LECTURE 18: One-Dimensional Problems: The Potential Barrier

Goals of the 1-d lectures: learn how to solve Schrodinger's equation for some simple problems

What I expect you to learn:

- How to solve the potential barrier
- What is the quantum tunneling effect
- Some physical manifestations of this effect

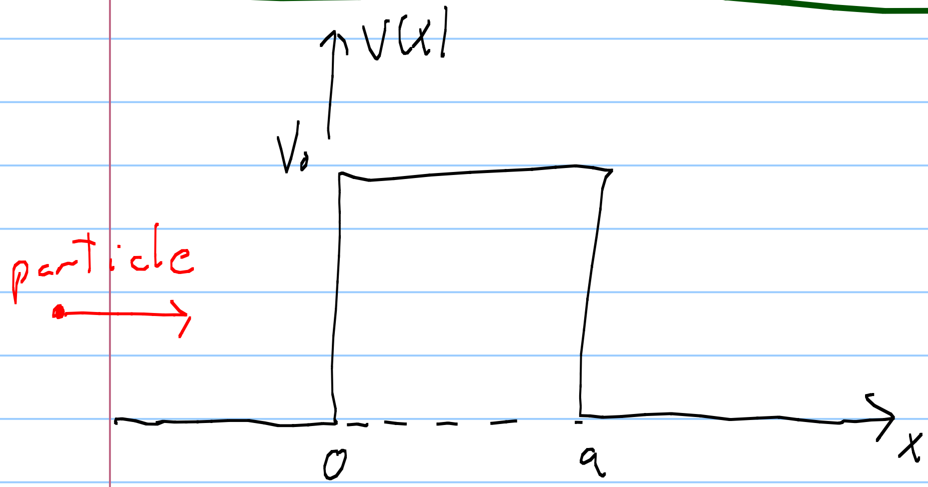
(Roughly corresponds to sections 4.4 of textbook)

Midterm: Wednesday Oct 25th at 11:00 in CLASS

NOTE: No tutorials this week but Rob will give an exam review:

Today- MP 137 (18:00-20:00)

THE POTENTIAL BARRIER



$$V(x) = 0 \;, \quad x < a$$
$$V(x) = V_0 \;, \quad 0 < x < a$$
$$V(x) = 0 \;, \quad x > a$$

IN CLASSICAL MECHANICS, WE WOULD WOULD OBSERVE:

CASE 1: $E < V_0 \rightarrow$ particle bounces off

CASE 2: $E > V_0 \rightarrow$ particle keeps going (slows down over barrier)

NOW THE QM SOLUTION:

IN REGIONS $x < 0$ AND $x > a$, THE PARTICLE IS FREE SO WE CAN WRITE:

$$Ae^{ik_1 x} + Be^{-ik_1 x} \quad \text{for} \quad x < 0$$
$$Ce^{ik_1 x} + De^{-ik_1 x} \quad \text{for} \quad x > a$$

THE POTENTIAL BARRIER

k_1 in the above is given by $\sqrt{\frac{2mE}{\hbar^2}}$

WITH INCIDENT PARTICLES COMING FROM THE LEFT ONLY, WE CAN SET $D = 0$

So we have: $\psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x < 0 \\ Ce^{ik_1x} & x > a \end{cases}$

THE PROBABILITY CURRENT IS GIVEN BY:

$$\vec{J} = \frac{\hbar}{2im}\left(\psi^* \frac{d\psi}{dx} - \frac{d\psi^*}{dx}\psi\right)$$

FOR $\psi(x)$, This gives (**From previous lecture**):

$$\frac{\hbar k_1}{m}\left[|A|^2 - |B|^2\right] \quad x < 0$$

$$\frac{\hbar k_1}{m}|C|^2 \quad x > a$$

THE POTENTIAL BARRIER

AS WE SAW BEFORE, WE CAN OBTAIN THE REFLECTION AND TRANSMISSION COEFFICIENTS FROM A, B, AND C:

$$R = \frac{|B|^2}{|A|^2} \qquad T = \frac{|C|^2}{|A|^2}$$

THE HARD PART IS TO DETERMINE C AND B IN TERMS OF A. FOR THIS, WE NEED TO LOOK AT WHAT HAPPENS IN THE BARRIER.

