LECTURE 17: One-Dimensional Problems: The Constant Potential and The Potential Step

Goals of the 1-d lectures: learn how to solve Schrodinger's equation for some simple problems

What I expect you to learn:

- How we interpret the free particle wave-function

- How to solve the potential step problem

(Roughly corresponds to sections 4.1-4.3 of textbook)

Midterm: Wednesday Oct 25th at 11:00 in CLASS

NOTE: No tutorials next week but Rob will give an exam review: Monday MP 137 (18:00-20:00)

$$\boxed{V(x) = V_0}$$

With $V(x) = V_0$, we have $\frac{dV(x)}{dx} = F = 0$

→ a free particle

WE CAN GET THE SAME PHYSICS IF WE CHANGE THE VALUE OF V_0 (i.e we "re-gauge" V_0 → RELATED TO "GLOBAL GAUGE INVARIANCE"). SO, FOR SIMPLICITY, WE SET $V_0 = 0$.

SCHRÖDINGER'S THEN READS

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

we know that : $k = \left(\frac{2m}{\hbar^2} E\right)^{1/2}$

$$\rightarrow \frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0$$ (we've seen this before)

the general solution is : $A e^{ikx} + B e^{-ikx}$

or $A \cos kx + B \sin kx$

$\boxed{V(x) = V_0 \quad (\text{cont})}$

note that :

- k must be real → why?
- → $E \geq 0$
- E is continuous (not quantised as in the infinite well)

- there are two solutions that satisfy $E = \frac{\hbar^2 k^2}{2m}$ → corresponds e^{ikx} and e^{-ikx},
  → "doubly degenerate"

Momentum Eigenfunctions:

NOTE THAT WITH $k = p/\hbar$ WE HAVE THAT:

$$p_{op} \psi(x) = p\, \psi(x) \quad \longrightarrow \quad -i\hbar \frac{\partial}{\partial x} \psi(x) = p\, \psi(x)$$

$\boxed{p}$ is continuous eigenvalue of operator $\boxed{-i\hbar \frac{\partial}{\partial x}}$

$$\boxed{V(x) = V_0 \text{ (cont)}}$$

We can write the Time-dependent solution for the free particle as:

$$\psi(x,t) = \left(A e^{ikx} + B e^{-ikx}\right) e^{-iEt/\hbar}$$

or : $A e^{i(kx-\omega t)} + B e^{-i(kx+\omega t)}$

Let's suppose a particle is Travelling is the $+x$ direction. We the have:

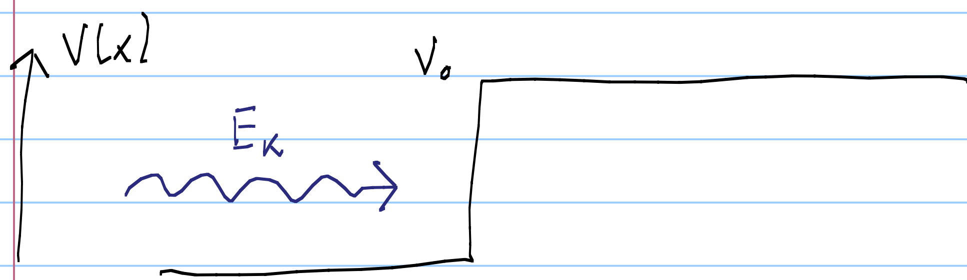
$$\psi(x,t) = A e^{i(kx-\omega t)}$$

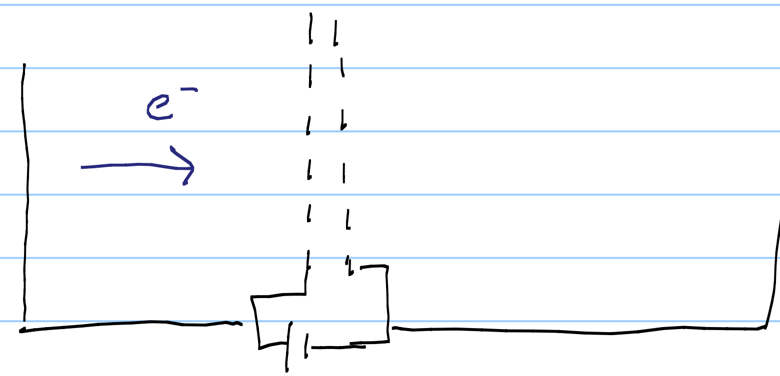
NOTE THAT:

→ corresponds to well-known value of k (p)
→ $\Delta k = 0 \Rightarrow \Delta x \to \infty$: particle could be anywhere
→ $\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx$ diverges! but as long as we do not integrate to ∞, we can use this solution (approx.*)

THE POTENTIAL STEP

"A JOURNEY OF A THOUSAND MILES BEGINS WITH A SINGLE STEP" CONFUCIUS





FOR A CLASSICAL PARTICLE:

if $E < V_0$, particle bounces off
if $E > V_0$, particle keeps going, with less energy (slows down)

FOR A QUANTUM PARTICLE, WE'LL SEE THAT THINGS ARE MORE INTERESTING...

