

LECTURE 1 and 2: Introduction to Gauge Theories

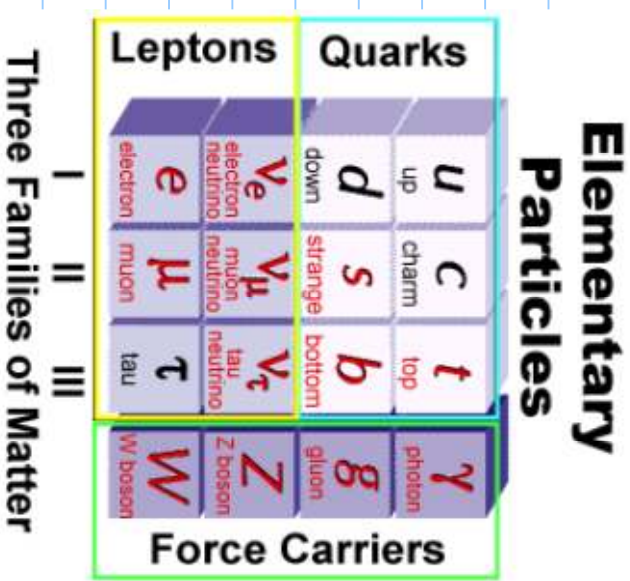
Overview:

- Overview of Group Theory
- Abelian Gauge Theory
- Non-Abelian Gauge Theory

The Standard Model

A quantum field theory based on $SU(3) \times SU(2) \times U(1)$ gauge symmetries.

Lagrangian invariant under a continuous group of local transformations (gauge transformations). They form a Lie group whose generators have associated vector fields (gauge fields)



To obtain a better understanding of this theory, we'll spend some time studying what is a gauge theory and this requires some knowledge of group theory. I'll give a brief introduction to group theory and Lie groups in particular. These topics (Group Theory and construction of Gauge Theories) deserve a course unto themselves so I encourage you to explore the beautiful mathematics behind our physical theories in more detail during the course of your studies. I suggest H. Georgi's "Lie Algebras in Particle Physics" as a good reference text.

Introduction to Group Theory

③

"Group Theory is the study of symmetry" Georgi
The study of symmetry in physical systems is extremely useful.

Noether's Theorem relates symmetries to conserved quantities

Space Translation \rightarrow Momentum

Rotation \rightarrow Ang. Momentum

Time \rightarrow Energy

Symmetry \rightarrow conserved quantities \rightarrow invariance
In field theory, the analogs are the identities

WARD-TAKAHASHI

Understanding how SM fields can transform requires we take some time to review some group theory

Group Theory Intro (cont.)

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A group is a set of elements (a, b, c, \dots) with a law to combine two elements in an ordered way such that

- 1- For every $a, b \in G$, the product $ab \in G$
- 2- $(ab)c = a(bc) \rightarrow$ associativity
- 3- G has a unique element e such that for all $a \in G$: $ae = ea = a \rightarrow$ identity
- 4- For all $a \in G$, there is a unique element a^{-1} such that $a^{-1}a = aa^{-1} = e \rightarrow$ inverse

if $ab = ba$: Abelian group \rightarrow commutative
if $ab \neq ba$: non-Abelian group \rightarrow non-commutative

examples: following set forms Abelian group $(1, i, -1, -i)$ using multiplication for complex numbers as "combination law"

Note That we can associate matrices to elements above:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Group Theory

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NOTE THAT IN THE CONTEXT OF FIELD THEORY, SYMMETRY TRANSFORMATIONS FORM A GROUP.

EXAMPLE: THEORY WITH ACTION $S[\phi]$

UNDER A SET OF TRANSFORMATIONS $\{g, h, k, \dots\}$ THE FIELDS CHANGE TO $\phi_g, \phi_h, \phi_k, \dots$ WHICH LEAVES THE ACTION INVARIANT:
 $S[\phi] = S[\phi_g] = S[\phi_h] = \dots$

ASSUME THAT THE SET $\{g, h, k, \dots\}$ CONTAINS ALL SYMMETRY TRANSFORMATIONS OF THE ACTION.

