LECTURE 10: Move Wave Packets and The Heisenberg Uncertainty Principle

Goal of the lecture: Complete our introductory discussion of wave packets and introduce Heisenberg's Uncertainty Principle

What I expect you to learn:

- How wave packets propagate in time
- How to do calculations with a Gaussian wave packet
- What is Heisenberg's Uncertainty Principle

(This roughly corresponds to section 2.4 and 2.5 of textbook)

Recap of Wave Functions + Fourier Transforms

$$\underset{(3)}{\psi(x,t)} = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \underset{(1)}{e^{-i(p\cdot x - E\cdot t)/\hbar}} \; \underset{(2)}{\varphi(p,t)} \, dp \qquad (1)$$

(1) a plane wave of wave number $K = \frac{p}{\hbar}$ and of frequency $\omega = \frac{E}{\hbar}$

(2) Amplitude for plane wave with momentum $p = \hbar K$

(3) a - Sum of plane waves having momenta between $-\infty$ and ∞. Amplitudes of plane waves given by function $\varphi(p,t)$

b - probability amplitude for finding particle a position x at time t

Recap of Wave Functions + Fourier Transforms

$$\varphi(p,T) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \Psi(x,T)\, dx \qquad (2)$$

(2) represents

a: probability amplitude for finding particle with momentum p at time T

b: the amplitude function of (1)

c: Fourier transform of (1)

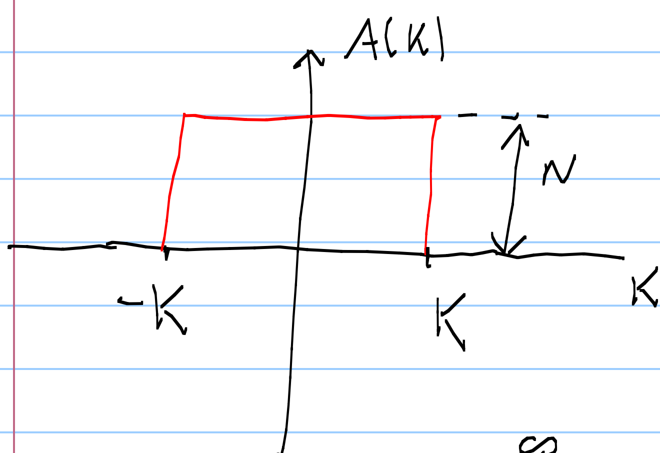
We saw examples in lecture 9 of how to calculate $\Psi(x)$ if we have $\varphi(p)$ and vice versa. **We continue with these today**

Then we'll turn to another important example: the Gaussian wave packet.

Fourier Transform Examples

Example 1:

We have a wave packet for which:



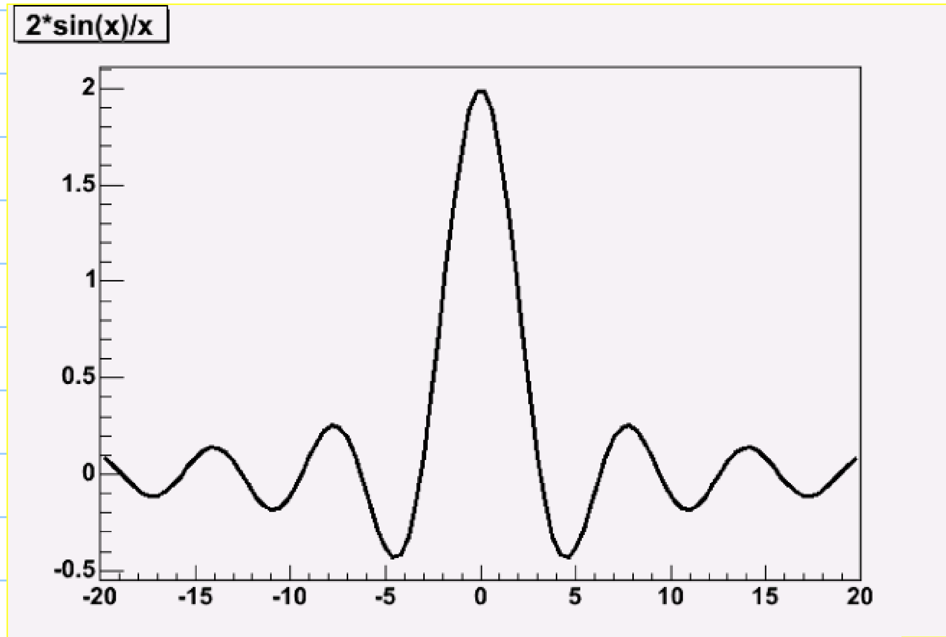
$A(k) = N$ for $-K \le k \le K$

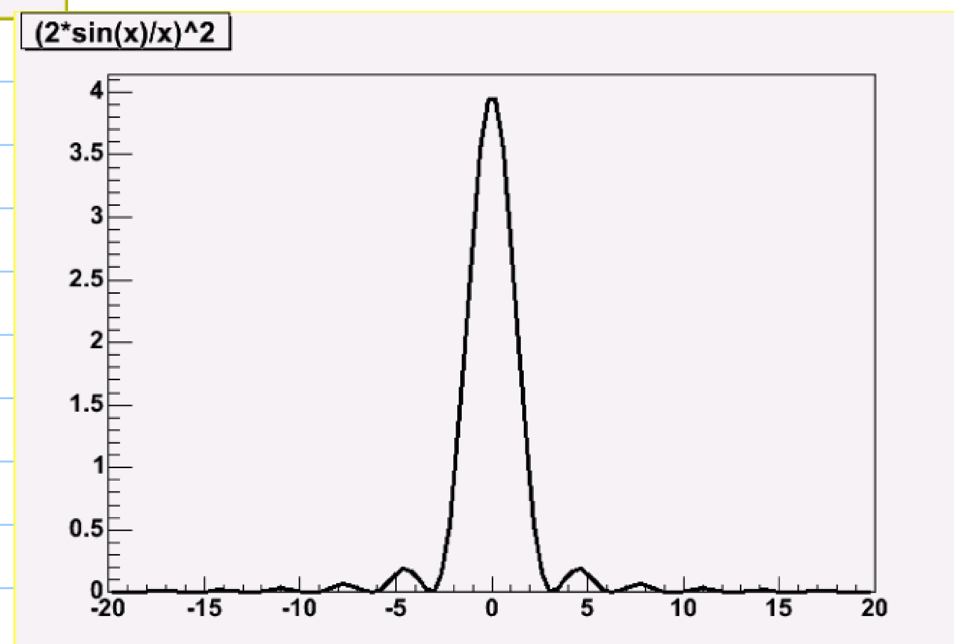
$= 0 \qquad k > |K|$

→ let's calculate $\psi(x,0)$

$$\psi(x,0) = \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk = \int_{-K}^{+K} N e^{ikx}\, dk$$

$$= \frac{N}{ix}\left(\underbrace{e^{iKx}}_{\text{big } K} - \underbrace{e^{-iKx}}_{\text{big } K}\right) = 2N\,\frac{\sin Kx}{x} :$$


2*sin(x)/x
2
1.5
1
0.5
0
-0.5
-20
-15
-10
-5
0
5
10
15
20


(2*sin(x)/x)^2
4
3.5
3
2.5
2
1.5
1
0.5
0
-20
-15
-10
-5
0
5
10
15
20

Fourier Transform Examples

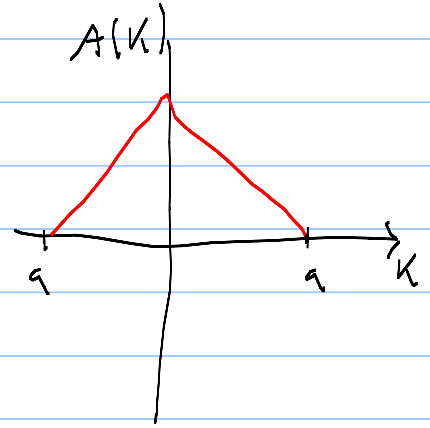
Example 2:

We have a wave packet for which:

$$A(K) = \begin{cases} A \cdot (a - |K|) , & |K| \leq a \\ 0 , & |K| > a \end{cases}$$

real

positive



$\rightarrow$ Find A such that $A(K)$ is normalised:

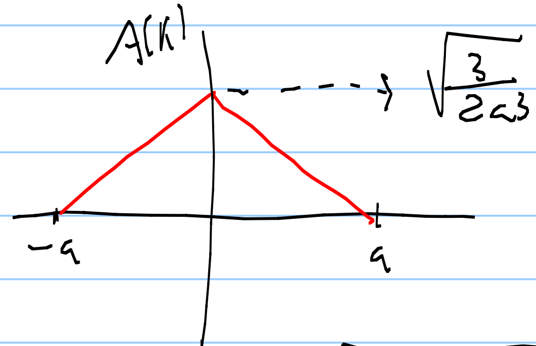
$$1 = \int_{-\infty}^{\infty} |A(K)|^2 \, dK = A^2 \int_{-a}^{0} (a+K)^2 \, dK + A^2 \int_{0}^{a} (a-K)^2 \, dK$$

$$= 2A^2 \int_0^a (a-K)^2 \, dK = 2A^2 \int_0^a (a^2 - 2aK + K^2) \, dK$$

$$= 2A^2 \cdot \left(a^2 K - a K^2 + \frac{K^3}{3} \right) \Bigg|_0^a = 2A^2 \cdot \left(a^3 - a^3 + \frac{a^3}{3} \right) = \frac{2}{3} a^3 \cdot A^2$$

Example 2 continued:

$$1 = \frac{2a^3}{3} A^2 \Rightarrow A = \sqrt{\frac{3}{2a^3}}, \quad A(K) = \sqrt{\frac{3}{2a^3}}\,(a - |K|)$$



The Transform is then:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(K)\, e^{iKx}\, dK$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{3}{2a^3}} \left[\int_{-a}^{0} (a+K)\, e^{iKx}\, dK + \int_{0}^{a} (a-K)\, e^{iKx}\, dK \right]$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{3}{2a^3}} \left[\underbrace{\int_{-a}^{0} K\, e^{iKx}\, dK}_{①} - \underbrace{\int_{0}^{a} K\, e^{iKx}\, dK}_{②} + \underbrace{a \int_{-a}^{a} e^{iKx}\, dK}_{③} \right]$$

① $= \frac{a}{ix} e^{-iax} + \frac{1}{x^2}\left(1 - e^{-iax}\right)$

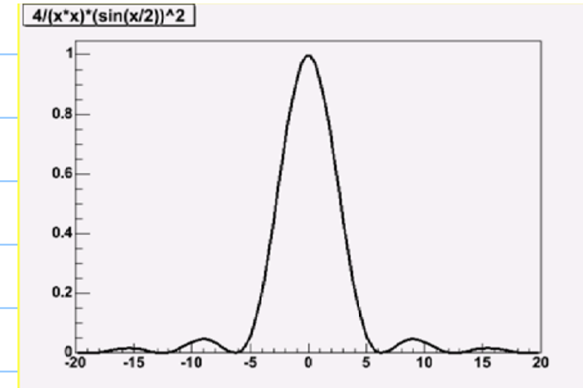
② $= \frac{a}{ix} e^{iax} + \frac{1}{x^2}\left(e^{iax} - 1\right)$

$-$② $= -\frac{a}{ix} e^{iax} + \frac{1}{x^2}\left(1 - e^{iax}\right)$

③ $= \frac{a}{ix}\left(e^{iax} - e^{-iax}\right)$

Example 2 continued:

$$① + ② + ③ = \frac{4}{x^2} \sin^2\left(\frac{ax}{2}\right)$$



Example 3:

Normalise $A(k)$ with $A(k) = A \exp\left[-a^2 (k-k_0)^2/4\right]$

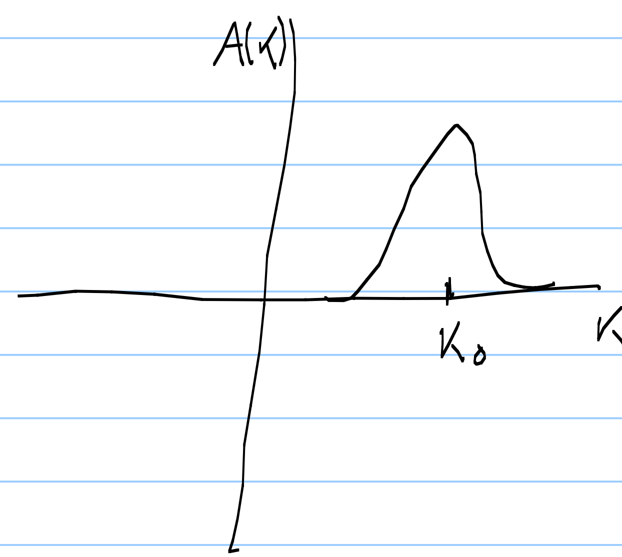
First let's find A: $1 = \int_{-\infty}^{\infty} |A(k)|^2 \, dk$

$$= |A^2| \int_{-\infty}^{\infty} dk \exp\left[-\frac{a^2}{2}(k-k_0)^2\right]$$

with $z = k - k_0$, we have

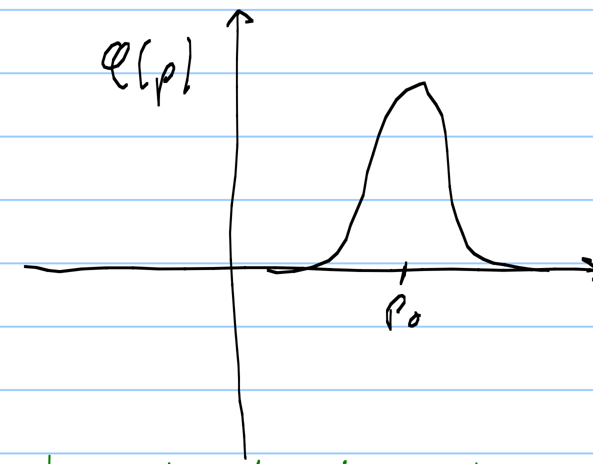
$$\int_{-\infty}^{\infty} e^{-a^2 z^2/2} \, dz = \sqrt{\frac{2\pi}{a}}$$

$$\Rightarrow A = \left[a^2/2\pi\right]^{1/4}$$



THE GAUSSIAN WAVE PACKET

$$\varphi(p) = C \exp\left[-\frac{(p-p_0)^2}{2(\Delta p)^2} \right]$$



Δp represents the width of the Gaussian distribution. We define it such that $|\varphi(p)|^2$ is $1/e$ of its value when $p = p_0 \pm \Delta p$ → standard deviation "σ"

Let's calculate C:

$$\int_{-\infty}^{\infty} |\varphi(p)|^2 \, dp = 1 \Rightarrow |C|^2 \int_{-\infty}^{\infty} \exp\left[\frac{-(p-p_0)^2}{\Delta p^2} \right] dp = 1$$

We have the formula: $$\int_{-\infty}^{\infty} e^{-\alpha u^2} e^{-\beta u} \, du = \sqrt{\frac{\pi}{\alpha}} \, e^{\beta^2/4\alpha}$$

with $u = p - p_0$, $du = dp$, $\alpha = \frac{1}{\Delta p^2}$, $\beta = 0$

we get: $|C|^2 \sqrt{\pi \Delta p^2} \Rightarrow C = (\pi)^{-1/4} \Delta p^{-1/2}$

THE GAUSSIAN WAVE PACKET (CONT.)

we have: $\varphi(p) = (\pi)^{-1/4} \Delta p^{-1/2} \exp\left[-\frac{(p-p_0)^2}{2(\Delta p)^2} \right]$

