_{e_3} U_{\mu_3}^* \left[e^{i(\varphi_3-\varphi_1)}-1\right]\right)$$
$$2\,\mathrm{Re}\left(U_{e_2}^* U_{\mu_2} U_{e_3} U_{\mu_3}^* \left[e^{i(\varphi_3-\varphi_2)}-1\right]\right)$$

For $P(\nu_e \to \nu_e) =$
$$1 + 2|U_{e_1}|^2|U_{e_2}|^2\,\mathrm{Re}\left(\left[e^{i(\varphi_2-\varphi_1)}-1\right]\right)$$
$$+ 2|U_{e_1}|^2|U_{e_3}|^2\,\mathrm{Re}\left(\left[e^{i(\varphi_3-\varphi_1)}-1\right]\right)$$
$$+ 2|U_{e_2}|^2|U_{e_3}|^2\,\mathrm{Re}\left(\left[e^{i(\varphi_3-\varphi_2)}-1\right]\right)$$

we have $\mathrm{Re}\left(e^{i(\varphi_j-\varphi_i)}-1\right) = \cos(\varphi_j-\varphi_i) - 1 = -2\sin^2\left(\frac{\varphi_j-\varphi_i}{2}\right) = -2\sin^2\Delta_{ji}$

$$\Delta_{ji} = \frac{\varphi_j-\varphi_i}{2} = \frac{(m_j^2-m_i^2)L}{4E_\nu}$$

$$\rightarrow P(\nu_e \to \nu_e) = 1 - 4|U_{e_1}|^2|U_{e_2}|^2\sin^2\Delta_{21} - 4|U_{e_1}|^2|U_{e_3}|^2\sin^2\Delta_{31}$$
$$- 4|U_{e_2}|^2|U_{e_3}|^2\sin^2\Delta_{32}$$

Note that $\Delta_{31} = \Delta_{32} + \Delta_{21}$ $\longrightarrow$ 2 independent mass^2 differences

Experimentally, we know that $(m_2^2-m_1^2) \ll (m_3^2-m_1^2) \approx (m_3^2-m_2^2)$

CP and T Violation

We had:
$$P(\nu_e \to \nu_\mu) = 2\,\mathrm{Re}\left(U_{e_1}^* U_{\mu_1} U_{e_2} U_{\mu_2}^* \left[e^{i(\varphi_2-\varphi_1)}-1\right]\right)$$
$$2\,\mathrm{Re}\left(U_{e_1}^* U_{\mu_1} U_{e_3} U_{\mu_3}^* \left[e^{i(\varphi_3-\varphi_1)}-1\right]\right)$$
$$2\,\mathrm{Re}\left(U_{e_2}^* U_{\mu_2} U_{e_3} U_{\mu_3}^* \left[e^{i(\varphi_3-\varphi_2)}-1\right]\right)$$

For $P(\nu_\mu \to \nu_e)$ we would change e and μ labels

T reversal symmetry implies $P(\nu_e \to \nu_\mu) = P(\nu_\mu \to \nu_e)$

this is true if matrix elements are real.

For T reversal : $\nu_e \to \nu_\mu \longrightarrow \nu_\mu \to \nu_e$

For CP Trans. : $\nu_e \to \nu_\mu \longrightarrow \bar{\nu}_e \to \bar{\nu}_\mu$

For CPT Trans : $\nu_e \to \nu_\mu \longrightarrow \bar{\nu}_\mu \to \bar{\nu}_e$

So if PMNS is not all real, then

$$P(\nu_e \to \nu_\mu) \neq P(\nu_\mu \to \nu_e)$$
$$P(\nu_e \to \nu_\mu) \neq P(\bar{\nu}_e \to \bar{\nu}_\mu)$$

Neglecting CP violation for now, we can simplify some more:

$$P(\nu_e \to \nu_\mu) = -4U_{e1}U_{\mu 1}U_{e2}U_{\mu 2}\sin^2\Delta_{21} - 4U_{e1}U_{\mu 1}U_{e3}U_{\mu 3}\sin^2\Delta_{31} - 4U_{e2}U_{\mu 2}U_{e3}U_{\mu 3}\sin^2\Delta_{32}$$

with $\Delta_{ji}: \dfrac{(m_j^2 - m_i^2)L}{4E} = \dfrac{\Delta m_{ji}^2 L}{4E}$

with $\Delta_{31} \approx \Delta_{32}$

$$P(\nu_e \to \nu_\mu) \approx -4U_{e1}U_{\mu 1}U_{e2}U_{\mu 2}\sin^2\Delta_{21} + 4U_{e3}^2 U_{\mu 3}^2 \sin^2\Delta_{32}$$