→ $g \cdot h$ implies successive application on the fields which will leave action unchanged \Rightarrow product is in the set

ALSO IDENTITY IS IN THE SET AND SO IS THE INVERSE: $S[\phi] = S[\phi_I] = S[\phi \cdot g \cdot g^{-1}] = S[\phi \cdot g^{-1}]$

→ THE FULL SET OF SYMMETRY TRANSFORMATIONS CONSTITUTES A GROUP

Group THEO. (cont.)

⑥

Def.: A subgroup S of a group G contains some elements that form a group by themselves.
 e and G are Trivial subgroups

Def.: The order of a group is the number of elements in G
 \rightarrow can be infinite

Def.: A representation of G is a mapping D of the elements of G onto a set of linear operators with the following properties:

- $D(e) = 1$, where 1 is the identity operator in the space on which the linear operators act
- $D(g_1)D(g_2) = D(g_1g_2)$ i.e. group combination or multiplication law is mapped onto the multiplication in the space in which the operators act.

Group Theory (cont.)

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Example: Z_3

MULT. Table:

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

→ abelian

Representation example:

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$[D(g)]_{ij} = \langle e_i | D(g) | e_j \rangle \rightarrow$ with basis vectors $|e_i\rangle$ associated with e, a, b

Another representation (realization):

$$\begin{aligned} D(e) &= 1 \\ D(a) &= e^{2\pi i/3} \\ D(b) &= e^{4\pi i/3} \end{aligned}$$

Group Theory (cont.)

⑧

Def: a rep. is said to be reducible if it has an invariant subspace \rightarrow The action of any $D(g)$ on any vector in the subspace is still in the subspace.

In Terms of a projection operator P :

$$P D(g) P = D(g) P$$

Example: For Z_3 $P = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\text{we get } D(g) P = P$$

Def: a completely reducible rep. has its matrix elements in the following form

$$\begin{pmatrix} D_1(g) & 0 & \dots \\ 0 & D_2(g) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

i.e. block diagonal

Group Theory (cont.)

Note: a rep. in block diagonal form is said to be in the direct sum of the subrep.

$$D_1 \oplus D_2 \oplus \dots$$

Def.: A set of generators is a set of group elements such that the repeated application of generators on themselves can generate the whole group

Example:

generators: p, q
 mult. rules: $p^3 = e$, $q^2 = e$, $(pq)^2 = e$

\Rightarrow group elements: $\{e, p, p^2, q, pq, qp^2\}$

note that $pqp = q^{-1} = q$

because $q^2 = e$

Lie Groups

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We are interested in continuous groups with elements labelled by continuously variable real parameters

$$\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_r, \quad g(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_r) \equiv g(\alpha)$$

For a continuous group, condition 1 takes the form:

$$g(\alpha)g(\beta) = g(\gamma(\alpha, \beta))$$

↳ continuous functions of α, β

→ if γ is an analytic function then we have a Lie Group

→ implies that it can be expressed as a power series. We can use a power series expansion to move from one element to another (within its neighborhood).

Lie proved that the properties of the elements which can be reached continuously from I are determined from elements in the neighbourhood of I

Lie Groups (cont.)

see Atiyah and Hey

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Consider a group of Transformations defined by:

$$X_i = F_i(x_1, x_2, \dots, x_N; \alpha_1, \alpha_2, \dots, \alpha_r)$$

X_i : "coordinates" \rightarrow or field components
 α_j : parameters of the Transformations

$\alpha = 0$ is the identity Transformation $\Rightarrow X_i = F_i(x, 0)$

$$dx_i = \sum_{\nu=1}^r \frac{\partial F_i}{\partial \alpha_\nu} d\alpha_\nu \quad (\text{partial deriv. evaluated at } (x, 0))$$

$$\begin{aligned} F \rightarrow F + dF &= F + \sum_{i=1}^N \frac{\partial F}{\partial x_i} dx_i = F + \sum_{i=1}^N \left[\sum_{\nu=1}^r \frac{\partial F_i}{\partial \alpha_\nu} d\alpha_\nu \right] \frac{\partial F}{\partial x_i} \\ &\equiv \left[1 - \sum_{\nu=1}^r \alpha_\nu i X_\nu \right] F \quad \text{with:} \end{aligned}$$

$$\hat{X}_\nu = i \sum_{i=1}^N \frac{\partial F_i}{\partial \alpha_\nu} \frac{\partial}{\partial x_i}$$

\rightarrow generator of infinitesimal Transfo.

Lie Groups (cont.)

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For finite Transformations: $\left[1 - \sum_{v=1}^r dx_v i \hat{X}_v \right]$
becomes $\exp[-i \alpha \cdot \hat{X}]$ $\alpha \cdot \hat{X} \equiv \sum_v \alpha_v \hat{X}_v$

Theorem states that commutator of any two generators is a linear combination of the generators:

$$[\hat{X}_\alpha, \hat{X}_\beta] = \sum_{\gamma} c_{\alpha\beta}^{\gamma} \hat{X}_\gamma$$

→ structure constants of the group
computation relations called
algebra of the group.

We can see this using the Baker-Campbell-Hausdorff Formula. BUT first we'll take a step back

Group Theory (cont.)

LET'S TAKE ONE-PARAMETER LIE GROUP WITH ELEMENTS $g(\xi)$.
BECAUSE OF ANALYTIC CONDITIONS, WE CAN CHOOSE A
PARAMETRIZATION THAT SATISFIES:

$$g(\xi^1) g(\xi^2) = g(\xi^1 + \xi^2)$$

$$\Rightarrow g(0) = I \quad \text{and} \quad g(\xi)^{-1} = g(-\xi)$$

IN THE NEIGHBOURHOOD OF THE IDENTITY WE CAN WRITE

$$g(\xi) = I + \xi T + O(\xi^2)$$

$T \rightarrow$ operator that generates infinitesimal transformations

$$\begin{aligned} \text{going finite: } g(\xi) &= \left\{ g(\xi/n) \right\}^n = \lim_{n \rightarrow \infty} \left\{ I + \frac{\xi T}{n} \dots \right\}^n \\ &= \exp(\xi T) \end{aligned}$$

For n -parameter $g(\xi_1, \xi_2, \dots) = \exp(\xi_a T^a)$
 \hookrightarrow generators

Group Theory (cont.)

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CONSIDER PRODUCT OF TWO ELEMENTS FOR LIE GROUP G :

$$g(\xi_1, \xi_2, \xi_3, \dots) \cdot g(\zeta_1, \zeta_2, \zeta_3, \dots) = \exp(\xi_a T^a) \exp(\zeta_b T^b)$$

Baker - Campbell - Hausdorff

$$= \exp\left\{ \xi_a T^a + \zeta_b T^b + \frac{1}{2} \xi_a \zeta_b [T^a, T^b] + \frac{1}{12} (\xi_a \zeta_b \zeta_c + \zeta_a \zeta_b \xi_c [T_a, [T_b, T_c]] + \dots) \right\}$$

SINCE G IS A GROUP WE MUST HAVE:

$$g(\xi_1, \dots, \xi_n) \cdot g(\zeta_1, \dots, \zeta_n) = \exp(\eta_a T^a)$$

→ POSSIBLE IF ANY COMMUTATOR CAN BE WRITTEN AS LINEAR COMBINATION OF GENERATORS

→ GENERATORS MUST CLOSE UNDER COMPUTATION

$$[T^a, T^b] = f_{ab}^c T^c$$

Group Theory (cont.)

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EXAMPLES:

1 - ONE DIM. TRANSLATIONS: $f(x) \rightarrow f(x + \xi)$

INF. TRANSFORMATION: $f(x) + \xi \frac{d}{dx} f(x) + O(\xi^2)$

generator: $Tf(x) = \frac{d}{dx} f(x)$

FINITE TRANSFO.: $f(x) \rightarrow \exp\left(\xi \frac{d}{dx}\right) f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n \frac{d^n}{dx^n} f(x)$

↳ Taylor series

2 - two dim. ROTATION: $g(\xi)$ changes θ into $\theta - \xi$

$x = r \cos \theta$, $y = r \sin \theta$: $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} + \xi \begin{pmatrix} -y \\ x \end{pmatrix} + O(\xi^2)$

generator is then $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ note that $T^2 = -I$

Group Theory (cont.)

$$\begin{aligned}
 g(\xi) &= \exp(\xi t) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n t^n \\
 &= \sum_{n=0}^{\infty} \left\{ \frac{1}{(2n)!} (-\xi^2)^n I + \frac{1}{(2n+1)!} (-1)^n \xi^{2n+1} t \right\} \\
 &= \cos \xi t + \sin \xi t = \begin{pmatrix} \cos \xi & \sin \xi \\ -\sin \xi & \cos \xi \end{pmatrix}
 \end{aligned}$$

→ Two-dim. rotation

→ The group above is $SO(2)$

" $O(2)$ ": orthogonal 2×2 matrices

" S ": "special" → $\det = 1$

" $U(N)$ ": unitary $N \times N$ matrices → N^2 generators

" $SU(N)$ ": "special" → have $(N^2 - 1)$ generators with $\det = 1$

Let's take a look at $SU(2)$...