THE POTENTIAL STEP (CONT)

$\boxed{E < V_0}$

$$\boxed{\begin{array}{ll} V(x) = 0 & x < 0 \\ V(x) = V_0 & x > 0 \end{array}}$$

FOR $x < 0$ we have: $\dfrac{d^2\psi}{dx^2} + K_1^2 \psi = 0$ (1)

$$K_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

FOR $x > 0$ we have: $\dfrac{d^2\psi}{dx^2} - K_2^2 \psi(x) = 0$

$$K_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \quad (2)$$

THE SOLUTION TO (1) is:

$$\psi(x) = A e^{iK_1 x} + B e^{-iK_1 x} \quad (3)$$

THE SOLUTION TO (2) is:

$$\psi(x) = C e^{K_2 x} + D e^{-K_2 x} \quad (4)$$

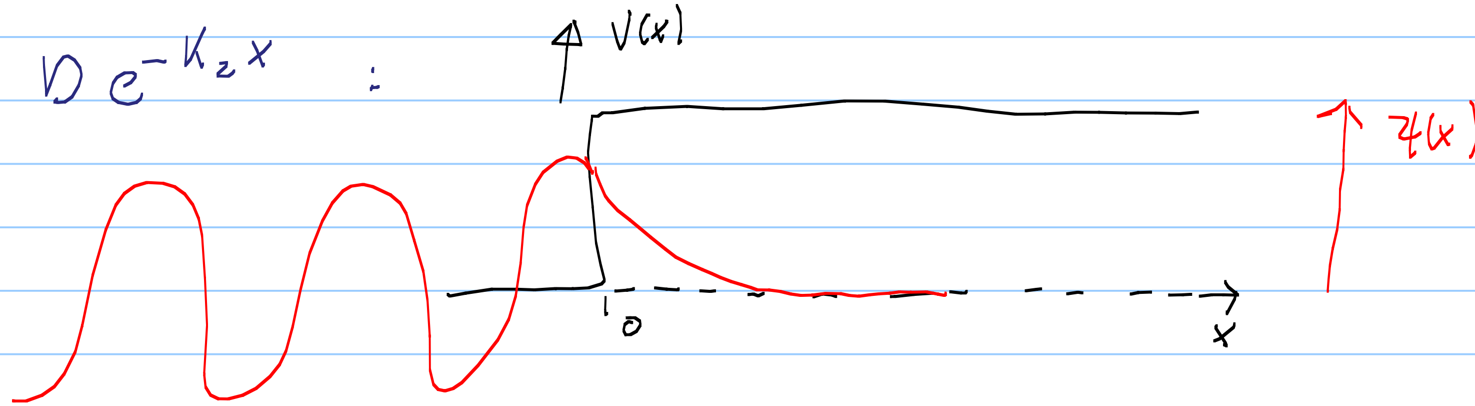
WE NEED $\psi(x)$ AND $\dfrac{d\psi(x)}{dx}$ finite + continuous

$\Rightarrow C = 0$

THE POTENTIAL STEP (CONT)

WITH $C = 0$, we have for $x > 0$:

$\psi(x) = D e^{-k_2 x}$:



→ WE HAVE A DISCONTINUITY IN $V(x)$ $\Rightarrow$ $\frac{d^2\psi}{dx^2}$ will ALSO BE DISCONTINUOUS

→ $\psi(x)$ AND $\frac{d\psi(x)}{dx}$ CAN REMAIN CONTINUOUS IF:

- $\psi(x)$ AT $x = 0$: $A + B = D$ (5)
- $\frac{d\psi}{dx}$ AT $x = 0$: $A i k_1 - B i k_2 = -k_2 D$ (6)

FROM (5) AND (6), WE GET:

$$A = \frac{1 + i k_2/k_1}{2} D \quad (7), \quad B = \frac{1 - i k_2/k_1}{2} D \quad (8)$$

THE POTENTIAL STEP (CONT)