Can also simplify:

$$P(\nu_e \to \nu_e) \approx 1 - 4U_{e1}^2 U_{e2}^2 \sin^2\Delta_{21} - 4(1 - U_{e3}^2)U_{e3}^2 \sin^2\Delta_{32}$$

Parametrization of PMNS Matrix

Use $\theta_{12}, \theta_{23}, \theta_{13}$ and a complex phase δ. Use $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\text{atmospheric}} \times \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix} \times \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{solar}}$$

Measurements : Δm^2_{21} , Δm^2_{32} → absolute value of masses

θ_{12} , θ_{13} , θ_{23}

δ

Experiments:

- sources of neutrinos :
  - nuclear reactors ($\bar{\nu}_e$)
  - cosmic rays hitting atmosphere → produces shower of neutrinos
  - Accelerator beams
  - the sun
- Detection : charged and neutral weak current on atomic electrons and nucleons

Reactor Experiments

Reactors produce large flux of $\bar{\nu}_e$ and this flux is proportional to reactor power output which is well known

Using $\Delta_{31} \simeq \Delta_{32}$ we have

$$P(\bar{\nu}_e \to \bar{\nu}_e) \simeq 1 - 4|U_{e1}|^2|U_{e2}|^2 \sin^2\Delta_{21} - 4|U_{e3}|^2\left[|U_{e1}|^2 + |U_{e2}|^2\right]\sin^2\Delta_{32}$$

Use unitarity relations (see Appendix A) and get:

$$P(\bar{\nu}_e \to \bar{\nu}_e) \simeq 1 - 4|U_{e1}|^2|U_{e2}|^2 \sin^2\Delta_{21} - 4|U_{e3}|^2\left[1 - |U_{e3}|^2\right]\sin^2\Delta_{32}$$

in terms of PMNS parametrization of previous slide:

$$P(\bar{\nu}_e \to \bar{\nu}_e) \simeq 1 - 4(c_{12}c_{13})^2(s_{12}c_{13})^2 \sin^2\Delta_{21} - 4s_{13}^2(1 - s_{13}^2)\sin^2\Delta_{32}$$
$$= 1 - \cos^4(\theta_{13})\sin^2(2\theta_{12})\sin^2\left(\frac{\Delta m_{21}^2 L}{4E_{\bar{\nu}}}\right) - \sin^2(2\theta_{13})\sin^2\left(\frac{\Delta m_{32}^2 L}{4E_{\bar{\nu}}}\right)$$

$\theta_{12} \sim 30°$, $\theta_{23} \sim 45°$, $\theta_{13} \sim 10°$, $\Delta m_{21}^2 \sim 8 \times 10^{-5}\,\text{eV}$, $\Delta m_{32}^2 \sim 3 \times 10^{-3}\,\text{eV}$

Daya Bay → close range (~ 0.5 Km), Kamland → longer range (~ 100 Km)

Long baseline experiments

Examples: Minos, T2K, and future HyperK, Dune

For Minos, we have a pure μ neutrino beam and we measure the disappearence of ν_μ. Note that we anticipate that $\nu_\mu \to \nu_\tau$ will dominate but we are below threshold to produce τ leptons.

Again with $\Delta_{31} \simeq \Delta_{32}$

$$P(\nu_\mu \to \nu_\mu) \approx 1 - 4|U_{\mu 1}|^2 |U_{\mu 2}|^2 \sin^2 \Delta_{21} - 4|U_{\mu 3}|^2 (1 - |U_{\mu 3}|^2) \sin^2 \Delta_{32}$$

$\to$ with $L \sim 700\,\text{Km}$ and $E_\nu > 1\,\text{GeV}$, first term Δ_{21} is very small

$$\to P(\nu_\mu \to \nu_\mu) \simeq 1 - 4|U_{\mu 3}|^2 (1 - |U_{\mu 3}|^2) \sin^2 \Delta_{32}$$

giving: $1 - 4\sin^2(\theta_{23}) \cos^2(\theta_{13}) [1 - \sin^2(\theta_{23}) \cos^2(\theta_{13})] \sin^2 \Delta_{32}$

$$= 1 - [\sin^2(2\theta_{23}) \cos^4(\theta_{13}) + \sin^2(2\theta_{13}) \sin^2(\theta_{23})] \sin^2 \Delta_{32}$$