SU(2)

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Consider a system with two fermion fields q_1 and q_2 and demand that it possesses symmetry under transformations that mix them together:

$$\begin{aligned} q_1 &\rightarrow q_1' = \alpha q_1 + \beta q_2 \\ q_2 &\rightarrow q_2' = \gamma q_1 + \delta q_2 \end{aligned}$$

The α, β, \dots are complex

Keep normalization: $\langle q_1, 1q_1 \rangle = \langle q_1, 1q_1 \rangle = 1$ etc.

$$\Rightarrow | \alpha |^2 + | \beta |^2 = | \gamma |^2 + | \delta |^2 = 1$$

Keep orthogonality: $\langle 2q_1, 2q_2 \rangle = 0$

$$\Rightarrow \alpha^* \delta + \beta^* \gamma = 0$$

in 2D component form:

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \rightarrow q' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

SU(2) (cont)

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$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ is unitary} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A 2x2 Unitary matrix has 4 free parameters and we can write:

$$U = e^{i\alpha_0} + i\alpha_1 \tau_1 + i\alpha_2 \tau_2 + i\alpha_3 \tau_3$$

τ_i are the Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$U = e^{i\alpha_0} \cdot e^V \text{ with } V \text{ a member of } \text{SU}(2)$$

You can show that V has unit det.

SU(2) (cont.)

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For infinitesimal SU(2) Transfo.:

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = (1 + i\alpha \cdot \frac{\sigma}{2}) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$\frac{i}{2} \begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -\alpha_3 \end{pmatrix}$$

$$dq_1 = \frac{i}{2} \alpha_3 q_1 + \left(\frac{i\alpha_1 + \alpha_2}{2} \right) q_2$$

$$dq_2 = \frac{-i\alpha_3}{2} q_2 + \left(\frac{i\alpha_1 - \alpha_2}{2} \right) q_1$$

$$\hat{X}_\nu = i \sum_{i=1}^3 \frac{\partial f_i}{\partial x_\nu} \frac{d}{dx_i}$$

$$\frac{\partial f_1}{\partial x_1} = i q_2, \quad \frac{\partial f_1}{\partial x_2} = \frac{q_2}{2}, \quad \frac{\partial f_1}{\partial x_3} = i q_1$$

$$\frac{\partial f_2}{\partial x_1} = i q_1, \quad \frac{\partial f_2}{\partial x_2} = -\frac{q_1}{2}, \quad \frac{\partial f_2}{\partial x_3} = -\frac{q_2}{2}$$

$$\hat{X}_1 = -\frac{1}{2} \left\{ q_2 \frac{d}{dx_1} + q_1 \frac{d}{dx_2} \right\}, \quad \hat{X}_2 = \frac{i}{2} \left\{ q_2 \frac{d}{dx_1} - q_1 \frac{d}{dx_2} \right\}$$

$$\hat{X}_3 = \frac{1}{2} \left\{ -q_1 \frac{d}{dx_1} + q_2 \frac{d}{dx_2} \right\}$$

$$[\hat{X}_i, \hat{X}_j] = i \epsilon_{ijk} \hat{X}_k$$

Example 2: $SU(3)$

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$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}^T = \left(1 + \frac{i}{2} \eta \cdot \lambda\right) \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

Now we have 8 parameters, and λ are the

Gell-Mann matrices e.g. $\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$

$$\lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix}$$

One can determine the generators as in example 2.

$SU(3)$ algebra: $[\hat{G}_a, \hat{G}_b] = i f_{abc} \hat{G}_c$

$$F_{123} = 1, \quad F_{147} = 1/2, \quad F_{458} = \frac{\sqrt{3}}{2} \quad \text{etc.}$$

More when we study QCD

Lie Groups (cont.)

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We've seen particular matrix representations of $SU(2)$ and $SU(3)$. One can use other representations: we just need to respect the group's element multiplication. For $SU(2)$,

the triplet representation, which will act on a triplet of fields, must respect the commutation relations. For example, one can write the matrices using $(T^i)_{jk} = -i\epsilon_{ijk}$. You can check that the 3×3 matrices satisfy the $SU(2)$ commutation relations.

