LECTURE 19: One-Dim. Problems: The Potential Barrier (Cont.)

Goals of the 1-d lectures: learn how to solve Schrodinger's equation for some simple problems

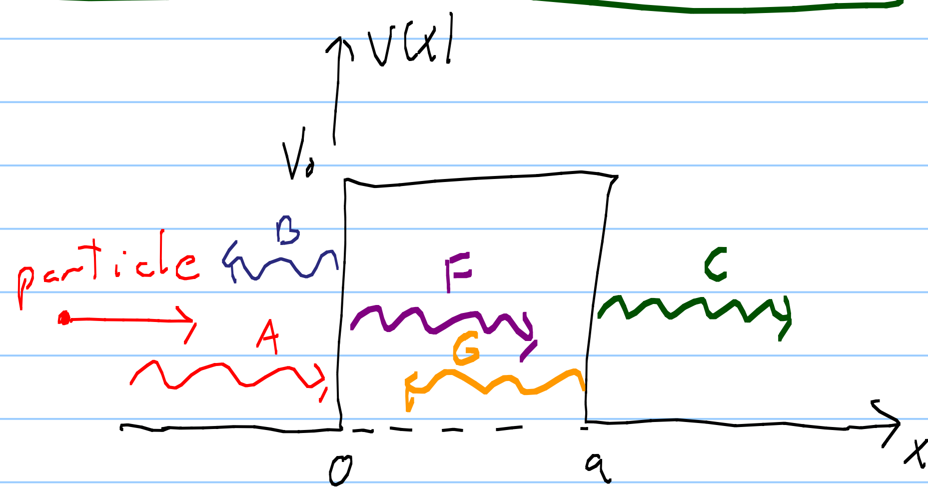
What I expect you to learn:

- What is the quantum tunneling effect
- Two examples of the manifestation of this effect

(Roughly corresponds to sections 4.4 of textbook, and pp. 386-409 of French and Taylor)

THE POTENTIAL BARRIER

RECAP



$$V(x) = 0 \, , \quad x < a$$
$$V(x) = V_0 \, ; \quad 0 < x < a$$
$$V(x) = 0 \, ; \quad x > a$$

IN REGIONS $x < 0$ AND $x > a$, THE PARTICLE IS FREE SO WE CAN WRITE:

$$A e^{ik_1 x} + B e^{-ik_1 x} \quad \text{for} \quad x < 0$$
$$C e^{ik_1 x} + \cancel{D} e^{-ik_1 x} \quad \text{for} \quad x > a$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

SO WE WRITE FOR $0 < x < a$: $\psi(x) = F e^{k_2 x} + G e^{-k_2 x}$

$$k_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

THE POTENTIAL BARRIER — RECAP

AS WE SAW BEFORE, WE CAN OBTAIN THE REFLECTION AND TRANSMISSION COEFFICIENTS FROM A, B, AND C:

$$R = \frac{|B|^2}{|A|^2} \qquad T = \frac{|C|^2}{|A|^2}$$

THE HARD PART IS TO DETERMINE C AND B IN TERMS OF A. FOR THIS, WE NEED TO LOOK AT WHAT HAPPENS IN THE BARRIER.

CASE 1: $E < V_0$

WE OBTAINED THE FOLLOWING SOLUTION FOR THE TRANSMISSION COEFFICIENT:

$$T = \left[1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 k_2 a \right]^{-1}$$

THE POTENTIAL BARRIER

RECAP

$$T = \left[1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 K_2 a \right]^{-1}$$

TAYLOR EXPANSION OF $\sinh z = z + \frac{1}{6} z^3$

FOR SMALL VALUES OF $K_2 a$, $\sinh^2 K_2 a = K_2^2 a^2 = \frac{2m(V_0 - E)}{\hbar^2} a^2$

if E close to V_0, $K_2 a$ will be small and in this case:

$$T = \left[1 + \frac{m V_0 a^2}{2\hbar^2} \right]^{-1}$$

→ notice that $T \to 0$ when:

- $\hbar \to 0$
- V_0 is large
- a is large

THE POTENTIAL BARRIER

$$T = \left[1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 k_2 a \right]^{-1}$$

if $k_2 a$ is large $\rightarrow$ $\sinh z = \frac{e^z - e^{-z}}{2} \rightarrow \frac{e^z}{2}$

$$\rightarrow T = \left[1 + \frac{V_0^2}{4E(V_0 - E)} \frac{e^{2k_2 a}}{4} \right]^{-1}$$

$$\approx \left[\frac{V_0^2}{16E(V_0 - E)} e^{2k_2 a} \right]^{-1} = \frac{16E(V_0 - E)}{V_0^2} e^{-2k_2 a}$$

$\rightarrow$ T DECREASES EXPONENTIALLY with a

An Example: 3 eV electron incident on a 10 eV barrier with a width of 4 Å (~ 1-2 layers of oxyde seperating two sheets of metal). What fraction of these electrons will go through the barrier?

$$k_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} = 1.4 \times 10^{10} \, m^{-1}$$

THE POTENTIAL BARRIER

$2K_2 a = 10.8$, $e^{-2K_2 a} \simeq 2 \times 10^{-5}$

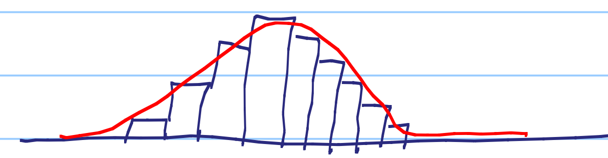
$\frac{16E(V_0 - E)}{V_0^2} = \frac{16 \cdot 4}{10^2}(6) = 3.84$

$\rightarrow T \simeq 7.7 \times 10^{-5}$

MORE REALISTIC POTENTIALS

NOTE THAT (UNDER CERTAIN CONDITIONS) ONE CAN APPROXIMATE AN IRREGULAR SHAPED POTENTIAL BY A SERIES OF SQUARE POTENTIALS → WKB APPROX.

WENTZEL-KRAMERS-BRILLOUIN



TUNNELLING EFFECT APPROXIMATION

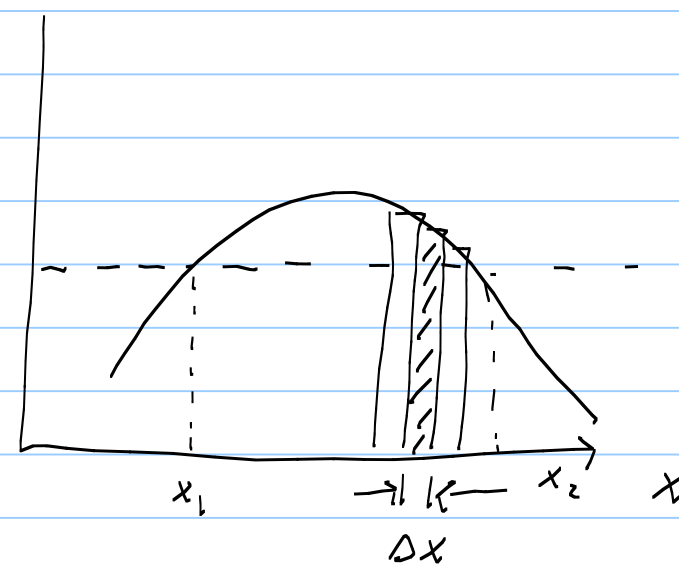
WE SAW THAT FOR LARGE VALUES OF $K_2 a$, WE COULD WRITE:

$$T = \frac{|C|^2}{|A|^2} \simeq \frac{16\,E(V_0-E)}{V_0^2}\, e^{-2K_2 a} \qquad K_2 = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

or

$$\frac{|C|}{|A|} \simeq C_E \exp\left[-\left(\frac{2m(V_0-E)}{\hbar}\right)^{1/2} a\right]$$

→ Assume reflection at $x_1 \simeq 100\%$
⇒ contribution from reflection at $x_2 \simeq 0\%$



→ We will approximate $\psi(x)$ inside the barrier as one negative exponential

→ We do not have to take into account the reflections in the "Δx" potential barriers

TUNNELLING EFFECT APPROXIMATION

FOR A "Δx" POTENTIAL BARRIER WE HAVE

$$\frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2}\left(V(x) - E\right)\psi = 0 = \frac{d^2\psi}{dx^2} - K(x)^2\psi$$

THE ATTENUATION OF $\psi(x) \propto \frac{C}{A}$ AFTER GOING THROUGH Δx WILL BE GIVEN BY : $\psi(x+\Delta x) \approx \psi(x)\cdot e^{-K(x)\Delta x}$

LET'S DO A TAYLOR EXPANSION FOR BOTH SIDES :

$$\psi(x) + \frac{d\psi}{dx}\Delta x \approx \psi(x)\left[1 - K(x)\Delta x\right]$$

$$F(x+\Delta x) = \sum_{n=0}^{\infty} \frac{\Delta x^n}{n!}\left(\frac{d}{dx}\right)^n F(x)$$

$$\Rightarrow \frac{d\psi}{dx} \simeq -K(x)\psi$$

$$\Rightarrow \frac{1}{\psi}\frac{d\psi}{dx} \simeq -K(x)$$

Integrate over entire barrier : $\int_{x_1}^{x_2} \frac{1}{\psi}\frac{d\psi}{dx}\,dx = \int_{x_1}^{x_2} -K(x)\,dx$

TUNNELLING EFFECT APPROXIMATION

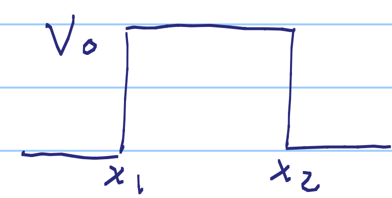
$$\ln\left(\frac{\psi(x_2)}{\psi(x_1)}\right) = \int_{x_1}^{x_2} \sqrt{\frac{2m(V(x) - E)}{\hbar}}\, dx$$

$$T = \left[\frac{\psi(x_2)}{\psi(x_1)}\right]^2 = \exp\left\{-2\int_{x_1}^{x_2} \sqrt{\frac{2m(V(x)-E)}{\hbar}}\, dx\right\}$$

WE'LL LOOK AT 3 CASES:

1 - $V(x) = V_0$
2 - $V(x) = V_0 - \text{constant} \cdot x$
3 - $V(x) = \text{constant} \cdot \frac{1}{x}$

1 - $V(x) = V_0$



$$T = \exp -2\sqrt{\frac{2m(V_0-E)}{\hbar}} \cdot \underbrace{(x_2 - x_1)}_{\equiv a} \rightarrow \exp -2K_2 a$$

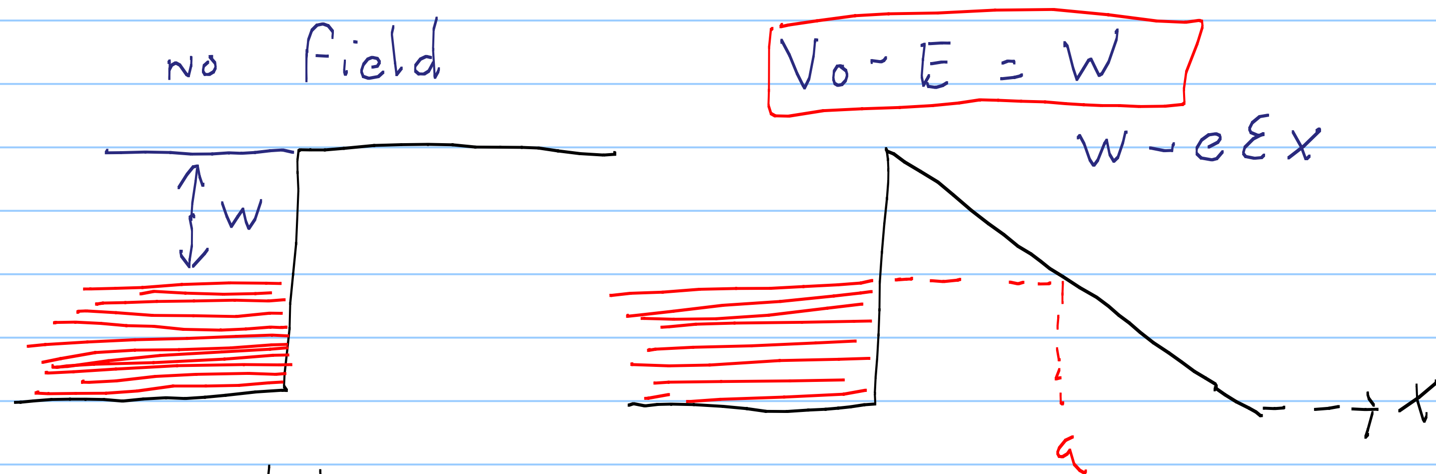
TUNNELLING EFFECT APPROXIMATION

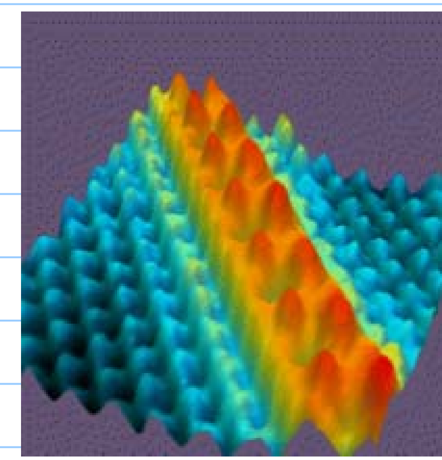
2 – $V(x) = 0 \quad (x<0)$

$V(x) = V_0 - e\mathcal{E}x \quad x \geq 0$

WE SAW IN THE PHOTOELECTRIC EFFECT THAT ELECTRONS NEEDED SOME MINIMUM ENERGY TO ESCAPE FROM THE METAL SURFACE (W → work Function)

ELECTRONS CAN ALSO BE REMOVED FROM A SURFACE BY APPLYING AN ELECTRIC FIELD $\mathcal{E}$

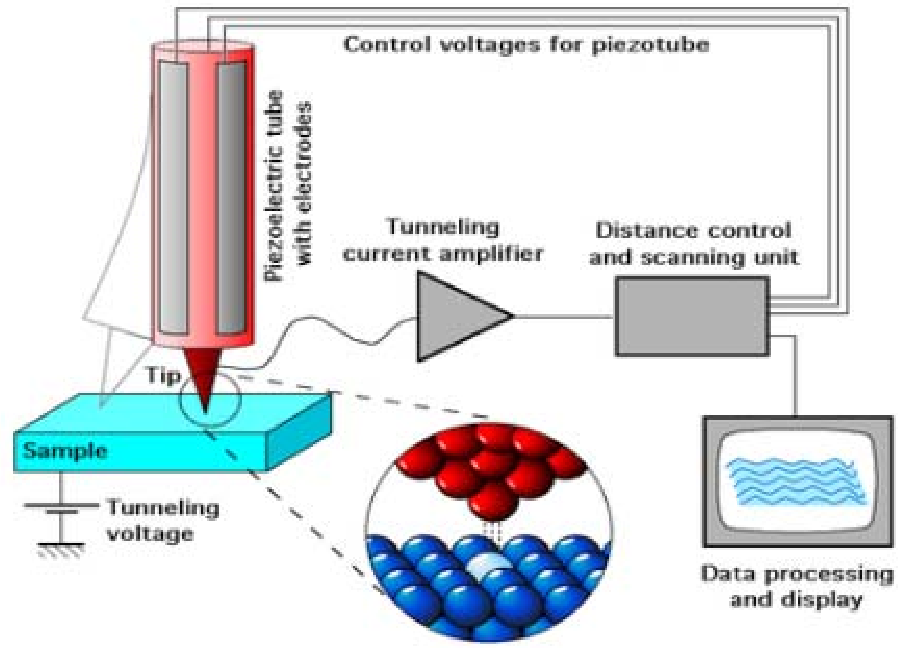


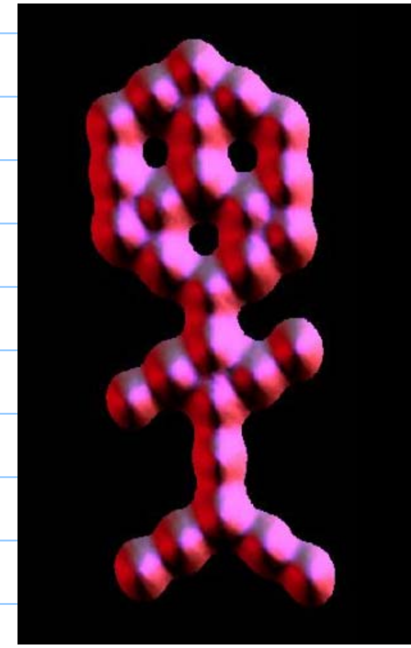


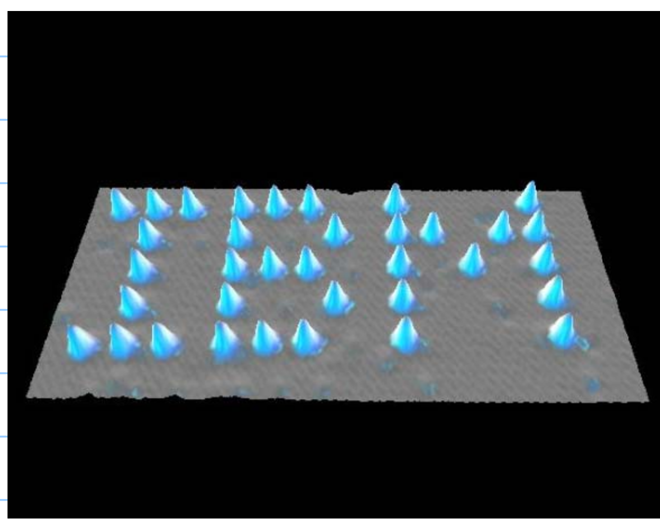
The width $a = \frac{W}{e\mathcal{E}}$

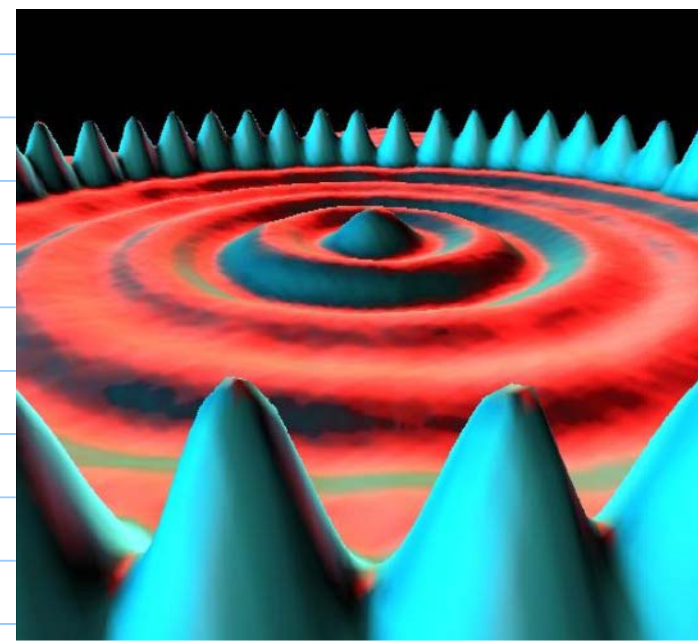
THE SCANNING TUNNELING MICROSCOPE OPERATES USING THIS PRINCIPLE

STM


Control voltages for piezotube
Piezoelectric tube with electrodes
Tunneling current amplifier
Distance control and scanning unit
Tip
Sample
Tunneling voltage
Data processing and display







TUNNELLING EFFECT APPROXIMATION

$$T = \exp\left\{-2\int_{x_1}^{x_2} \frac{\sqrt{2m(V(x)-E)}}{\hbar}\,dx\right\}$$

$$= \exp\left\{\frac{-2\sqrt{2m}}{\hbar}\int_0^a \sqrt{V_0 - e\mathcal{E}x - E}\,dx\right\}$$

$$V_0 - E = W \quad , \quad a = W/e\mathcal{E}$$

$$\rightarrow T \simeq \exp\left\{-2\frac{\sqrt{2me\mathcal{E}}}{\hbar}\int_0^a \sqrt{a-x}\,dx\right\}$$

$$T \simeq \exp\left\{-\frac{4}{3}\frac{\sqrt{2me\mathcal{E}}}{\hbar}\cdot\left(\frac{W}{e\mathcal{E}}\right)^{3/2}\right\}$$

with $W = 4\,eV \rightarrow 6.4\times 10^{-19}\,J$, $m = 0.9\times 10^{-30}\,Kg$

we have : $\frac{4}{3}\frac{\sqrt{2m}}{\hbar}\frac{W^{3/2}}{e} \sim 6\times 10^{10}\,V/m$

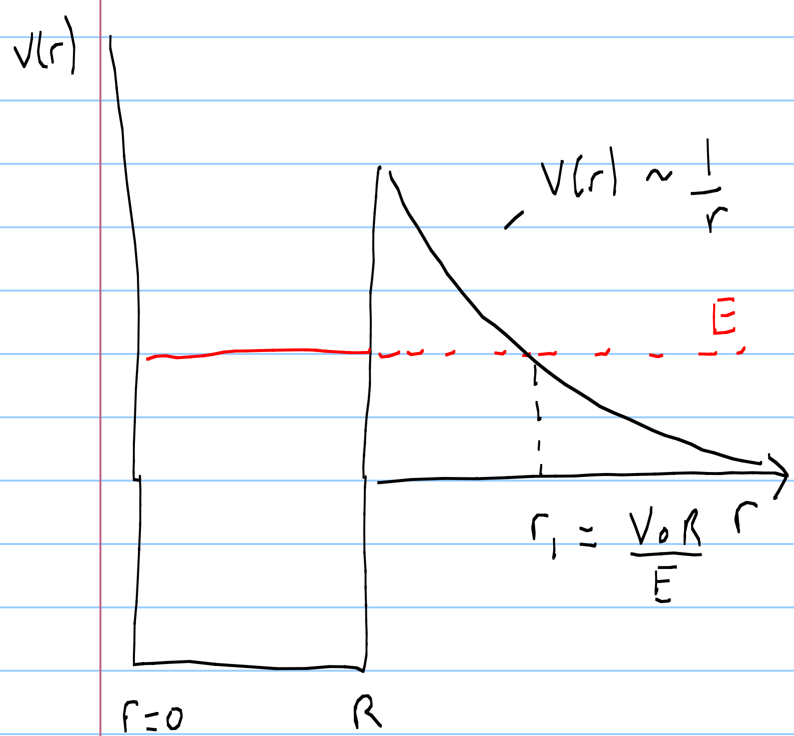
with $\mathcal{E} = 10^9\,V/m \rightarrow a \sim 40\,\mathring{A}$

↳ below that T vanishingly small

TUNNELLING EFFECT APPROXIMATION

3- ALPHA RADIATION CONSISTS OF THE NUCLEUS OF A HELIUM ATOM: $2p + 2n$

ONE CAN GET A GOOD UNDERSTANDING OF THIS PROBLEM BY TREATING THE α AS TRAPPED IN A POTENTIAL WELL



→ MEAN LIFETIMES VARY FROM μs to billions of years

→ energy of α's are within a factor of 2!

→ was understood in terms of tunneling effect

TUNNELLING EFFECT APPROXIMATION

WE HAVE A SPHERICALLY SYMMETRIC SYSTEM THAT IS MORE EASILY DEALT WITH THE SCHRODINGER EQUATION EXPRESSED IS SPHERICAL COORDINATES. WE'LL COME BACK TO THIS LATER → FOR NOW WE'LL JUST USE THE END PRODUCT: WE CAN SEPARATE THE WAVE FUNCTION INTO A RADIAL PART AND A PART THAT DEPENDS ON THE ANGLES θ AND φ.

→ WE WILL WRITE $\psi = u/r$ (we'll see later why this is)

THE PROB. CURRENT BECOMES

$$\psi^* \cdot \frac{d\psi}{dr} - \psi \frac{d\psi^*}{dr} = \frac{1}{r^2}\left(u^* \frac{du}{dr} - u \frac{du^*}{dr}\right)$$

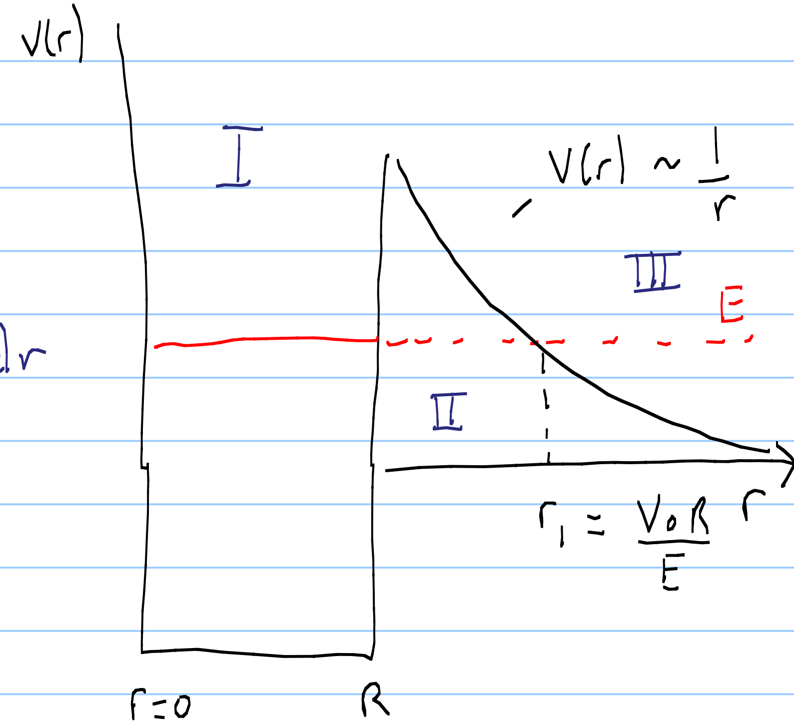
$$j(r) = \frac{-i\hbar}{2mr^2}\left(u^* \frac{du}{dr} - u\frac{du^*}{dr}\right) \rightarrow \vec{J}(r) = -\frac{2\pi i\hbar}{m}\left(u^*\frac{du}{dr} - u\frac{du^*}{dr}\right)$$

where, over the sphere, $\vec{J}(r) = 4\pi r^2 j(r)$

TUNNELLING EFFECT APPROXIMATION

In region I : $A e^{ikr} + B e^{-ikr}$

" III : $C e^{ik_3 r}$



$$P = \int_0^R (\psi^* \psi)\, 4\pi r^2\, dr = 4\pi \int_0^R (u^* u)\, dr$$

with $|A| \simeq |B|$

$$P = 4\pi\, 2\, |A|^2 R$$

$$J_{III} = -4\pi \frac{i\hbar}{2m} |C|^2 (2ik_3)$$

$$\rightarrow \frac{dP}{dT} = -J(R) \Rightarrow 2R \frac{d}{dT}|A|^2 = -\frac{\hbar k_3}{m} |C|^2$$

$$\frac{d|A|^2}{|A|^2} = \frac{-\hbar k_3}{m \cdot 2R} \frac{|C|^2}{|A|^2} dT \rightarrow \frac{dP}{P} = \frac{-\hbar k_3}{m \cdot 2R} \frac{|C|^2}{|A|^2} dT = -\gamma\, dT$$

TUNNELLING EFFECT APPROXIMATION

$$\frac{dP}{P} = \frac{-\hbar K_3}{m \cdot 2R}\frac{|C|^2}{|A|^2}\, dt = -\gamma\, dT, \qquad P(t) = P(0)\, e^{-\gamma T}$$

(T: Time)

with $T = \frac{K_3}{K_1}\frac{|C|^2}{|A|^2}$ (Trans.), $\gamma = \frac{\hbar K_1}{2m} \frac{T}{2R}$ (T → Transmission)

AN ALPHA PARTICLE OUTSIDE THE NUCLEAR POTENTIAL WELL WILL FEEL THE ELECTROSTATIC REPULSION OF THE NUCLEUS:

$$V(r) = k\frac{q_1 q_2}{r}, \quad q_1 = 2e, \quad q_2 = (Z-2)e$$

($k = \frac{1}{4\pi\varepsilon_0}$)

$V(r)$ will be maximum at $r = R$: $V_0 = \frac{kZ(Z-2)e^2}{R}$

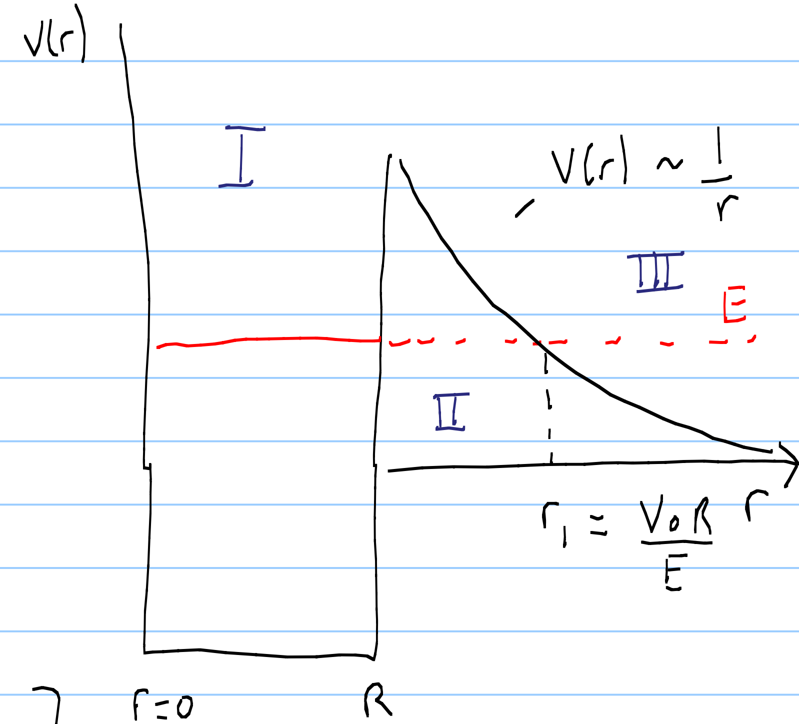
($V_0 \sim 25$ MeV for $Z-2 = 90$, $R = 10^{-14}$ m)

TUNNELLING EFFECT APPROXIMATION

$$E = k\,2(Z-2)\,\frac{e^2}{r_1}$$

$$r_1 = \frac{V_0}{E}\,R$$

$$V(r) = \frac{R}{r}\,V_0$$



$$T \approx \exp\left[-2\int_R^{r}\frac{\sqrt{2m\,(V(r)-E)}}{\hbar}\,dr\right]$$

$$= \exp\left[\frac{-2\sqrt{2mE}}{\hbar}\int_R^{r_1}\left(\frac{r_1}{r}-1\right)^{1/2}dr\right]$$

with $R \ll r_1$ $(E \ll V_0)$ $\approx \frac{\pi}{2}\frac{V_0}{E}R - 2\left(\frac{V_0}{E}\right)^{1/2}R$

$$T \approx \exp\left[\frac{-2\sqrt{2mE}}{\hbar}\,R\left[\frac{\pi}{2}\frac{V_0}{E} - 2\left(\frac{V_0}{E}\right)^{1/2}\right]\right]$$