USING (7) AND (8), WE GET

$$\frac{B}{A} = \frac{1 - iK_2/K_1}{1 + iK_2/K_1} = \frac{1 - i\sqrt{V_0/E - 1}}{1 + i\sqrt{V_0/E - 1}} \quad (9)$$

$$\boxed{K_1 = \sqrt{\frac{2mE}{\hbar^2}} \qquad K_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}}$$

$$\frac{D}{A} = \frac{2}{1 + iK_2/K_1} = \frac{2}{1 + i\sqrt{V_0/E - 1}} \quad (10)$$

$$\rightarrow \psi(x) = A e^{iK_1 x} + B e^{-iK_1 x} \qquad x < 0$$

$$= A \left[e^{iK_1 x} + \frac{1 - i\sqrt{V_0/E - 1}}{1 + i\sqrt{V_0/E - 1}} e^{-iK_1 x} \right]$$

$$\psi(x) = D e^{K_2 x} \qquad x > 0$$

$$= \frac{2A}{1 + i\sqrt{V_0/E - 1}} \cdot e^{iK_2 x}$$

THE POTENTIAL STEP (CONT)

DEFINE THE RELECTION COEFFICIENT AS THE INTENSITY OF THE REFLECTED PROBABILITY CURRENT OVER THE INCIDENT INTENSITY : $R = \frac{|B|^2}{|A|^2}$

$\rightarrow$ FROM (9) $\rightarrow$ $E < V_0 \Rightarrow R = 1$

FROM LECTURE 14, THE PROB. CURRENT FOR $Ae^{ikx} + Be^{-ikx}$

$$\vec{J} = \frac{\hbar}{2im}\left(\psi^* \frac{d\psi}{dx} - \frac{d\psi^*}{dx}\psi\right)$$

$$= \frac{\hbar}{2im}\Big[(A^* e^{-ikx} + B^* e^{ikx})\, ik(Ae^{ikx} - Be^{-ikx}) - $$
$$-ik(A^* e^{-ikx} - B^* e^{ikx})(Ae^{ikx} + Be^{-ikx})\Big]$$

$$= \frac{\hbar k}{2m}\Big[A^*A - B^*B - A^*Be^{-2ikx} + B^*Ae^{2ikx} + A^*A$$
$$-B^*B + A^*Be^{-2ikx} - B^*Ae^{2ikx}\Big]$$

$$\boxed{= \frac{\hbar k}{m}\left(|A|^2 - |B|^2\right)} = 0$$

THE POTENTIAL STEP (CONT)

→ For $x < 0$ with $E < V_0$, we have obtained the classical result: $R = 1$. HOWEVER, WE HAVE INTERFERENCE BETWEEN THE INCIDENT AND REFLECTED WAVE.

FOR $x > 0$ we have $\psi(x) = D e^{-K_2 x}$

$$P(x) = |D^2| e^{-2K_2 x}$$

→ THE is A FINITE PROBABILITY OF FINDING THE PARTICLE IN THE REGION $x > 0$

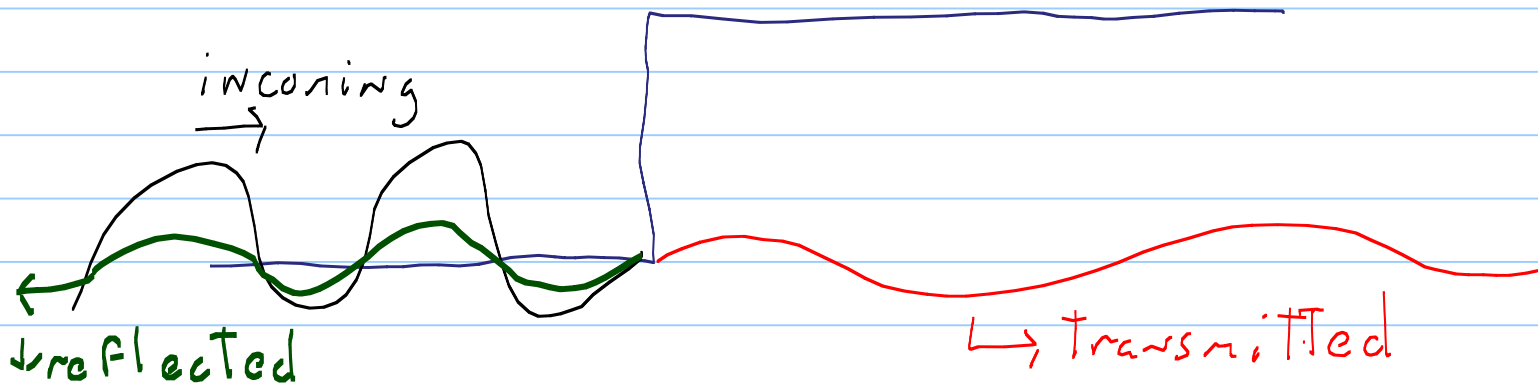
→ NOTE THAT WITH $\Delta x \sim \frac{1}{K_2}$