Fixed L, measure oscillation probability vs E_ν (spectrum distortion)

Solar neutrinos

We get $\frac{\Delta m_{31}^2 L}{4E} \approx \frac{\Delta m_{32}^2 L}{4E} \gg 1$

$\rightarrow$ oscillate quickly relative Δm_{12}^2 term which leads to an average value for both terms

this leads to $P(\nu_e \rightarrow \nu_e) \simeq 1 - \sin^2 2\theta_{12} \sin^2\left(\frac{\Delta m_{12}^2 L}{4E}\right)$

Neutrino masses

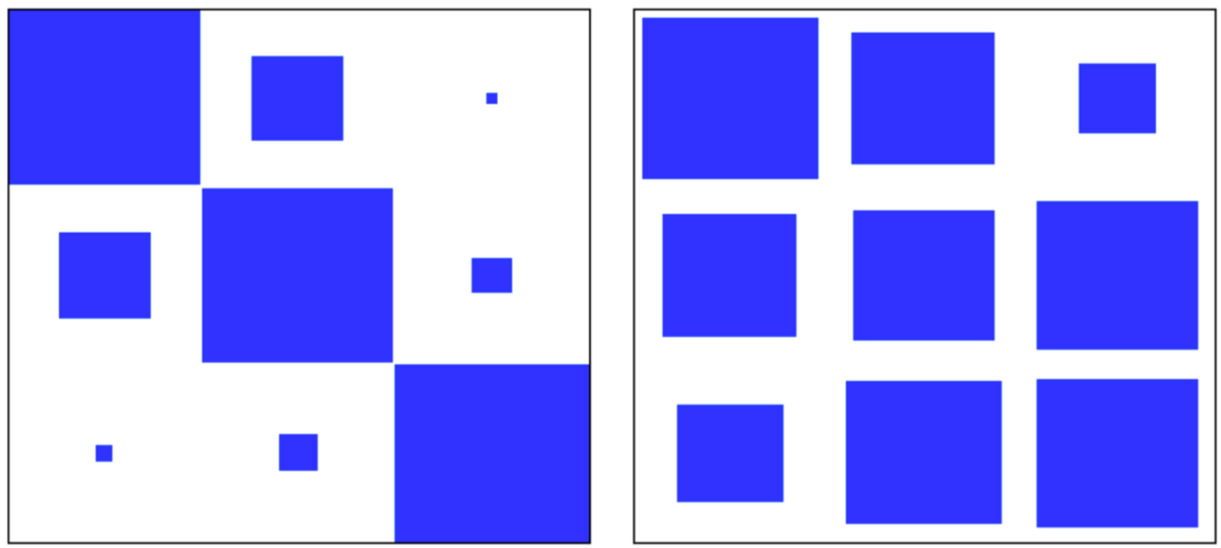
$\rightarrow$ Dirac and Majorana masses covered in future presentation

$\rightarrow$ current limits from cosmology : $\sum_i m_{\nu_i} < 0.17\,\text{eV}$ @ 95% CL

from tritium decay : $m_{\nu_e} < 1.1\,\text{eV}$ @ 90% CL

$\hookrightarrow$ KATRIN from 2019

Neutrino Mixing



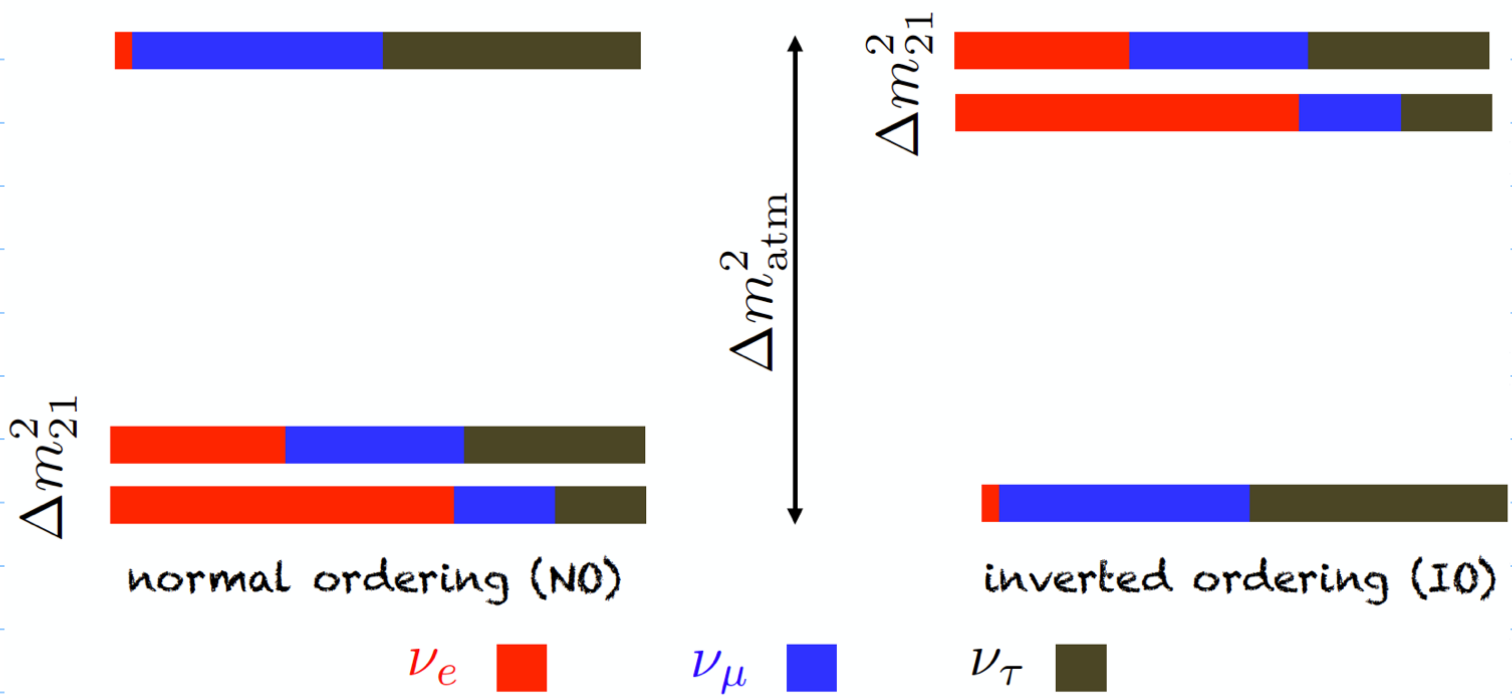
$$\begin{pmatrix} 0.8 & 0.5 & 0.16 \\ -0.4 & 0.5 & -0.7 \\ -0.4 & 0.5 & 0.7 \end{pmatrix}$$

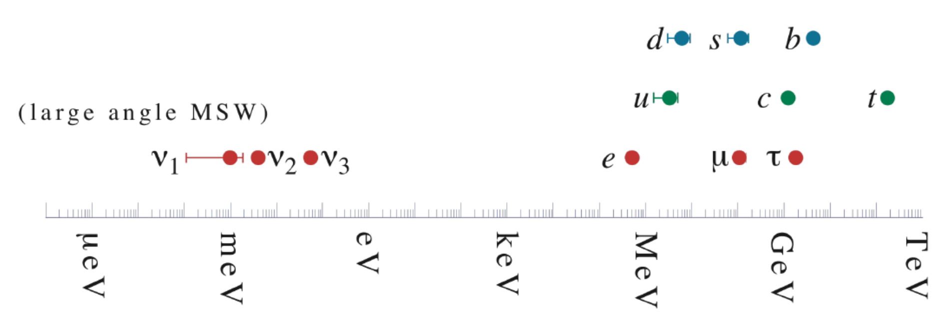
NuFIT 4.1 (2019)

$$|U|_{3\sigma}^{\text{w/o SK-atm}} = \begin{pmatrix} 0.797 \to 0.842 & 0.518 \to 0.585 & 0.143 \to 0.156 \\ 0.244 \to 0.496 & 0.467 \to 0.678 & 0.646 \to 0.772 \\ 0.287 \to 0.525 & 0.488 \to 0.693 & 0.618 \to 0.749 \end{pmatrix}$$

$$|U|_{3\sigma}^{\text{with SK-atm}} = \begin{pmatrix} 0.797 \to 0.842 & 0.518 \to 0.585 & 0.143 \to 0.156 \\ 0.243 \to 0.490 & 0.473 \to 0.674 & 0.651 \to 0.772 \\ 0.295 \to 0.525 & 0.493 \to 0.688 & 0.618 \to 0.744 \end{pmatrix}$$

Masses and Mixing





Latest global Fit performed in 2019 (NuFit)

		Normal Ordering (best fit)	
		bfp $\pm 1\sigma$	3σ range
with SK atmospheric data	$\sin^2\theta_{12}$	$0.310^{+0.013}_{-0.012}$	$0.275 \to 0.350$
	$\theta_{12}/^\circ$	$33.82^{+0.78}_{-0.76}$	$31.61 \to 36.27$
	$\sin^2\theta_{23}$	$0.563^{+0.018}_{-0.024}$	$0.433 \to 0.609$
	$\theta_{23}/^\circ$	$48.6^{+1.0}_{-1.4}$	$41.1 \to 51.3$
	$\sin^2\theta_{13}$	$0.02237^{+0.00066}_{-0.00065}$	$0.02044 \to 0.02435$
	$\theta_{13}/^\circ$	$8.60^{+0.13}_{-0.13}$	$8.22 \to 8.98$
	$\delta_{CP}/^\circ$	221^{+39}_{-28}	$144 \to 357$
	$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.39^{+0.21}_{-0.20}$	$6.79 \to 8.01$
	$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.528^{+0.029}_{-0.031}$	$+2.436 \to +2.618$

Assignment Due April 3rd (last assignment)

1- **13.5** The general unitary transformation between mass and weak eigenstates for two flavours can be written as

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta \exp(i\delta_1) & \sin\theta \exp\left(i\left[\frac{\delta_1+\delta_2}{2} - \delta\right]\right) \\ -\sin\theta \exp\left(i\left[\frac{\delta_1+\delta_2}{2} + \delta\right]\right) & \cos\theta \exp(i\delta_2) \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.$$

a) Show that the matrix in the above expression is indeed unitary.

b) Show that the three complex phases δ_1, δ_2 and δ can be eliminated from the above expression by the transformation

$$\ell_\alpha \to \ell_\alpha e^{i(\theta_e+\theta'_\alpha)}, \quad \nu_k \to \nu_k e^{i(\theta_e+\theta'_k)} \quad \text{and} \quad U_{\alpha k} \to U_{\alpha k} e^{i(\theta'_\alpha-\theta'_k)},$$

without changing the physical form of the two-flavour weak charged current

$$-i\frac{g_W}{\sqrt{2}}(\bar{e}, \bar{\mu})\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.$$

2- **13.4** Derive equation (13.30) in the following three steps:

a) By writing the oscillation probability $P(\nu_e \to \nu_\mu)$ as

$$P(\nu_e \to \nu_\mu) = 2\sum_{i<j} \Re e\left\{U^*_{ei}U_{\mu i}U_{ej}U^*_{\mu j}\left[e^{i(\phi_j-\phi_i)} - 1\right]\right\},$$

→ Continued on next page

Assignement (cont.)

and writing $\Delta_{ij} = (\phi_i - \phi_j)/2$, show that

$$P(\nu_e \to \nu_\mu) = -4 \sum_{i<j} \mathfrak{Re}\{U^*_{ei} U_{\mu i} U_{ej} U^*_{\mu j}\} \sin^2 \Delta_{ij} + 2 \sum_{i<j} \mathfrak{Im}\{U^*_{ei} U_{\mu i} U_{ej} U^*_{\mu j}\} \sin 2\Delta_{ij} .$$

b) Defining $-J \equiv \mathfrak{Im}\{U^*_{e1} U_{\mu 1} U_{e3} U^*_{\mu 3}\}$, use the unitarity of the PMNS matrix to show that

$$\mathfrak{Im}\{U^*_{e1} U_{\mu 1} U_{e3} U^*_{\mu 3}\} = -\mathfrak{Im}\{U^*_{e2} U_{\mu 2} U_{e3} U^*_{\mu 3}\} = -\mathfrak{Im}\{U^*_{e1} U_{\mu 1} U_{e2} U^*_{\mu 2}\} = -J .$$

c) Hence, using the identity

$$\sin A + \sin B - \sin(A+B) = 4 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{A+B}{2}\right),$$

show that

$$P(\nu_e \to \nu_\mu) = -4 \sum_{i<j} \mathfrak{Re}\{U^*_{ei} U_{\mu i} U_{ej} U^*_{\mu j}\} \sin^2 \Delta_{ij} + 8J \sin\Delta_{12} \sin\Delta_{13} \sin\Delta_{23} .$$

d) Hence show that

$$P(\nu_e \to \nu_\mu) - P(\overline{\nu}_e \to \overline{\nu}_\mu) = 16\, \mathfrak{Im}\{U^*_{e1} U_{\mu 1} U_{e2} U^*_{\mu 2}\} \sin\Delta_{12} \sin\Delta_{13} \sin\Delta_{23} .$$

e) Finally, using the current knowledge of the PMNS matrix determine the maximum possible value of $P(\nu_e \to \nu_\mu) - P(\overline{\nu}_e \to \overline{\nu}_\mu)$.

3- How can we determine whether we have a "normal" hiearchy vs an inverted hiearchy using long baseline experiments e.g. T2K ?

Unitarity Conditions

Appendix A

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \cdot \begin{pmatrix} U_{e1}^* & U_{\mu 1}^* & U_{\tau 1}^* \\ U_{e2}^* & U_{\mu 2}^* & U_{\tau 2}^* \\ U_{e3}^* & U_{\mu 3}^* & U_{\tau 3}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U_{e1} U_{e1}^* + U_{e2} U_{e2}^* \quad U_{e3} U_{e3}^* = 1$$

$$U_{\mu 1} U_{\mu 1}^* + U_{\mu 2} U_{\mu 2}^* \quad U_{\mu 3} U_{\mu 3}^* = 1$$

$$U_{\tau 1} U_{\tau 1}^* + U_{\tau 2} U_{\tau 2}^* \quad U_{\tau 3} U_{\tau 3}^* = 1$$

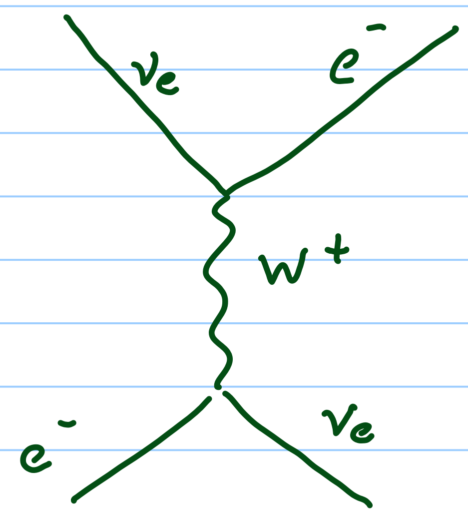
$$U_{e1} U_{\mu 1}^* + U_{e2} U_{\mu 2}^* \quad U_{e3} U_{\mu 3}^* = 0$$

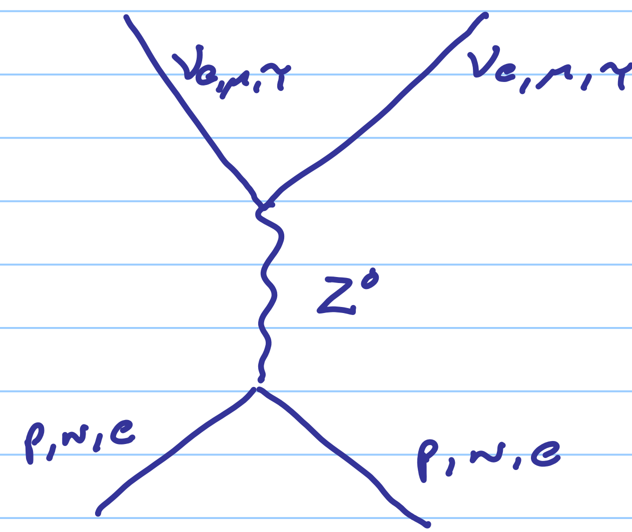
$$U_{e1} U_{\tau 1}^* + U_{e2} U_{\tau 2}^* \quad U_{e3} U_{\tau 3}^* = 0$$

$$U_{\mu 1} U_{\tau 1}^* + U_{\mu 2} U_{\tau 2}^* \quad U_{\mu 3} U_{\tau 3}^* = 0$$

Neutrino oscillations in MATTER

Appendix B





$$H_{cc} = \frac{G_F}{\sqrt{2}} [\bar{e}\gamma_\mu(1-\gamma_5)\nu_e][\bar{\nu}_e\gamma^\mu(1-\gamma_5)e] \qquad \text{Fierz} \longrightarrow$$

$$= \frac{G_F}{\sqrt{2}} [\bar{e}\gamma_\mu(1-\gamma_5)e][\bar{\nu}_e\gamma^\mu(1-\gamma_5)\nu_e] \qquad \text{(low E neutrinos)}$$

$$H_{eff}(\nu_e) = \langle H_{cc}\rangle_e \equiv \bar{\nu}_e V_e \nu_e$$

↳ integrated over all e variables

Unpolarized medium with zero total momentum, relevant term is $\langle \bar{e}\gamma_0 e\rangle = \langle e^+ e\rangle = N_e$ → number density

Neutrino oscillations in MATTER (cont.)

$\rightarrow \quad (V_e)_{cc} \equiv V_{cc} = \sqrt{2}\, G_F N_e$

$(V_e)_{NC} = -G_F N_n/\sqrt{2}$ (protons, electrons cancel off)

$(V_m)_{NC} = (V_i)_{NC} = -\dfrac{G_F N_n}{\sqrt{2}}$

More convenient to work in flavour basis because effective potentials are diagonal in this basis.

For the two-flavour case, in the absence of matter:

$$i\left(\frac{d}{dt}\right)|\nu_m\rangle = H_m|\nu_m\rangle \quad , \quad H_m \text{ is diagonal}$$

$$i\left(\frac{d}{dt}\right)|\nu_F\rangle = H_F|\nu_F\rangle = UH_mU^\dagger|\nu_F\rangle$$

$$E_i \simeq p + m_i^2/2E$$

$$i\left(\frac{d}{dt}\right)\begin{pmatrix}\nu_e\\ \nu_m\end{pmatrix} = \begin{pmatrix}\left(p+\frac{m_1^2+m_2^2}{4E}\right) - \Delta m^2\cos 2\theta_0 & \frac{\Delta m^2}{4E}\sin 2\theta_0 \\ \frac{\Delta m^2}{4E}\sin 2\theta_0 & \left(p+\frac{m_1^2+m_2^2}{4E}\right) - \Delta m^2\cos 2\theta_0\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_m\end{pmatrix}$$

Neutrino oscillations in MATTER (cont.)

Extra terms on the diag. can only modify the common phase of the neutrino states $\Rightarrow$ we can omit them

We get
$$i\frac{d}{dt}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \begin{pmatrix} -\frac{\Delta m^2}{4E}\cos 2\theta_0 & \frac{\Delta m^2}{4E}\sin 2\theta_0 \\ \frac{\Delta m^2}{4E}\sin 2\theta_0 & \frac{\Delta m^2}{4E}\cos 2\theta_0 \end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}$$

With matter present:
$$i\frac{d}{dt}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \begin{pmatrix} -\frac{\Delta m^2}{4E}\cos 2\theta_0 \boxed{+\sqrt{2}G_F N_e} & \frac{\Delta m^2}{4E}\sin 2\theta_0 \\ \frac{\Delta m^2}{4E}\sin 2\theta_0 & \frac{\Delta m^2}{4E}\cos 2\theta_0 \end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}$$

Note the $G_F N_n$ term is common to both and on diag.
so we can discard it (overall phase)

Neutrino oscillations in MATTER (cont.)

In matter with constant density (N_e = const), diag. of Hamiltonian gives following eigenstates:

$$\nu_A = \nu_e \cos\theta + \nu_\mu \sin\theta$$

$$\nu_B = -\nu_e \sin\theta + \nu_\mu \cos\theta$$

$$\tan 2\theta = \frac{\frac{\Delta m^2}{2E} \sin 2\theta_0}{\frac{\Delta m^2}{2E} \cos 2\theta_0 - \sqrt{2}\, G_F N_e}$$

$$P(\nu_e \to \nu_\mu ; L) = \sin^2 2\theta \, \sin^2\left(\pi \frac{L}{l_m}\right)$$

l_m (oscillation length in matter $= \dfrac{2\pi}{E_A - E_B}$

$$E_A - E_B = \sqrt{\left(\frac{\Delta m^2}{2E} \cos 2\theta_0 - \sqrt{2}\, G_F N_e\right)^2 + \left(\frac{\Delta m^2}{2E}\right) \sin^2 2\theta_0}$$