For $T=0$, $\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi(p)\, e^{ipx/\hbar}\, dp$

$$\Rightarrow \psi(x) = \frac{(\pi)^{-1/4} \Delta p^{-1/2}}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} \exp\left[\frac{-(p-p_0)^2}{2(\Delta p)^2} + \frac{ipx}{\hbar} \right] dp$$

$$= -\left[\frac{1}{\sqrt{2}\Delta p}(p-p_0) - \frac{ix\Delta p}{\hbar\sqrt{2}} \right]^2 - \frac{x^2 \Delta p^2}{\hbar^2 \, 2} + ixp_0$$

with $y = \frac{1}{\sqrt{2}\Delta p}(p-p_0) - \frac{ix\Delta p}{\hbar\sqrt{2}}$, $dy = \frac{dp}{\sqrt{2}\Delta p}$

$$\Rightarrow \psi(x) = \frac{(\pi^{-1/4})\Delta p^{-1/2}}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-y^2} e^{ixp_0} e^{-\frac{x^2 \Delta p^2}{2\hbar^2}} \cdot dy \cdot \sqrt{2}\Delta p$$

Gaussian Packet (cont.)

$$\psi(x) = \frac{\sqrt{\Delta_p}\cdot\pi^{-1/4}}{\sqrt{\pi\hbar}}\, e^{ixp_0}\, e^{-\frac{x^2\Delta p^2}{2\hbar^2}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2}\,dy}_{=\sqrt{\pi}}$$

$$\psi(x) = \frac{\pi^{-1/4}}{\sqrt{\hbar}}\sqrt{\Delta_p}\; e^{ip_0x}\; e^{-\frac{x^2\Delta_p^2}{2\hbar^2}}$$

$\psi(x)$ is a Gaussian distribution times a phase factor e^{ip_0x}

$\hookrightarrow$ this phase goes away when we square $\psi(x)$

- $|\psi(x)|^2$ will peak at $x=0$
- The width of $|\psi(x)|^2$ (as defined before) is $\boxed{\Delta x = \frac{\hbar}{\Delta p}}$.

$\rightarrow$ For this $\varphi(p)$, $\psi(x)$ we have

$$\boxed{\Delta x\Delta p = \hbar}$$

Gaussian Wave packet (cont)

$$\psi(x) = \frac{\pi^{-1/4}}{\sqrt{\hbar}} \sqrt{\Delta_p}\; e^{ip_0 x}\; e^{-\frac{x^2 \Delta_p^2}{2\hbar^2}}$$

$\Delta x \Delta p = \hbar$ implies that the is a limit to how well I can know Δx and Δp at a given Time

→ If I reduce Δp, Δx will increase

→ If I reduce Δx, Δp will increase

→ in the limit where $\Delta p = \delta(p)$ $\psi(x)$ becomes a plane wave: $\Delta x \to \infty$

From math standpoint: if the "interval of amplitudes" is small, The resulting sum will have large spacial extension

GAUSSIAN WAVE PACKETS (cont)

- we've dealt with the case of $t=0$
  we can generalise $\varphi(p)$ to

$$\varphi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar}\, \psi(x,t)\, dx$$

Fourier Transform of $\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi(p,t)\, e^{-ipx}\, dp$

momentum space wave function

How does our Gaussian wave packet evolve in time?

Recall our Taylor expansion:

$$E(p) = E(p_0) + (p-p_0)\left.\frac{dE(p)}{dp}\right|_{p=p_0} + \frac{1}{2}(p-p_0)^2 \left.\frac{d^2E(p)}{dp^2}\right|_{p=p_0} + \dots$$

$$= \frac{p_0^2}{2m} + (p-p_0)\frac{p_0}{m} + \frac{(p-p_0)^2}{2m} + \dots$$

neglect: $\Delta p^2 t \ll 1$

Gaussian Wave Packets (cont)

it gave us:

$$\psi(x,t) = \underbrace{e^{\frac{i}{\hbar}(p_0 x - E(p_0)t)}}_{\text{plane wave}} \cdot \underbrace{\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{i(p x - p_0)(x - v_g t)/\hbar} \varphi(p)\,dp}_{\text{envelope function}}$$

Let's now include the quadratic term we neglected

$$\psi(x,t) = \underbrace{e^{i/\hbar (p_0 x - E(p_0)t)} \frac{1}{2\pi\hbar}}_{=C} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}(p-p_0)(x-v_g t)} e^{-i(p-p_0)^2 \underset{\text{constant}}{\alpha} t} \varphi(p)\,dp$$

Let's set $q = (p - p_0)$

$$\psi(x,t) = C \int_{-\infty}^{\infty} e^{\frac{i}{\hbar} q(x - v_g t)} \underbrace{e^{-iq^2 \alpha t}} \varphi(p)\,dp \qquad (3)$$

↳ Affects width of wave packet. Without it, wave packet does not change shape:

$$\psi(x,t) = \psi_0(x - v_g t)$$

GAUSSIAN WAVE PACKETS (cont)

We won't integrate (3) here but I invite you to give it a try e.g. lift some weights at the math gym...

the answer for $|\psi(x,t)|^2$ looks like this:

$$|\psi(x,t)|^2 = \frac{a_1}{\Delta x(t)} \exp\left[\frac{-a_2(x - v_g t)^2}{\Delta x(t)^2}\right]$$ with

$\Delta x(t) = a_3\sqrt{1 + a_4 t^2}$, the a_n are constants

→ wave packet moves with velocity v_g

→ wave packet width increases with time

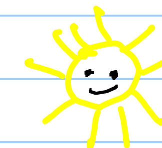
→ wave packet undergoes <u>dispersion</u>

Note: for photons: $E = pc$, $\frac{dE}{dp} = v_g = c$, $\frac{E}{p} = v_p = c$

⇒ there is no dispersion for photons (in vacuum)

Gaussian Wave Packets (cont)

Back To our Gaussian wave packet:



Integration yields:

$$\psi(x,T) = \pi^{-1/4}\left[\frac{\Delta p/\hbar}{1+i\Delta_p^2 T/m\hbar}\right]^{1/2}\cdot \exp\left(\frac{ip_0x/\hbar-(\Delta p/\hbar)^2x^2/2-ip_0^2T/2m\hbar}{1+i\Delta_p^2T/m\hbar}\right)$$

$$|\psi(x,T)|^2 = \frac{1}{\sqrt{\pi}}\,\frac{\Delta p/\hbar}{[1+\Delta_p^4T^2/m^2\hbar^2]^{1/2}}\cdot \exp\left[\frac{(\Delta p/\hbar)^2(x-v_gt)^2}{1+(\Delta p)^4t^2/m^2\hbar^2}\right]$$

The width is given by:

$$\Delta x(t) = \frac{\hbar}{\Delta p}\left[1+\frac{(\Delta p)^4t^2}{m^2\hbar^2}\right]^{1/2} \qquad (4)$$

→ if Δp is small or m is large, Δx will evolve slowly

Gaussian Wave Packets (cont)

Let's plug some numbers into (4)

1- electron whose position is known to within 10^{-10} m: packet will have doubled in size in 10^{-16} sec

2- particle of 1 gram with position known to within $1\,\mu m$: packet will have doubled in size in 10^{19} sec. $\longrightarrow$ > age of universe

Note that once a measurement is performed, the wave packet "collapses" and Δx is then determined by precision of measurement.

→ more on this in a couple of lectures

THE HEISENBERG UNCERTAINTY PRINCIPLE

We've seen that for Gaussian wave packets, the product of the width of the packet and the width of the distribution of the momenta has to be greater than h

From a mathematical standpoint, this is related to a property of Fourier transforms: greater width of momentum distribution for the waves implies smaller width of the distribution of the sum of waves

From a physical standpoint: a particle's position and its momentum cannot be known simultaneously to arbitrary accuracy. This is more than a statement on the practical limits of experimental apparatus: it's an inherent property of nature

Also note that the Uncertainty Principle does not limit the precision with which, say, momentum can be measured, it is a statement about both x and p.