$$\Delta x \Delta p \geq \hbar \quad \rightarrow \quad \Delta p \geq \hbar K^2 = \sqrt{2m(V_0 - E)}$$

$$\Delta E = \frac{(\Delta p)^2}{2m} \sim V_0 - E$$

THE POTENTIAL STEP (CONT)

$\boxed{E > V_0}$



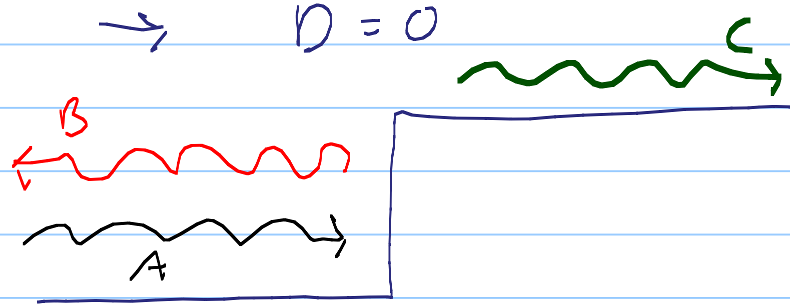
$$x < 0: \frac{d^2 \psi(x)}{dx^2} + k_1^2 \psi(x) = 0 \quad , \quad k_1 = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$$

$$x > 0 \quad \frac{d^2 \psi(x)}{dx^2} + k_2^2 \psi(x) = 0 \quad , \quad k_2 = \left(\frac{2m}{\hbar^2}(E - V_0)\right)^{1/2}$$

$$\psi(x) = \begin{matrix} A e^{ik_1 x} + B e^{-ik_1 x} \\ C e^{ik_2 x} + D e^{-ik_2 x} \end{matrix}$$

THE POTENTIAL STEP (CONT)

→ we'll consider particles coming from the left → $D = 0$



Continuity requirements at $x = 0$ ⇒

$$A + B = C$$
$$iK_1 A - iK_1 B = iK_2 C$$

$$\frac{B}{A} = \frac{K_1 - K_2}{K_1 + K_2} \quad (11) \qquad , \qquad \frac{C}{A} = \frac{2K_1}{K_1 + K_2} \quad (12)$$

ex: $K_1 A - K_1 B = K_2 (A + B)$ → divide by A

→ $K_1 - K_1 \frac{B}{A} = K_2 + K_2 \frac{B}{A}$

$K_1 - K_2 = \frac{B}{A}(K_1 + K_2)$, $\frac{B}{A} = \frac{K_1 - K_2}{K_1 + K_2}$

$\boxed{\text{THE POTENTIAL STEP (CONT)}}$

$x < 0$ $\qquad j = \frac{\hbar k_1}{m}\left(|A|^2 - |B|^2\right)$

$x > 0$

$$\vec{j} = \frac{\hbar}{2im}\left(\psi^* \frac{d\psi}{dx} - \frac{d\psi^*}{dx}\psi\right), \qquad \psi(x) = C\, e^{ik_2 x}$$

$$j = \frac{\hbar}{2im}\left(C e^{-ik_2 x} \cdot +ik_2 C e^{+ik_2 x} - -ik_2 C e^{-ik_2 x} \cdot C e^{ik_2 x}\right)$$

$$= \frac{\hbar}{2im}\left(|C|^2 \cdot ik_2 + ik_2 |C|^2\right)$$

$$= \frac{\hbar k_2}{m}|C|^2$$

$$\rightarrow \quad \frac{\hbar k_1}{m}\left(|A|^2 - |B|^2\right) = \frac{\hbar k_2}{m}|C|^2$$

$$R = \frac{|B|^2}{|A|^2} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \qquad T = 1 - R$$

$$T = \frac{k_2}{k_1}\frac{|C|^2}{|A|^2}$$

THE POTENTIAL STEP (CONT)

Note that $R \neq 0$ has no classical equivalent except in wave mechanics

WHAT DO YOU THINK WILL HAPPEN IF THE PARTICLE COMES FROM THE RIGHT: