LECTURE 11: More on Heisemberg's Uncertainty Relations

Goal of the lecture: Complete discussion of Heisenberg's Uncertainty Principle

What I expect you to learn:

- What is the uncertainty relation for time and energy and how it differs from the position-momentum uncertainty
- How this principle applies to examples in the notes

(Corresponds to section 2.4 and 2.5 of textbook)

Recap on the Uncertainty relation for position and momentum

We've seen that for Gaussian wave packets, the product of the width of the packet and the width of the distribution of the momenta has to be greater than h

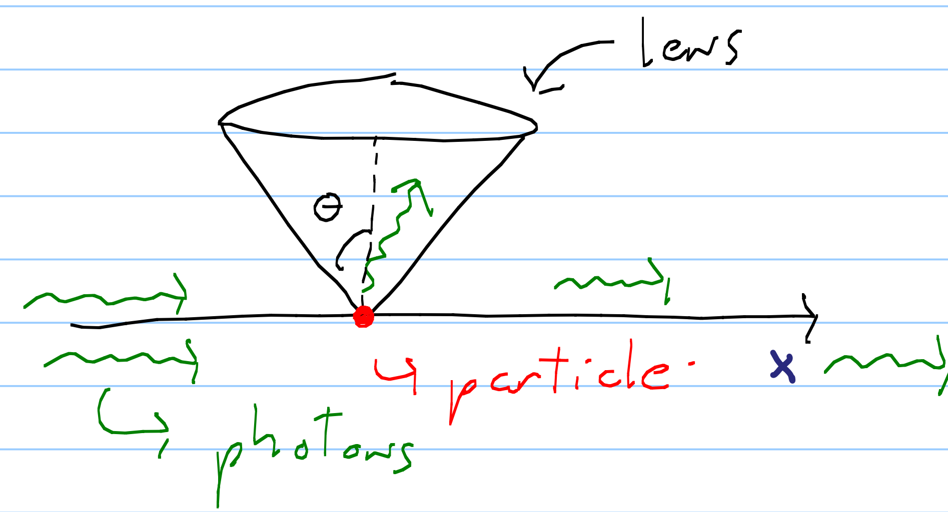
From a mathematical standpoint, this is related to a property of Fourier transforms: greater width of momentum distribution for the waves implies smaller width of the distribution of the sum of waves

From a physical standpoint: a particle's position and its momentum cannot be known simultaneously to arbitrary accuracy. This is more than a statement on the practical limits of experimental apparatus: it's an inherent property of nature

Also note that the Uncertainty Principle does not limit the precision with which, say, momentum can be measured, it is a statement about both x and p.

THE HEISENBERG UNCERTAINTY PRINCIPLE (cont.)

the γ-ray microscope:



λ : wavelength of photons

x position of particle can be resolved with precision:

$$\Delta x = \frac{\lambda}{\sin\theta} \quad (5)$$

→ recoil of particle of comparable magnitude to photon $= \frac{h}{\lambda}$

→ angle of scattered photon not known exactly

$$\Delta p_x \simeq h/\lambda \sin\theta \quad (6)$$

(5) and (6) give $\Delta x \, \Delta p_x \sim h$

Some microscopes:

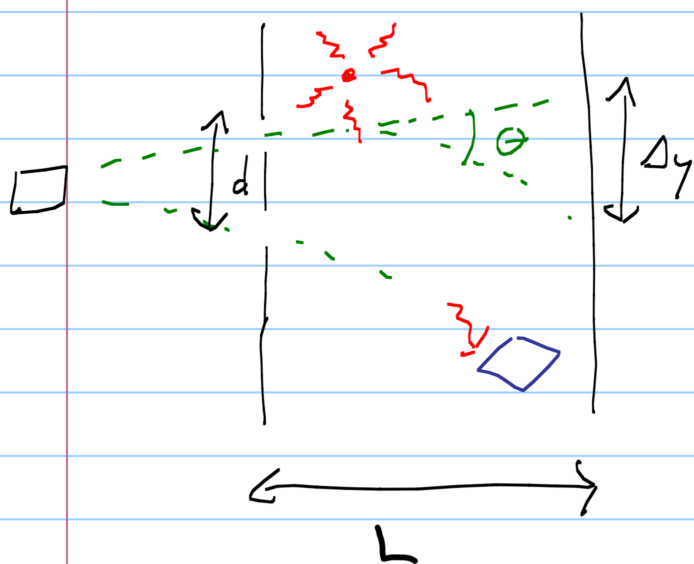
ordinary optical microscope

electron microscopes

$p\bar{p}$ collider

THE HEISENBERG UNCERTAINTY PRINCIPLE (cont.)

the double-slit experiment



- wavelength of light must be smaller than d,

$$\Rightarrow p_\gamma > \frac{h}{d}$$

$$(\text{or } \lambda < d)$$

electron will recoil and momentum transfer will be of order p_γ

$$\Rightarrow \Delta p \gtrsim \frac{h}{d} \quad \rightarrow \quad d\,\Delta p \gtrsim h$$

Electron will change by $\Delta\theta \simeq \frac{\Delta p}{p} \simeq \frac{h}{pd} \simeq \frac{\lambda_e}{d}$

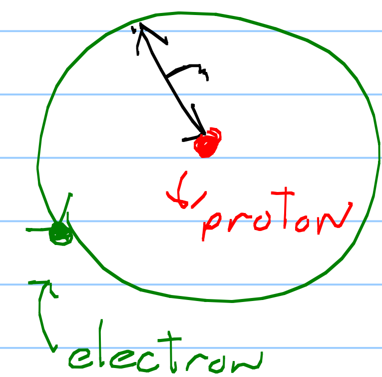
$$\Delta y = L \sin\theta \simeq L\,\Delta\theta = \boxed{\frac{L\lambda_e}{d}}$$

$\Rightarrow$ comparable to distance between maxima and minima

THE HEISENBERG UNCERTAINTY PRINCIPLE (cont.)

The stability of atoms:

Hydrogen atom: electron in "orbit" around proton



→ CONFINED TO BE WITHIN Δr OF THE PROTON

we use $\Delta r \Delta p \gtrsim \hbar$ and $E = \frac{p^2}{2m} - \frac{e^2}{(4\pi\varepsilon_0) r}$

substitute

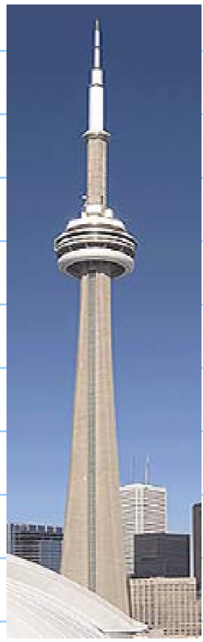
$\rightarrow \frac{\hbar^2}{2mr^2} - \frac{e^2}{(4\pi\varepsilon_0) r}$, minimum energy wrt r will be

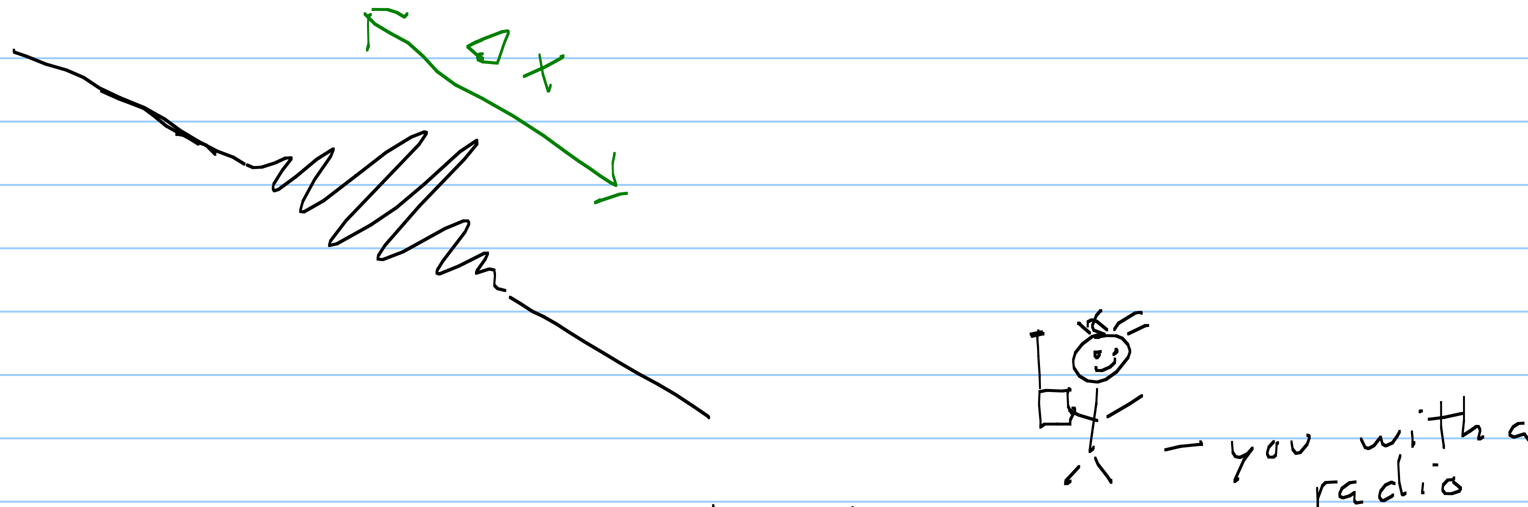
$\frac{dE}{dr} = 0$, $\frac{dE}{dr} = -\frac{\hbar^2}{mr^3} + \frac{e^2}{4\pi\varepsilon_0 r^2} \Rightarrow \frac{\hbar^2}{mr} = \frac{e^2}{4\pi\varepsilon_0}$

$\boxed{\Rightarrow r = \frac{4\pi\varepsilon_0 \hbar^2}{m e^2} = r_0}$ we found using Bohr's postulates!

UNCERTAINTY RELATION FOR TIME AND ENERGY

Consider the case of a radio Tower emitting em waves for a short time





Emits burst during interval ΔT

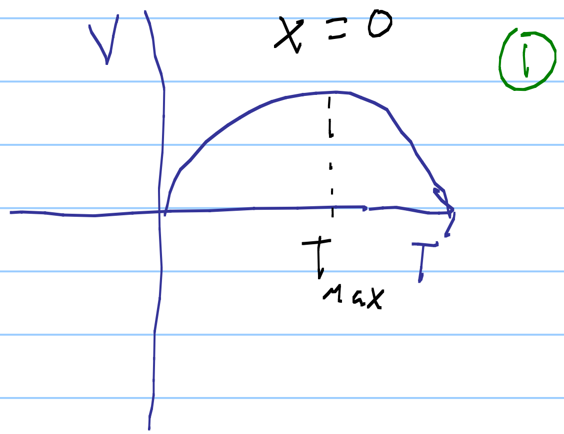
→ the short burst of radio waves will be contained within interval ΔT in time and $c\,\Delta T$ in space i.e. $\Delta x = c\,\Delta T$

→ the Δk interval can be converted into $\Delta \omega$ through the relation $c = \frac{\omega}{k} = \frac{\Delta \omega}{\Delta k}$ for em waves. We have: $\frac{\Delta \omega}{c} \cdot \Delta T c > \hbar$, $\boxed{\Delta \omega \Delta t > \hbar}$

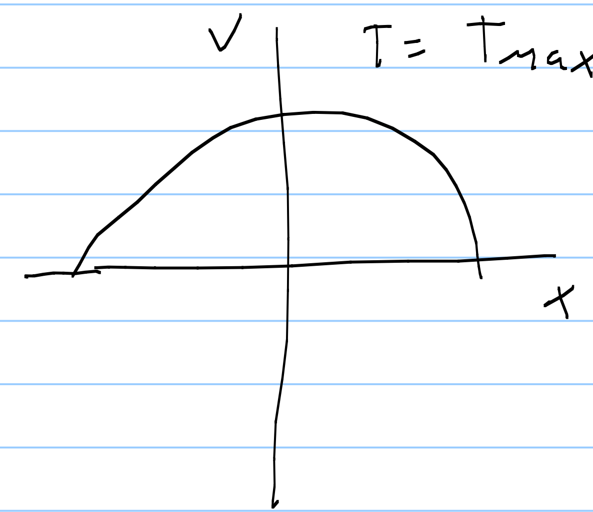
Uncertainty Relation for Time (cont.)

Although the previous example was done with Em waves, the result holds for matter waves. why?

Another way to picture what is going on: consider a half wave in time:



in space:



- For ②, we can obtain $A(K)$ using Fourier Transforms.
- The equivalent of $A(K)$ for ① is $A'(\omega)$:

$$\frac{\Delta \omega}{\Delta K} = V_g, \quad \Delta X = V_g \Delta T, \quad \Delta x \Delta K = \Delta \omega \Delta T$$

or

$$\Delta x \Delta p = \Delta E \Delta T$$

The Uncertainty Relation for Time and Energy (cont.)

The Fourier transforms of the wave function at a fixed position in space can be written as:

$$\Psi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\omega) e^{-i\omega t} \, d\omega$$

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(t) e^{i\omega t} \, dt$$

The wave function expressed as a pulse or "time packet" is therefore expressed as a superposition of monochromatic waves of frequency ω.

Note that the interpretation is different than in the position-momentum relation: the relation implies that if certain quantum states exist for a certain time, then their energy cannot be determined with arbitrary precision.

Also note that this is a probabilistic statement: nothing prevents the observer from determining the energy of one of these states to a high degree of precision.

The Uncertainty Relation for Time and Energy (cont.)

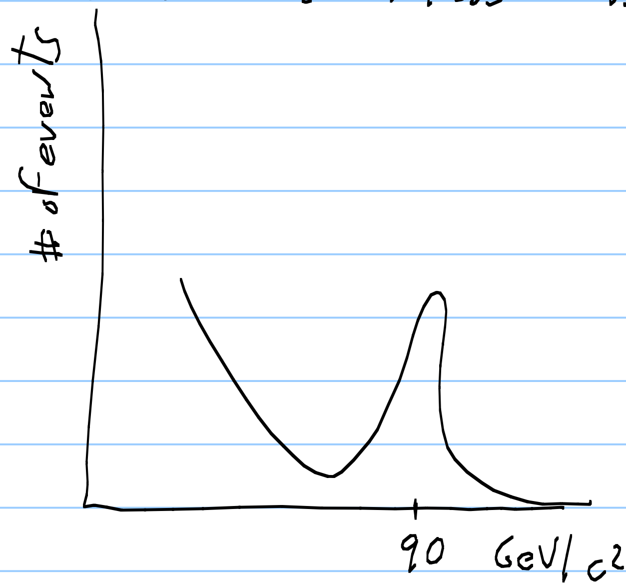
Example: atomic transitions and decay of subatomic particles

$\Delta E_b = \frac{\hbar}{\tau_b}$, τ : lifetime of atomic state

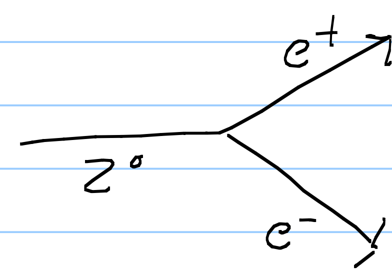
$$\Rightarrow \Delta E_{ab} = \hbar \left(\frac{1}{\tau_a} + \frac{1}{\tau_b} \right)$$

The width of a subatomic particle is related to its lifetime

lifetime of proton $> 10^{33}$ years! $\Delta E \Delta t \geq \hbar$ implies that its mass is very peaked about average value



Z Boson: lifetime $\sim 10^{-25}$ sec
mass has a measurable width:
$\sim 2\%$ of its mass



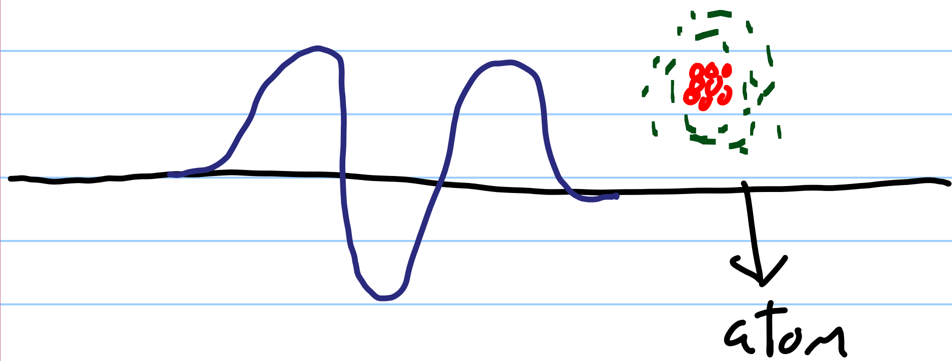
Other implications of $\Delta E \Delta t \gtrsim \hbar$

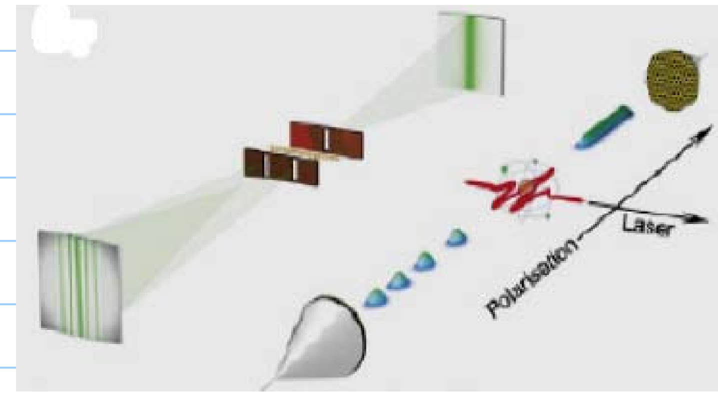
- Field theory description of particle interactions
    - we interact with a whole "zoo" of particles
    - also Casimir effect
- Quantum vaccum is where you'll find most violent physics
    - Planck mass, energy, time, length
    - Energy density of the Universe
    - Quantum gravity in the lab?

Double Slit Experiment in Time

Experiment by G. Paulus et al.

Use a short pulse:





Pulse can ionise an electron which is then collected by a detector which measures time of arrival.

Time of arrival (Energy) shows interference pattern