CASE 1: $E < V_0$

→ THIS IS SIMILAR TO THE POTENTIAL STEP WHEN WE HAD $E < V_0$: WHEN THE PARTICLE WAVE FUNCTION ENTERS THE BARRIER WE HAVE: $K_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$

SO WE WRITE FOR $0 < x < a$: $\psi(x) = F e^{K_2 x} + G e^{-K_2 x}$

THE POTENTIAL BARRIER

WE REQUIRE CONTINUITY OF $\psi(x)$ at $x=0$, $x=a$
AND CONTINUITY OF $\frac{d\psi(x)}{dx}$ at $x=0$, $x=a$

$x=0$

$$Ae^{ik_1 0} + Be^{-ik_1 0} = Fe^{k_2 0} + Ge^{-k_2 0}$$

$$\Rightarrow A + B = F + G \quad (1)$$

$\frac{d\psi}{dx}$:

$$Aik_1 e^{ik_1 0} - Bik_1 e^{-ik_1 0} = Fk_2 e^{k_2 0} - Gk_2 e^{-k_2 0}$$

$$\Rightarrow ik_1(A-B) = k_2(F-G) \quad (2)$$

$x=a$

$$Ce^{ik_1 a} = Fe^{k_2 a} + Ge^{-k_2 a} \quad (3)$$

$\frac{d\psi}{dx}$:

$$ik_1 Ce^{ik_1 a} = k_2\left(Fe^{k_2 a} - Ge^{-k_2 a}\right) \quad (4)$$

We need To eliminate F and G

THE POTENTIAL BARRIER

→ MULTIPLY (3) BY K_2 AND ADD (4)

$$
\begin{aligned}
& K_2 C e^{iK_1 a} = K_2 F e^{K_2 a} + K_2 G e^{-K_2 a} \\
+\ & iK_1 C e^{iK_1 a} = K_2 F e^{K_2 a} - K_2 G e^{-K_2 a}
\end{aligned}
$$

$$\Rightarrow \frac{C}{2} e^{iK_1 a} \left(1 + \frac{iK_1}{K_2}\right) e^{-K_2 a} = F \qquad (5)$$

→ MULTIPLY (3) BY K_2 AND SUBTRACT (4)

$$
\begin{aligned}
& K_2 C e^{iK_1 a} = K_2 F e^{K_2 a} + K_2 G e^{-K_2 a} \\
-\ & iK_1 C e^{iK_1 a} = K_2 F e^{K_2 a} - K_2 G e^{-K_2 a}
\end{aligned}
$$

$$= C e^{iK_1 a} (K_2 - iK_1) = +2 G e^{-K_2 a} \cdot K_2$$

$$\Rightarrow \frac{C}{2} e^{iK_1 a} e^{K a} \left(1 - \frac{iK_1}{K_2}\right) = G \qquad (6)$$

THE POTENTIAL BARRIER

Insert (5) and (6) into (1)

$$A + B = \frac{C}{2} e^{iK_1 a} e^{-K_2 a}\left(1 + \frac{iK_1}{K_2}\right) + \frac{C}{2} e^{iK_1 a} e^{K_2 a}\left(1 - \frac{iK_1}{K_2}\right)$$

$$= \frac{C}{2} e^{iK_1 a} \cdot \left[e^{-K_2 a} + e^{K_2 a} + \frac{iK_1}{K_2}\left(e^{-K_2 a} - e^{K_2 a}\right)\right]$$

$$\boxed{\text{note that: } \sinh x = \frac{e^{x} - e^{-x}}{2}, \quad \cosh x = \frac{e^{x} + e^{-x}}{2}}$$

$$A + B = C e^{iK_1 a}\left[\cosh a - \frac{iK_1}{K_2} \sinh a\right]$$

→ divide by A

$$1 + \frac{B}{A} = \frac{C}{A} e^{iK_1 a}\left[\cosh a - \frac{iK_1}{K_2} \sinh a\right] \quad (7)$$

THE POTENTIAL BARRIER

Insert 5 and 6 into 2 $\rightarrow$ $iK_1(A-B) = K_2(F-G)$

$$A - B = \frac{K_2}{iK_1}\left[\frac{C}{2}e^{iK_1a}e^{-K_2a}\left(1+\frac{iK_1}{K_2}\right) - \frac{C}{2}e^{iK_1a}e^{K_2a}\left(1-\frac{iK_1}{K_2}\right)\right]$$

$$= \frac{C}{2}e^{iK_1a}e^{-K_2a}\left(\frac{K_2}{iK_1}+1\right) - \frac{C}{2}e^{iK_1a}e^{K_2a}\left(\frac{K_2}{iK_1}-1\right)$$

$$= \frac{C}{2}e^{iK_1a}\left[e^{-K_2a} + e^{K_2a} + \frac{K_2}{iK_1}\cdot\left(e^{-K_2a} - e^{K_2a}\right)\right]$$

$$= Ce^{iK_1a}\left[\cosh K_2a - \frac{K_2}{iK_1}\sinh K_2a\right]$$

$$= Ce^{iK_1a}\left[\cosh K_2a + \frac{iK_2}{K_1}\sinh K_2a\right]$$

divide by A:

$$1 - \frac{B}{A} = \frac{C}{A}e^{iK_1a}\left[\cosh K_2a + \frac{iK_2}{K_1}\sinh K_2a\right] \quad (8)$$

THE POTENTIAL BARRIER

(7) + (8) =

$$2 = \frac{C}{A} e^{iK_1 a} \left[2\cosh K_2 a + \left(\frac{iK_2}{K_1} - \frac{iK_1}{K_2} \right) \sinh K_2 a \right]$$

$$2 = \frac{C}{A} e^{iK_1 a} \left[2\cosh K_2 a + \frac{i(K_2^2 - K_1^2)}{K_1 K_2} \sinh K_2 a \right]$$

$$\frac{C}{A} = 2e^{-iK_1 a} \left[2\cosh K_2 a + \frac{i(K_2^2 - K_1^2)}{K_1 K_2} \sinh K_2 a \right]^{-1}$$

$$\frac{|C|^2}{|A|^2} = 4 \left[4\cosh^2 K_2 a + \frac{(K_2^2 - K_1^2)^2}{K_1^2 K_2^2} \sinh^2 K_2 a \right]^{-1}$$

note that: $\cosh^2 a - \sinh^2 a = 1$

$$\frac{|C|^2}{|A|^2} = 4 \left[4 + \sinh^2 K_2 a \left(4 + \frac{(K_2^2 - K_1^2)^2}{K_1^2 K_2^2} \right) \right]^{-1}$$

THE POTENTIAL BARRIER

note that: $$4 + \frac{|K_2^2 - K_1^2|^2}{K_1^2 K_2^2} = \frac{4K_1^2K_2^2 + K_1^4 + K_2^4 - 2K_1^2K_2^2}{K_1^2 K_2^2}$$

$$= \frac{K_1^4 + K_2^4 + 2K_1^2K_2^2}{K_1^2K_2^2} = \frac{(K_1^2 + K_2^2)^2}{K_1^2 K_2^2}$$

$$\Rightarrow \frac{|C|^2}{|A|^2} = T = \left[1 + \frac{1}{4}\left(\frac{K_1^2 + K_2^2}{K_1 K_2}\right)^2 \sinh^2 K_2 a\right]^{-1}$$

$$K_1 = \sqrt{\frac{2mE}{\hbar^2}} \qquad K_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$$K_1^2 + K_2^2 = \frac{2mE}{\hbar^2} + \frac{2mV_0}{\hbar^2} - \frac{2mE}{\hbar^2} = \frac{2mV_0}{\hbar^2}$$

$$K_1 K_2 = \frac{2m}{\hbar^2}\sqrt{E(V_0 - E)} \Rightarrow \left(\frac{K_1^2 + K_2^2}{K_1 K_2}\right)^2 = \frac{V_0^2}{E(V_0 - E)}$$

$$\boxed{T = \left[1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 K_2 a\right]^{-1}}$$

THE POTENTIAL BARRIER

$$\boxed{T = \left[1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 K_2 a \right]^{-1}}$$

TAYLOR EXPANSION OF $\sinh z = z + \frac{1}{6} z^3$

FOR SMALL VALUES OF $K_2 a$, $\sinh^2 K_2 a = K_2^2 a^2 = \frac{2m(V_0 - E)}{\hbar^2} a^2$

if E close to V_0, $K_2 a$ will be small and in this case:

$$T = \left[1 + \frac{m V_0 a^2}{2\hbar^2} \right]^{-1}$$

→ notice that $T \to 0$ when:

- $\hbar \to 0$
- V_0 is large
- a is large

THE POTENTIAL BARRIER

$$\boxed{T = \left[1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 K_2 a \right]^{-1}}$$

if Ka is large $\rightarrow$ $\sinh z = \dfrac{e^z - e^{-z}}{2} \rightarrow \dfrac{e^z}{2}$

$$\rightarrow T = \left[1 + \frac{V_0^2}{4E(V_0-E)} \frac{e^{2K_2 a}}{4} \right]^{-1}$$

$$\approx \left[\frac{V_0^2}{16E(V_0-E)} e^{2K_2 a} \right]^{-1} = \frac{16E(V_0-E)}{V_0^2} e^{-2K_2 a}$$

$\rightarrow$ T DECREASES EXPONENTIALLY with a

An Example: 3 eV electron incident on a 10 eV barrier with a width of 4 Å (~ 1-2 layers of oxyde seperating two sheets of metal). What fraction of these electrons will go through the barrier?

$$K_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} = 1.4 \times 10^{10} \, m^{-1}$$

THE POTENTIAL BARRIER

$2\kappa_2 a = 10.8 \quad , \quad e^{-2\kappa_2 a} \simeq 2 \times 10^{-5}$

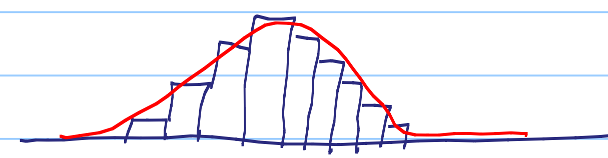
$$\frac{16E(V_0-E)}{V_0^2} = \frac{16 \cdot 4}{10^2}(6) = 3.84$$

$\rightarrow T \simeq 7.7 \times 10^{-5}$

MORE REALISTIC POTENTIALS

NOTE THAT (UNDER CERTAIN CONDITIONS) ONE CAN APPROXIMATE AN IRREGULAR SHAPED POTENTIAL BY A SERIES OF SQUARE POTENTIALS $\rightarrow$ WKB APPROX.

WENTZEL-KRAMERS-BRILLOUIN



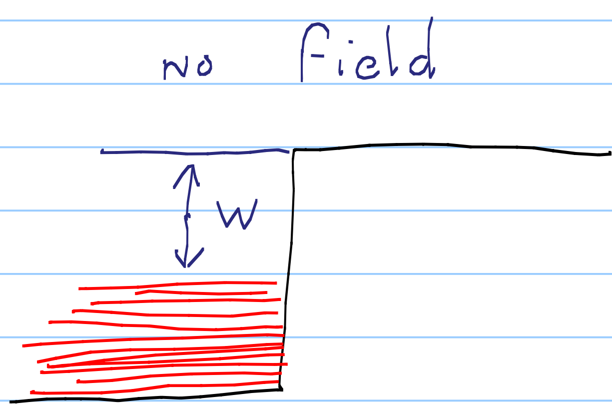
$\rightarrow$ we'll do an example for the next lecture