TUNNELLING EFFECT APPROXIMATION

$$T \approx \exp\left[-\frac{\pi\sqrt{2m}}{\hbar} \cdot \frac{V_0 R}{\sqrt{E}} + \frac{4\sqrt{2mV_0}}{\hbar} R \right]$$

this has the form

$$T(E) \simeq A\, e^{-C/E^{1/2}}$$

$$C = \frac{\pi\sqrt{2m}}{\hbar} V_0 R = \frac{\pi\sqrt{2m}}{\hbar} \, 2(Z-2)e^2 \cdot K$$

For $Z-2 \simeq 90$, $m = 6.6 \times 10^{-27}$ Kg

$$C = 360\sqrt{MEV}$$

$$\gamma = \frac{\hbar K_1}{m \cdot 2R} \cdot T$$

→ with C large, variation of γ dominated by exponential in $T(E)$.

TUNNELLING EFFECT APPROXIMATION

$$\log_{10} \gamma = \text{const} - \frac{C \log e}{E^{1/2}\,(\text{MEV})} = \text{const} - \frac{156}{E^{1/2}\,(\text{MEV})}$$

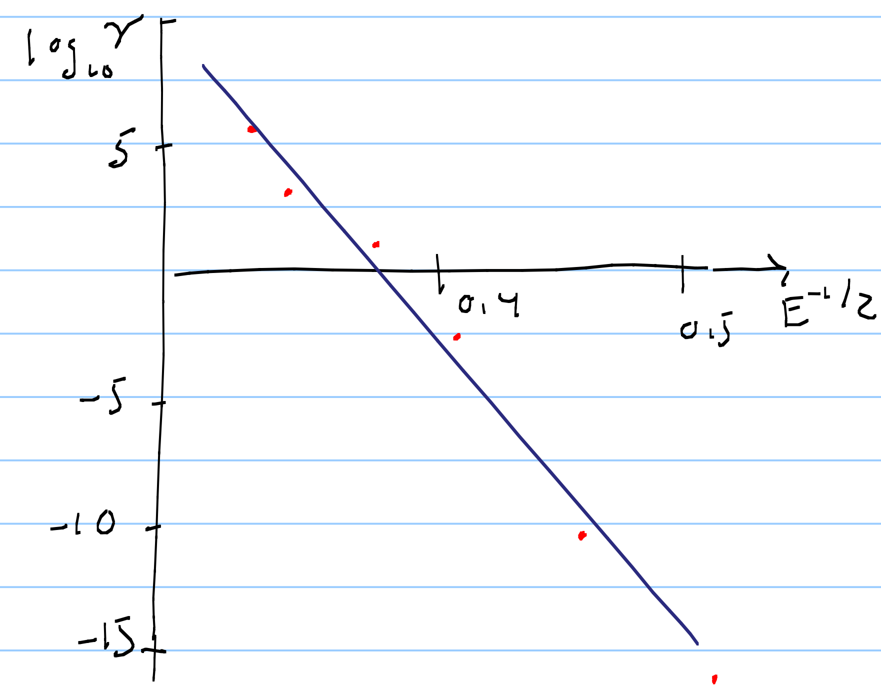
Half-life $\equiv T_{1/2}$: Time for which half the atoms have decayed.

$$e^{-\gamma T_{1/2}} = \frac{1}{2}, \quad \gamma = \frac{\ln 2}{T_{1/2}}$$

Th^{232} : $E = 4.05$ MeV,
$T_{1/2} = 1.39 \times 10^{10}$ years
$= 4.4 \times 10^{17}$ sec

Po^{212} : $E = 8.95$ MEV
$T_{1/2} = 3.0 \times 10^{-7}$ sec

Empirical slope agrees To within a few % of the data



THE POTENTIAL BARRIER

CASE 2: $E > V_0$

– THE INTERNAL REGION SOLUTIONS NOW READ:

$$\psi(x) = F e^{i k_2 x} + G e^{-i k_2 x} \qquad 0 < x < a$$

with $k_2 = \sqrt{\dfrac{2m(E-V_0)}{\hbar^2}}$

FOR THIS WE GET:

$$R = \frac{|B|^2}{|A|^2} = \left[1 + \frac{4E(E-V_0)}{V_0^2 \sin^2(k_2 a)} \right]^{-1}$$

$$T = \frac{|C|^2}{|A|^2} = \left[1 + \frac{V_0^2 \sin^2(k_2 a)}{4E(E-V_0)} \right]^{-1}$$

NOTE THAT $T = 1$ WHEN $k_2 a = \pi, 3\pi, \ldots$! why?

what is happening?