Gauge Invariance and E and M. ①

Why "gauge"?

Weyl was looking for geometric basis for both E and M and gravity by considering a space-time dependent change of scale.

Field invariants: gauge or standard of calibration

consider $F(x)$ that changes between x_n and $(x_n + dx_n)$

if space is uniform

$$F(x + dx) = F(x) + \lambda^n F dx_n$$

What if the unit of measure or calibration changes in going from x_n to $(x_n + dx_n)$?

$$F(x + dx) = (F(x) + \lambda^n F dx_n) \cdot (1 + S^v dx_v)$$

(2)

$$F(x+dx) = (F(x) + \lambda^m F dx_m) \cdot (1 + S^V dx_V) \\ = F(x) + (\lambda^m F(x) + F(x) S^m) dx_m + O(dx)^2$$

$$\Delta F = (\lambda^m + S^m) F dx_m \quad (\text{First order})$$

Modified differential operator

For E and M

$$P_m = (E, p_x, p_y, p_z)$$

$$\text{From QM } p^m \rightarrow i\lambda^m \quad (i\lambda^0, -iV)$$

$$(p^m - eA^m) \rightarrow i(\lambda^m + ieA^m)$$

$$\text{if } S^m \rightarrow ieA^m$$

$$(1 + ieA^m dx_m) \approx e^{ieA^m dx_m}$$

\rightarrow invariance under change of phase (Keft gauge invariance)

Phase invariance in QM

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$$\langle 0 \rangle = \int \psi^* O_{op} \psi$$

→ invariant under overall phase $\psi(x) \rightarrow e^{i\theta} \psi(x)$

but, relative phases do matter

Can we formulate QM with position dependant (local) phase rotations?

$$\psi(x) \rightarrow \psi'(x) = e^{i\alpha(x)} \psi(x)$$

yes, but at a price

$$\partial_m \psi \rightarrow \partial_m \psi' = e^{i\alpha(x)} [\partial_m \psi(x) + i (\partial_m \alpha(x)) \psi(x)]$$

picked up extra term ...

replace $\partial_m \rightarrow D_m \equiv \partial_m - ieA_m$

with $A_m(x) \rightarrow A'_m(x) = A_m(x) + \frac{1}{e} \partial_m \alpha(x)$

(4)

then

$$D_n \psi(x) \rightarrow e^{i\alpha(x)} D_n \psi(x)$$

quantities like $\psi^\dagger D_n \psi(x)$ are invariant under local phase transformations

what did we do? introduced a current

had Free Field theory \rightarrow Turned on EM interactions

$i\bar{\psi} \gamma^\mu \rightarrow i\bar{\psi} \gamma^\mu + e A^\mu$ at the centre of QED

note that a mass Term $m A^\mu A_\mu$ would not preserve local gauge invariance

(5)

then

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Klein Gordon Equation

we can start with: $E^2 - p^2 c^2 = m^2 c^4$

$$p^m p_m - m^2 c^2 = 0$$

$$p^m \rightarrow i \hbar \partial^m \quad : \quad -\hbar^2 \partial^m \partial_m \psi - m^2 c^2 \psi = 0$$

$\hbar = c = 1$ From now on ...

$$(\partial_\mu \partial^\mu + m^2) \psi = 0$$

Turn on QED: $i \partial^m \rightarrow i \partial^m + e A^m$
 $\partial^m \rightarrow \partial^m - i e A^m$

$$= (\partial_\mu \partial^\mu + m^2) \psi = -V \psi$$

$$V = -ie (\partial_\mu A^m + A^m \partial_\mu) - e^2 A^2$$

$\alpha \approx \frac{e^2}{4\pi} \approx 1/137 \rightarrow$ small \rightarrow pert. theory
 lowest order, omit e^2 term

Non-Abelian gauge theories

For the Abelian case we studied, $U(1)$, we saw that imposing local gauge transformations:

$$Q(x) \rightarrow e^{i\alpha(x)} Q(x) \text{ required addition of gauge-}$$

covariant derivative to keep theory invariant

under those transformations: $D_\mu \equiv \partial_\mu + i g A_\mu(x)$

$$A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \alpha$$

We now look at an example of a non-Abelian gauge theory: $SU(2)$ -isospin gauge theory

$$\psi = \begin{pmatrix} \text{proton} \\ \text{neutron} \end{pmatrix}$$

Global symmetry \rightarrow freedom to choose what we call a proton and neutron everywhere

Non-Abelian gauge theories ⑧

Consider a local gauge transformation for the field $\psi(x)$:

$$\psi(x) \rightarrow \psi(x)' = G(x)\psi(x)$$

with $G(x) \equiv \exp\left(\frac{i}{2}\gamma \cdot \alpha(x)\right)$

γ are the Pauli matrices

$$\alpha \equiv \alpha_1, \alpha_2, \alpha_3$$

$$2\gamma \cdot \alpha \rightarrow G(D_n \psi) + \underbrace{(D_n G)} \psi$$

↳ To take care of this term, let's introduce a gauge covariant derivative:

$$D_n \equiv \partial_n + ig B_n$$

with $B_n = \frac{1}{2}\gamma^a b_n^a = \frac{1}{2} \begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix}$

↳ 2×2 matrix

$$b_n = (b_1, b_2, b_3)$$

Non-Abelian gauge theories (cont.) (9)

How does B_n need to transform to cancel extra terms?

We want $D_n \psi \rightarrow D'_n \psi' = \underbrace{G(D_n \psi)}$

$$D'_n \psi' = (\partial_n + i g B'_n) \psi'$$

$$= G(\partial_n \psi) + (\partial_n G) \psi + i g D'_n (G \psi)$$

$$\equiv G(\partial_n \psi + i g B_n \psi)$$

$$= G(\partial_n \psi) + i g G(B_n \psi)$$

$$\Rightarrow i g B'_n (G \psi) = i g G(D_n \psi) - (\partial_n G) \psi$$

multiply from the right by G^{-1}

$$\rightarrow B'_n = G B_n G^{-1} + \frac{i}{g} (\partial_n G) G^{-1}$$

$$= G [B_n + i g G^{-1} (\partial_n G)] G^{-1}$$

Non-Abelian gauge theories (cont.) (10)

$$B_\mu' = G [B_\mu + i G^{-1} (\partial_\mu G)] G^{-1} \quad (1)$$

Looks complicated... let's try it with $G = e^{i\eta(x)}$

$$\begin{aligned} \text{We set: } A_\mu' &= e^{i\eta(x)} [A_\mu + \frac{i}{g} e^{-i\eta(x)} \cdot i g \partial_\mu e^{i\eta(x)}] e^{-i\eta(x)} \\ &= A_\mu - \partial_\mu \eta \quad (\text{ok it works...}) \end{aligned}$$

Consider infinitesimal gauge Transformation ($|\eta| \ll 1$)

$$G = 1 + \frac{i}{2} \eta \cdot \alpha \quad \text{using (1) above we get:}$$

$$\begin{aligned} G B_\mu G^{-1} &= \left[B_\mu + \frac{i}{2} \eta \cdot \alpha B_\mu \right] \cdot \left[1 - \frac{i}{2} \eta \cdot \alpha \right] \\ &= B_\mu + \frac{i}{2} \eta \cdot \alpha B_\mu - \frac{i}{2} B_\mu \eta \cdot \alpha + O(\alpha^2) \end{aligned}$$

$$\frac{i}{g} (\partial_\mu G) G^{-1} = \left[\frac{i}{g} \frac{i}{2} \partial_\mu (\alpha \cdot \eta) \right] \cdot \left(1 - \frac{i}{2} \eta \cdot \alpha \right) = -\frac{1}{2g} \partial_\mu (\alpha \cdot \eta)$$

Non-Abelian gauge theories (cont.)

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we have

$$D_\mu' = B_\mu + \frac{i}{2} \alpha \cdot \tau B_\mu - \frac{i}{2} B_\mu \alpha \cdot \tau - \frac{1}{25} \partial_\mu (\alpha \cdot \tau) + \dots$$

now $B_\mu = \frac{1}{2} \tau \cdot b_\mu$ so we get:

$$\tau \cdot b_\mu' = \tau \cdot b_\mu + \frac{i}{2} (\tau \cdot \alpha \tau \cdot b_\mu - \tau \cdot b_\mu \alpha \cdot \tau) - \frac{1}{5} \partial_\mu (\alpha \cdot \tau)$$

in component form: $\frac{i}{2} \alpha^j b_\mu^k (\tau^j \tau^k - \tau^k \tau^j) = \frac{i}{2} \alpha^j b_\mu^k [\tau^j, \tau^k]$

For $SU(2)$: $[\tau^j, \tau^k] = 2i \epsilon_{jkl} \tau^l$

second term becomes: $- \epsilon_{jkl} \alpha^j b_\mu^k \tau^l = - \alpha \times b_\mu \cdot \tau$

isospin components are linearly indep. so:

$$\tau \cdot b_\mu' = \tau \cdot b_\mu - \alpha \times b_\mu \cdot \tau - \frac{1}{5} \partial_\mu (\alpha \cdot \tau) \text{ becomes:}$$

$$b_\mu'^k = b_\mu^k - \epsilon_{jkl} \alpha^j b_\mu^l - \frac{1}{5} \partial_\mu \alpha^k$$

Non-Abelian gauge theories (cont.)

(12)

Note: new Term $[\epsilon_{ijk} a^i b^j c^k]$ is picked because gauge Transformations do not commute

Now we focus on field strength Tensor. We would like it To Transform as:

$$F_{\mu\nu}' = G F_{\mu\nu} G^{-1}$$

if we try

$$\partial_\nu B_\mu' - \partial_\mu B_\nu' \text{ for } F_{\mu\nu}'$$

we do not get

$$G (\partial_\nu B_\mu - \partial_\mu B_\nu) G^{-1}$$

Are we missing a commutator again?

$$F_{\mu\nu} \rightarrow F_{\mu\nu}^a \gamma^a$$

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} = -\frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu})$$

$$\text{and } \text{Tr} (\gamma^a \gamma^b) = 2 \delta^{ab}$$

Non-Abelian gauge theories (cont.)

(13)

if we try $F_{\mu\nu} = \frac{1}{ig} [D_\nu, D_\mu]$ for QED:

$$F_{\mu\nu} = \frac{1}{ig} [(2\nu + igA_\nu), (2\mu + igA_\mu)] \\ = \frac{1}{ig} [2\nu A_\mu - 2\mu A_\nu + ig \underbrace{[A_\nu, A_\mu]}_{=0}]$$

$$F_{\mu\nu} = \frac{1}{ig} [D_\nu, D_\mu] = 2\nu B_\mu - 2\mu B_\nu + ig [B_\nu, B_\mu]$$

$$GF_{\mu\nu} G^{-1} = F_{\mu\nu}'$$

Yang-Mills Lagrangian: $\mathcal{L} = \frac{1}{2} (\dot{\phi}^a)^2 - \frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu}$

in compact form: $F_{\mu\nu}^a = 2\nu B_\mu^a - 2\mu B_\nu^a + g \epsilon_{ijk} b_\mu^i b_\nu^j b_\mu^k$

$$b_\mu^i = b_\mu^a - \frac{1}{g} \partial_\mu a^a - \epsilon_{ijk} a^j b_\mu^k$$

For other groups, we'll replace ϵ_{ijk} by the group's structure constants f_{ijk}

Non-Abelian gauge theories (cont.)

$$g_1 = 1 + iw_1^2 T_2 \quad g_2 = 1 + iw_2^2 T_2, \quad g_1 g_2 = 1 + i(w_1^2 + w_2^2) T_2 + \dots$$

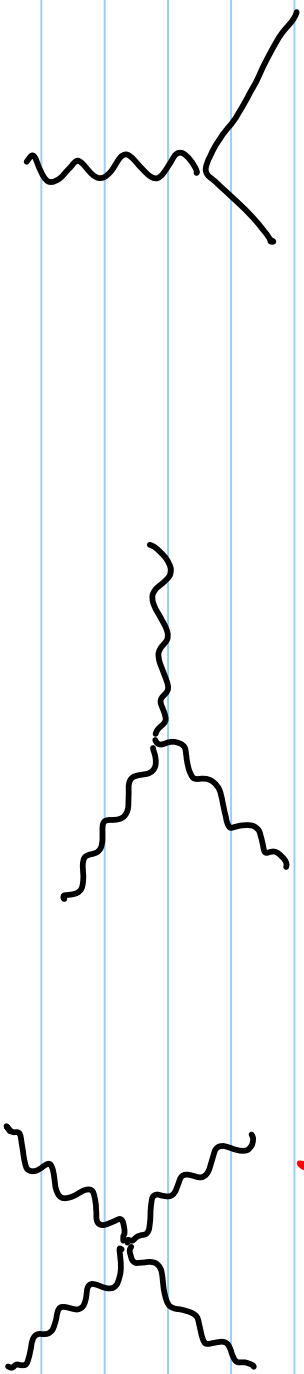
$$S_1 S_2 (S_2 S_1) = (1 + iw_1^2 T_2) (1 + iw_2^2 T_2) (1 - iw_2^2 T_2) (1 - iw_1^2 T_2) \\ = 1 - w_1^2 w_2^2 [T_2, T_2], \quad [T_2, T_2] = i f_{22}^r T_r$$

see Appendix B of Burgess and Moore

Also note that the non-Abelian theory has additional couplings:

$$F_2 (2B - 2B) + g_{BB}$$

$$f_2 (2B - 2B) + \underbrace{g (2B - 2B) BB}_{\text{propagator}} + \underbrace{g^2 BB BB}_{\text{triple}} + \underbrace{g^2 BB BB}_{\text{quartic}}$$



U(1) and SU(2) recap

(15)

$$U(1): \quad \varphi \rightarrow \varphi' = \exp(i\alpha(x)) \varphi$$

$$SU(2) \quad \varphi \rightarrow \varphi' = \exp\left(\frac{i}{2} \vec{\tau} \cdot \vec{\alpha}(x)\right) \varphi$$

$\hookrightarrow \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3$
 $\hookrightarrow 3$ Pauli Matrices

Both cases: $D_\mu \varphi \rightarrow G(D_\mu \varphi) + \underbrace{(D_\mu G)\varphi}$

To take care of extra Term introduce gauge covariant derivative:

$$U(1): \quad D_\mu = \partial_\mu + ieA_\mu$$

$$SU(2): \quad D_\mu = \partial_\mu + ig B_\mu \quad B_\mu = \frac{1}{2} \vec{\tau} \cdot \vec{b}_\mu \quad \hookrightarrow b_\mu^1, b_\mu^2, b_\mu^3$$

Both cases: $C'_\mu = G C_\mu G^{-1} + \frac{i}{g} (D_\mu G) G^{-1}$

$$U(1): \quad A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \alpha$$

$$SU(2): \quad b_\mu^1 \rightarrow b_\mu^2 = b_\mu^1 - \alpha \times b_\mu - \frac{1}{g} \partial_\mu \alpha$$

or $b_\mu^1 = b_\mu^2 + \alpha \times b_\mu - \frac{1}{g} \partial_\mu \alpha$

As an aside: Curved space, GR etc.

Covariant derivative:

$$\frac{\partial \vec{V}}{\partial x^b} = \frac{\partial V^\alpha}{\partial x^b} \vec{e}_\alpha + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^b}$$

we define $\frac{\partial \vec{e}_\alpha}{\partial x^b} = \Gamma_{\alpha b}^\mu \vec{e}_\mu$

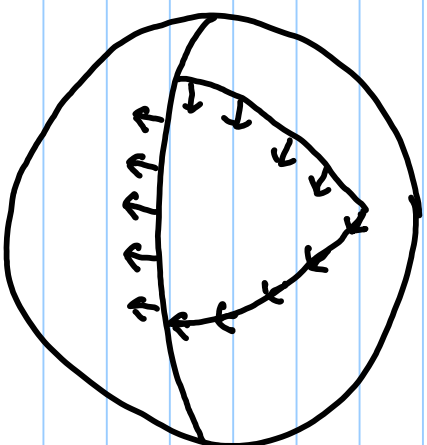
$\Gamma_{\alpha\beta}^\alpha$: Christoffel symbol, α^{th} component of $\frac{\partial \vec{e}_\beta}{\partial x^\alpha}$ (vector)

So we have:

$$\frac{\partial \vec{V}}{\partial x^b} = \frac{\partial V^\alpha}{\partial x^b} + \Gamma_{\alpha b}^\alpha V^\alpha$$

Parallel Transport

Curvature defined as the change of a vector that is "parallel Transported" along a closed loop



$$[\nabla_\alpha, \nabla_\beta] V^\mu = R_{\alpha\beta}^\mu V^\mu$$

→ Riemann curvature Tensor