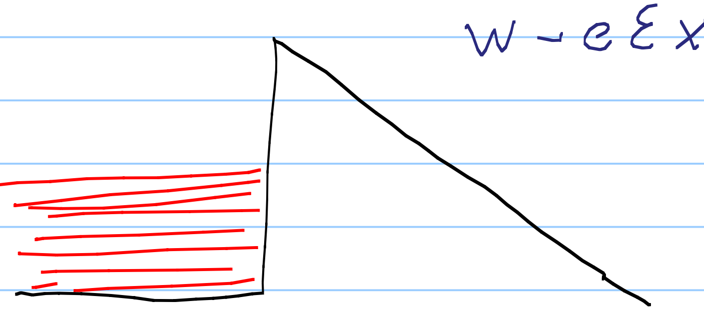
THE POTENTIAL BARRIER

SOME PHYSICAL MANIFESTATIONS OF THE TUNNELING EFFECT

WE SAW IN THE PHOTOELECTRIC EFFECT THAT ELECTRONS NEEDED SOME MINIMUM ENERGY TO ESCAPE FROM THE METAL SURFACE ($W \rightarrow$ work Function)

ELECTRONS CAN ALSO BE REMOVED FROM A SURFACE BY APPLYING AN ELECTRIC FIELD $\mathcal{E}$





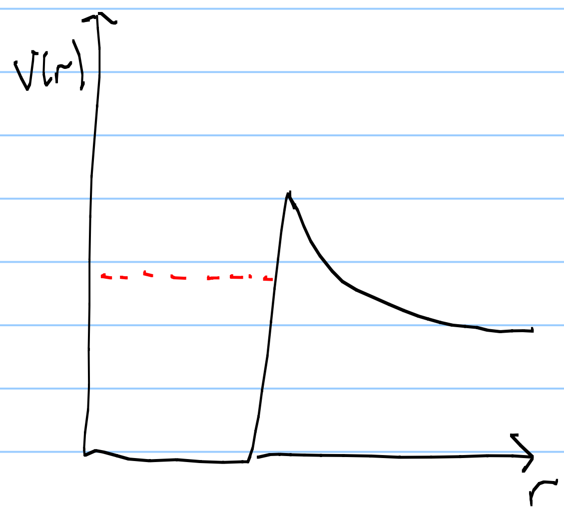
The width $a = \dfrac{W}{e\mathcal{E}}$

THE SCANNING TUNNELING MICROSCOPE OPERATES USING THIS PRINCIPLE

$\boxed{\text{THE POTENTIAL BARRIER}}$

ALPHA RADIATION CONSISTS OF THE NUCLEUS OF A HELIUM ATOM : $2p + 2n$

ONE CAN GET A GOOD UNDERSTANDING OF THIS PROBLEM BY TREATING THE α as TRAPPED IN A POTENTIAL WELL



→ MEAN LIFETIMES VARY FROM μs To billions of years

→ energy of α_s are within a Factor of 2!

→ Was understood in Terms of Tunneling effect

WE WILL SOLVE THIS NEXT CLASS

THE POTENTIAL BARRIER

CASE 2 : $E > V_0$

- THE INTERNAL REGION SOLUTIONS NOW READ:

$$\psi(x) = F e^{ik_2 x} + G e^{-ik_2 x} \qquad 0 < x < a$$

with $k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

FOR THIS WE GET:

$$R = \frac{|B|^2}{|A|^2} = \left[1 + \frac{4E(E-V_0)}{V_0^2 \sin^2(k_2 a)}\right]^{-1}$$

$$T = \frac{|C|^2}{|A|^2} = \left[1 + \frac{V_0^2 \sin^2(k_2 a)}{4E(E-V_0)}\right]^{-1}$$

NOTE THAT $T = 1$ WHEN $k_2 a = \pi, 3\pi, \ldots$! why?

what is happening?