LECTURE 20: One-Dim. Problems: The Finite Square Well

Goals of the 1-d lectures: learn how to solve Schrodinger's equation for some simple problems

What I expect you to learn:

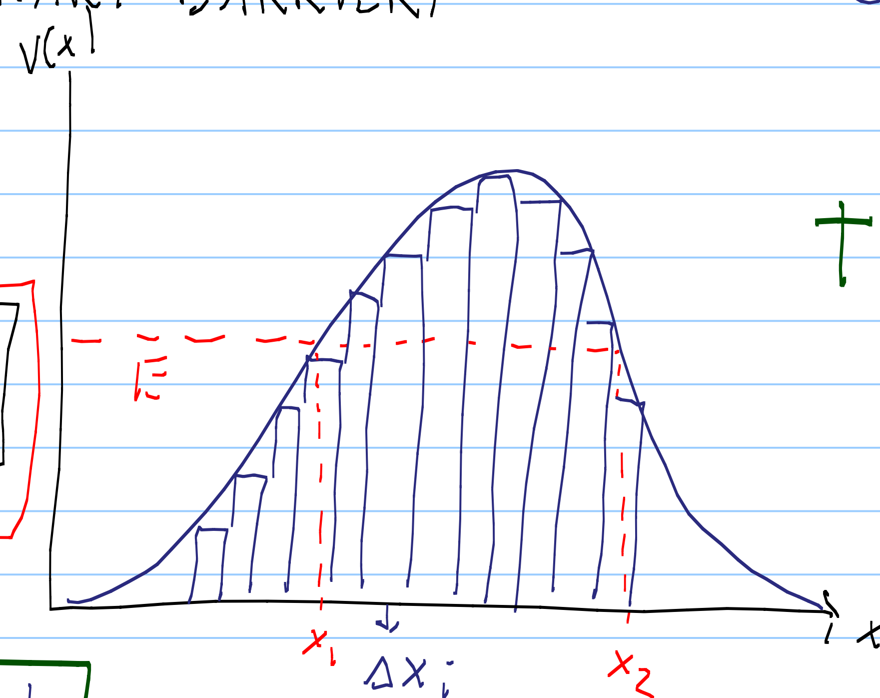
- How to solve the finite well problem
- Properties of the solutions when $E > 0$ (e.g. "resonant transmission" effect)
- Properties of the solutions when $E < 0$

(Corresponds to sections 4.6 of textbook)

TUNNELING EFFECT (ARBITRARY BARRIER)

We saw how to approximate the transmission coefficient by:

$$\boxed{T \sim \exp\left[\frac{-2}{\hbar}\int_{x_1}^{x_2} dx \sqrt{2m(V(x) - E)}\right]}$$



T?

RECAP:

$$\boxed{T = \left[1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 k_2 a\right]^{-1}}$$

if $k_2 a$ is large $\rightarrow$ $\sinh z = \frac{e^z - e^{-z}}{2} \rightarrow \frac{e^z}{2}$

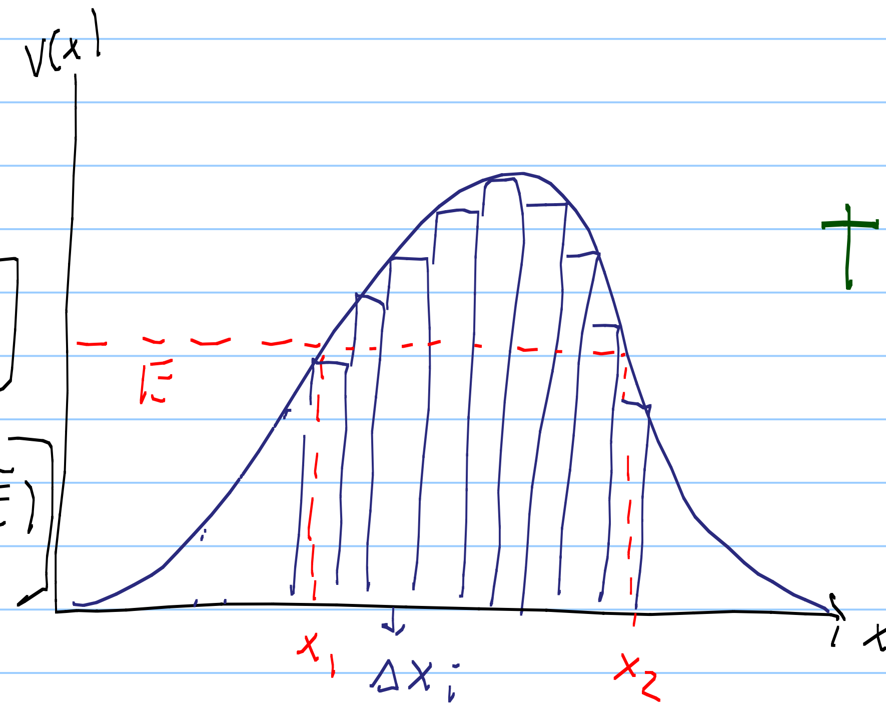
$$\rightarrow T = \left[1 + \frac{V_0^2}{4E(V_0 - E)} \frac{e^{2k_2 a}}{4}\right]^{-1}$$

$$\approx \left[\frac{V_0^2}{16E(V_0 - E)} e^{2k_2 a}\right]^{-1} = \boxed{\frac{16E(V_0 - E)}{V_0^2} e^{-2k_2 a}}$$

(3)

Transmission though width Δx_i:

$$T_i \sim \exp\left[-\frac{2\Delta x_i}{\hbar}\sqrt{2m(V(x_i)-E)}\right]$$



T?

$$T \sim \lim_{N\to\infty} \prod_{i=1}^{N} \exp\left[-\frac{2\Delta x_i}{\hbar}\sqrt{2m(V(x_i)-E)}\right]$$

$$= \exp\left[-\frac{2}{\hbar}\lim_{\Delta x_i\to 0}\sum_i \Delta x_i\sqrt{2m(V(x_i)-E)}\right]$$

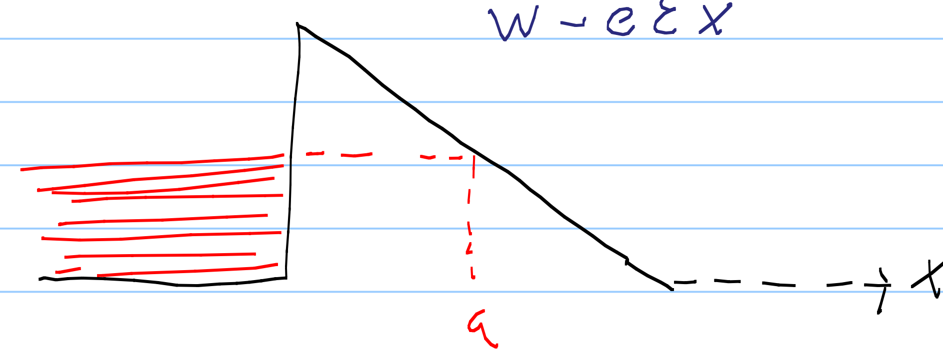
$$= \exp\left[-\frac{2}{\hbar}\int_{x_1}^{x_2} dx\sqrt{2m[V(x)-E]}\right]$$

Example: Field emission

$$V(x) = V_0 - e\mathcal{E}x$$

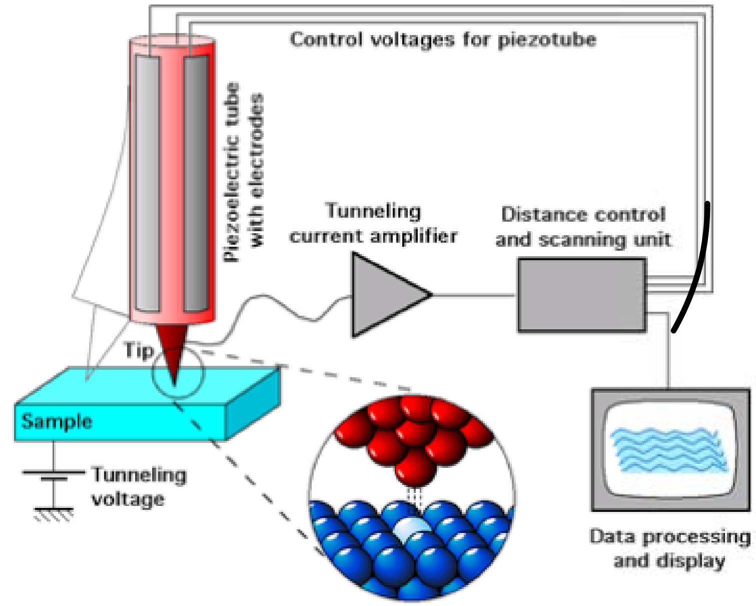
$$\boxed{V_0 - E = W}$$

$W - e\mathcal{E}x$



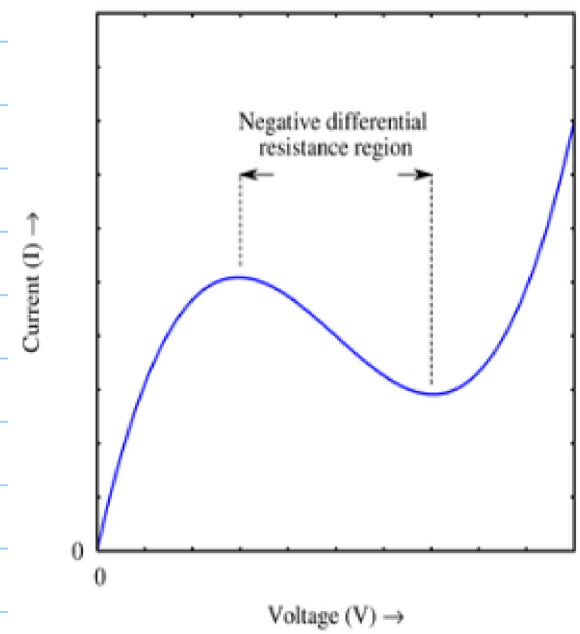
TUNNELING EXAMPLES:

Scanning Tunneling Microscope



TUNNELING DIODE (ESAKI DIODE)

- very Thin p-n junction
- current increases at first due to Tunneling effect
- current then falls because of increased potential step
- then diode operates as a regular diode



TUNNELING EXAMPLES (CONT.)

IS IT POSSIBLE TO CREATE A UNIVERSE IN THE LABORATORY BY QUANTUM TUNNELING?

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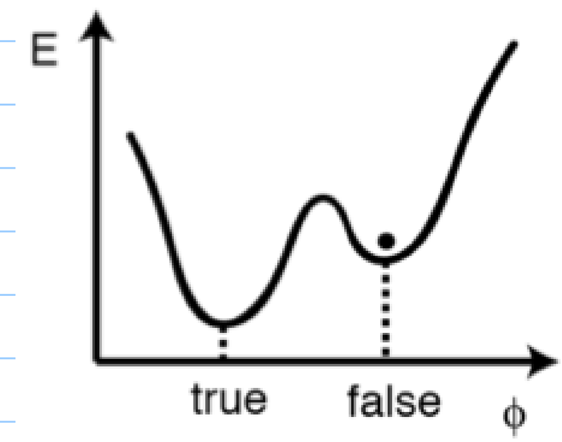
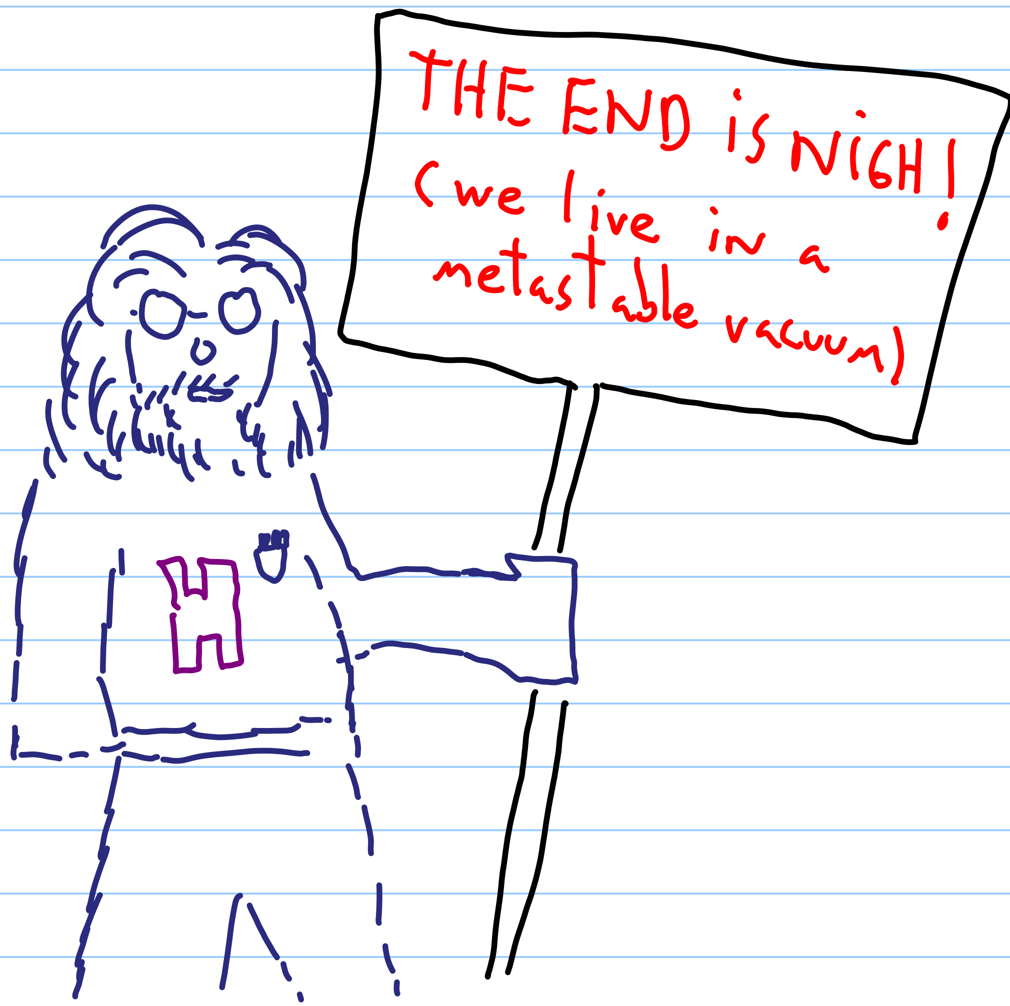
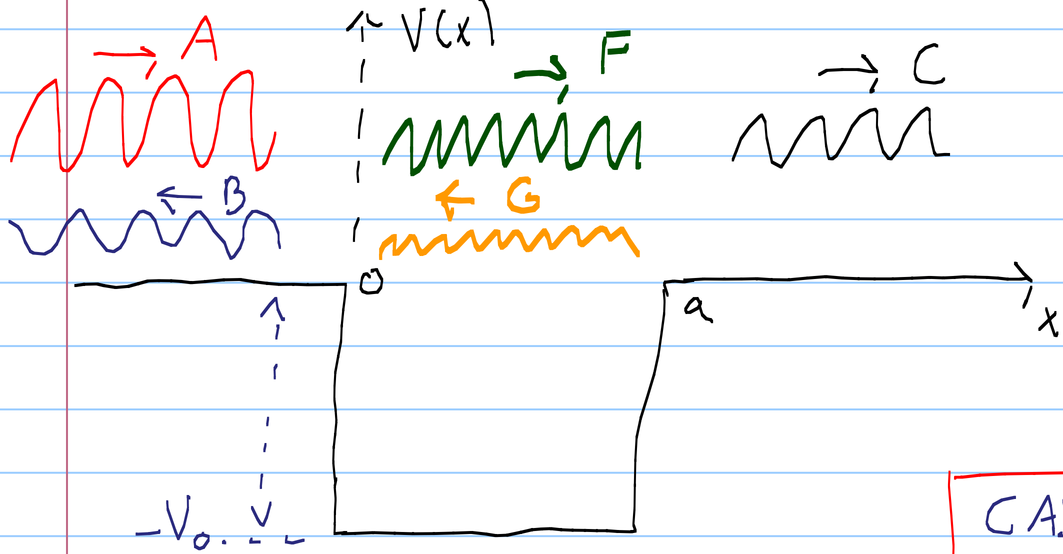
We explore the possibility that a new universe can be created by producing a small bubble of false vacuum. The initial bubble is small enough to be produced without an initial singularity, but classically it could not become a universe – instead it would reach a maximum radius and then collapse. We investigate the possibility that quantum effects allow the bubble to tunnel into a larger bubble, of the same mass, which would then classically evolve to become a new universe. The calculation of the tunneling amplitude is attempted, in lowest order semiclassical approximation (in the thin-wall limit), using both a canonical and a functional integral approach. The canonical approach is found to have flaws, attributable to our method of space-time slicing. The functional integral approach leads to a euclidean interpolating solution that is not a manifold. To describe it, we define an object which we call a "pseudomanifold", and give a prescription to define its action. We conjecture that the tunneling probability to produce a new universe can be approximated using this action, and we show that this leads to a plausible result.

TUNNELING, INFLATION, VACUUM STABILITY, ETC.





THE FINITE WELL



$$\boxed{\begin{aligned} V(x) &= 0 \,, & x &< a \\ V(x) &= V_0 \,; & 0 < x &< a \\ V(x) &= 0 \,; & x &> a \end{aligned}}$$

$$\boxed{\text{CASE 1: } E > 0}$$

IN REGIONS $x < 0$ AND $x > a$, THE PARTICLE IS FREE SO WE CAN WRITE:

$$A e^{ik_1 x} + B e^{-ik_1 x} \quad \text{for} \quad x < 0$$

$$C e^{ik_1 x} + \cancel{D} e^{-ik_1 x} \quad \text{for} \quad x > a$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

SO WE WRITE FOR $0 < x < a$: $\psi(x) = F e^{ik_2 x} + G e^{-ik_2 x}$

$$k_2 = \sqrt{\frac{2m(V_0 + E)}{\hbar^2}}$$

THE FINITE WELL (cont.)

RECALL THAT FOR THE POTENTIAL BARRIER WITH $E > V_0$, WE HAD:

$$A e^{ik_1 x} + B e^{-ik_1 x} \quad \text{for} \quad x < 0$$
$$C e^{ik_1 x} + \cancel{D} e^{-ik_1 x} \quad \text{for} \quad x > a$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

SO WE WRITE FOR $0 < x < a$: $\psi(x) = F e^{+ik_2 x} + G e^{-ik_2 x}$

$$k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

NOW WE HAVE: $k_2 = \sqrt{\frac{2m(E + V_0)}{\hbar^2}}$

$(E + V_0)$ and $(E - V_0)$ are both positive

$\Rightarrow$ $\boxed{\text{solutions are qualitatively similar}}$

→ so you are solving this in problem set #3 🙂

THE FINITE WELL (cont.)

IN THE EXPRESSIONS WE OBTAINED FOR THE POTENTIAL BARRIER WITH $E > V_0$, I REPLACE:

$$(E - V_0) \quad \text{by} \quad (E + V_0)$$

AND K_2 CHANGES FROM: $\sqrt{\frac{2m(E-V_0)}{\hbar^2}}$ TO $\sqrt{\frac{2m(E+V_0)}{\hbar^2}}$

AND OBTAIN:

$$R = \left[1 + \frac{4E(V_0+E)}{V_0^2 \sin^2(K_2 a)}\right]^{-1}$$

$$T = \left[1 + \frac{V_0^2 \sin^2(K_2 a)}{4E(V_0+E)}\right]^{-1}$$

* NOTE THAT IN GENERAL $T \neq 1$ (differs from classical result)

* NOTE AGAIN THAT FOR $K_2 a = n\pi$, $T = 1$, $R = 0$

* $T \to 0$ WHEN $E \to 0$

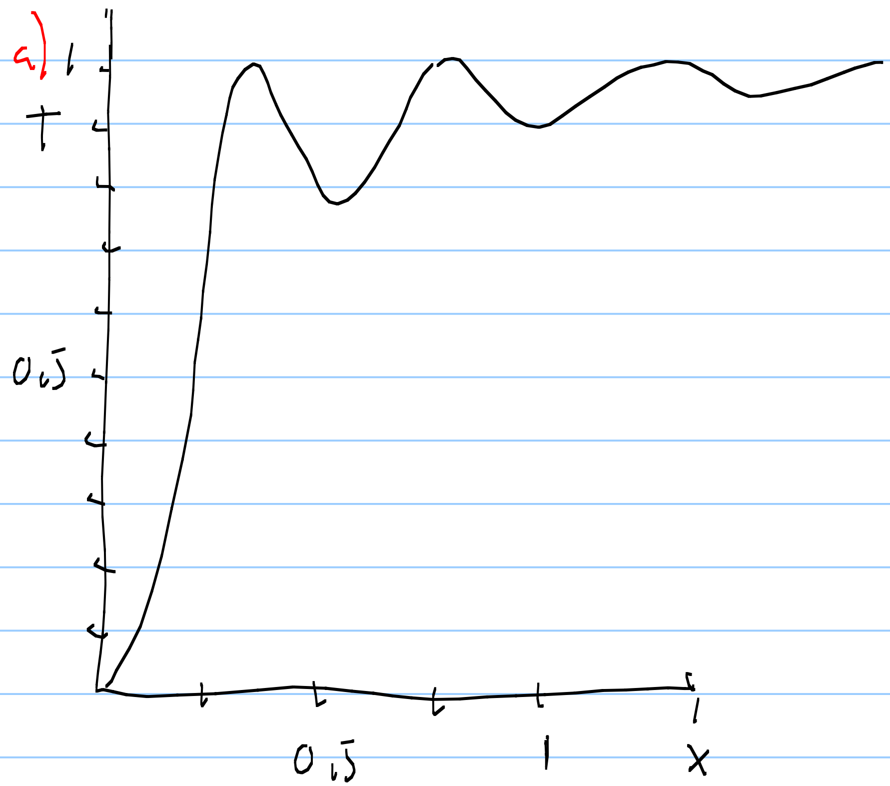
* $T \to 1$ WHEN $E \to \infty$

THE FINITE WELL (cont.)

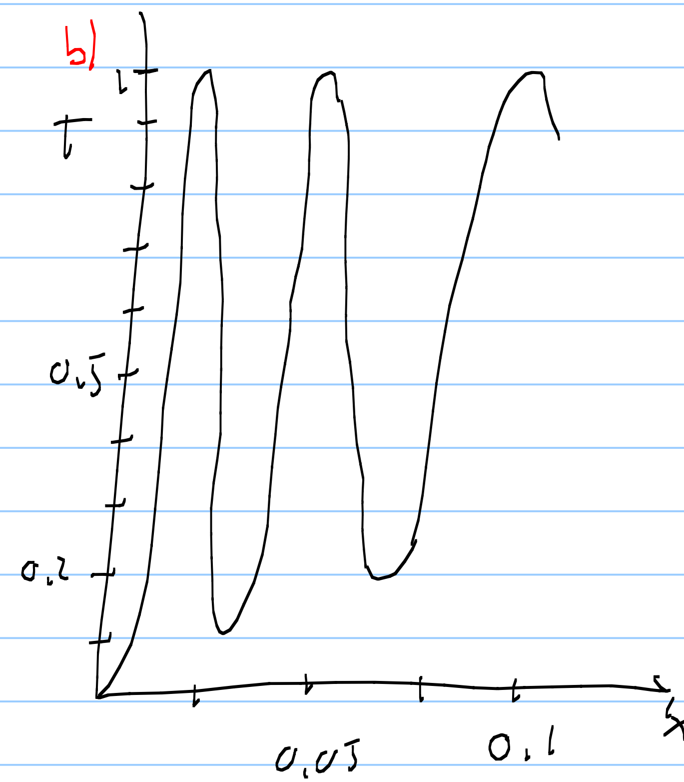
PLOT OF T VS E/V_0 $\quad \dfrac{2mV_0 a^2}{\hbar^2} = S_p \sim 10 \xrightarrow{} a, \quad 100 \xrightarrow{} b$

a) $T = \left[1 + \dfrac{\sin^2\left(\sqrt{10}\,(1+x)\right)}{4x(1+x)} \right]^{-1}, \quad x = E/V_0$

a)



b)



→ PEAKS AT $T=1$ WHEN: $k_2 a = \sqrt{\dfrac{2m(V_0+E)}{\hbar^2}}\, a = \sqrt{S_p(1+x)} = N\pi$

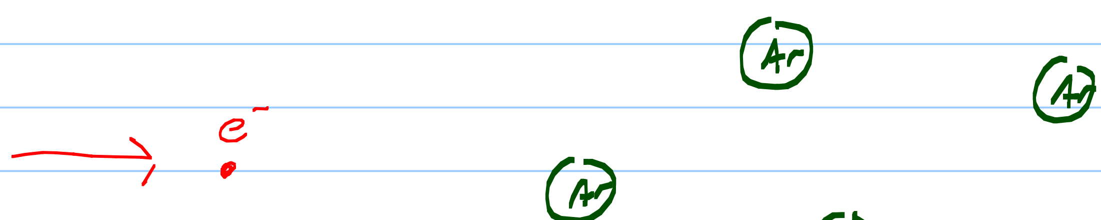
THE FINITE WELL (cont.)

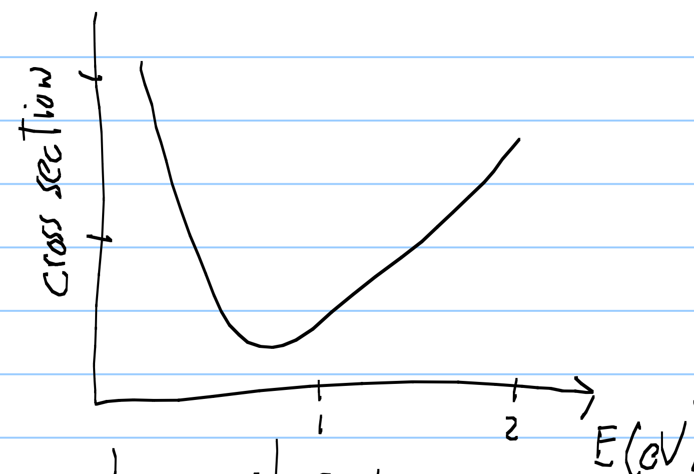
LET'S VISUALISE WHAT HAPPENS WHEN $R=0, T=1$:

http://www.abdn.ac.uk/physics/vpl/barrier/applet.htm

http://www.physics.brocku.ca/www/faculty/sternin/teaching/mirrors/qm/packet/wave-map.html

TRANSMISSION RESONANCE MANIFESTS ITSELF IN THE SCATTERING OF LOW ENERGY ELECTRONS BY NOBLE GAS ATOMS : RAMSAUER - TOWNSEND EFFECT



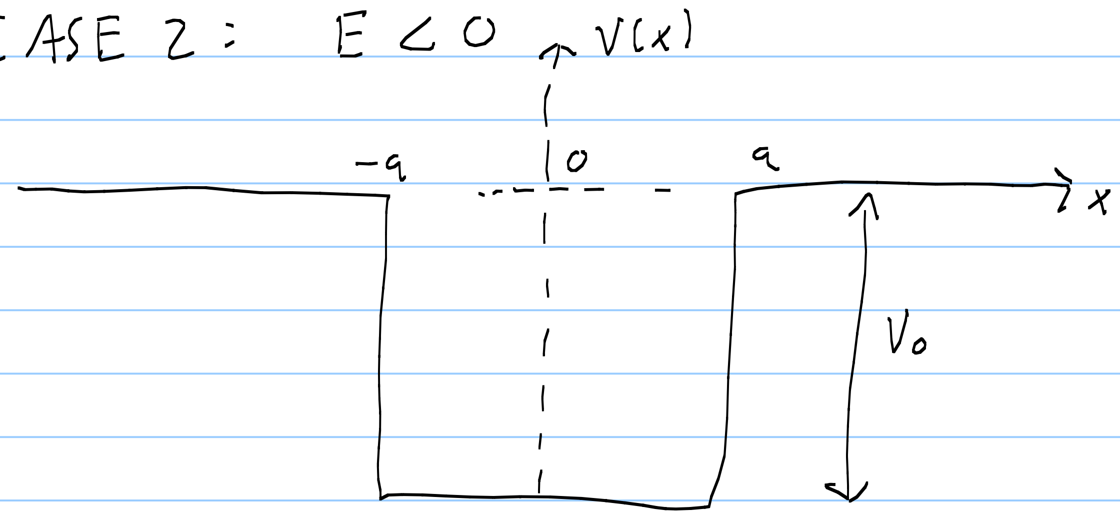


~ Scattering prob. drops (T becomes large) when electron wavelength is a multiple of the well's length

THE FINITE WELL (cont.)

CASE 2: $E < 0$

$$V(x) = -V_0, \quad |x| < a$$
$$= 0, \quad |x| > a$$



For $|x| < a$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - V(x))\psi, \quad \frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - V(x))\psi$$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\underbrace{(E + V_0)}_{\text{negative}}\psi = 0 \rightarrow \frac{d^2\psi}{dx^2} + \alpha^2\psi = 0 \qquad (1)$$

→ $(V_0 - |E|)$

THE FINITE WELL (cont.)

$\boxed{|x| > a}$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))\psi = 0 \quad , \quad V(x) = 0$$

E is negative

$$\rightarrow \frac{d^2\psi}{dx^2} - \frac{2mE}{\hbar^2} = 0 \rightarrow \frac{d^2\psi}{dx^2} - \beta^2\psi = 0 \quad (2)$$

Solutions to:

(1) : $A\cos\alpha x + B\sin\alpha x$

$B = 0$, even soln. $\qquad A = 0$, odd soln.

(2) : $C_1 e^{\beta x} + C_2 e^{-\beta x}$

$C_1 = C_2$ for even soln.

$C_1 = -C_2$ for odd soln.

we will use $C = C_2$

THE FINITE WELL (cont.)

Continuity relations

$x = a$, even solutions:

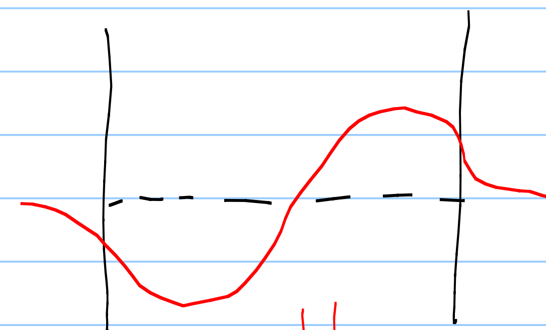
(3) $A \cos\alpha a = C e^{-\beta a}$ for $\psi(x)$

(4) $-\alpha A \sin\alpha a = -\beta C e^{-\beta a}$ for $\frac{d\psi(x)}{dx}$

(4) divided by (3) gives:

$$\alpha \tan\alpha a = \beta \qquad (5)$$

$x = a$, odd solutions



(6) : $\beta \sin\alpha a = C e^{-\beta a}$

(7) $\alpha\beta \cos\alpha a = -C\beta e^{-\beta a}$

(7)/(6) : $\alpha \cot\alpha a = -\beta$

set $\xi = \alpha a$, $\eta = \beta a$

$\rightarrow$ $\xi \tan\xi = \eta$ (even), $\xi \cot\xi = -\eta$ (odd)

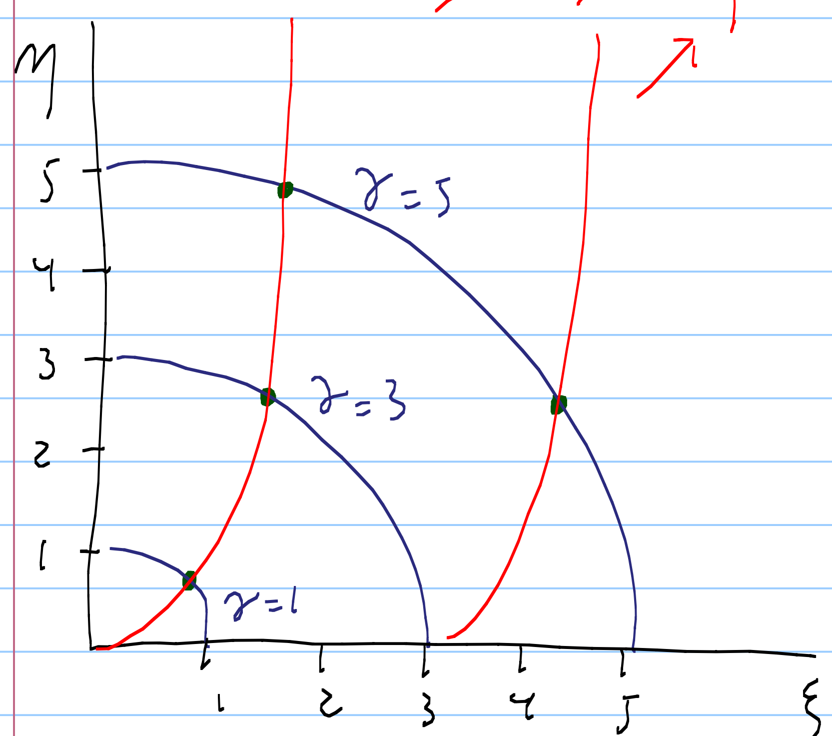
THE FINITE WELL (cont.)

$$\xi^2 = \alpha^2 a^2 = \frac{2m}{\hbar^2}(E+V_0)a^2 = \frac{2m}{\hbar^2}(V_0 - |E|)a^2$$

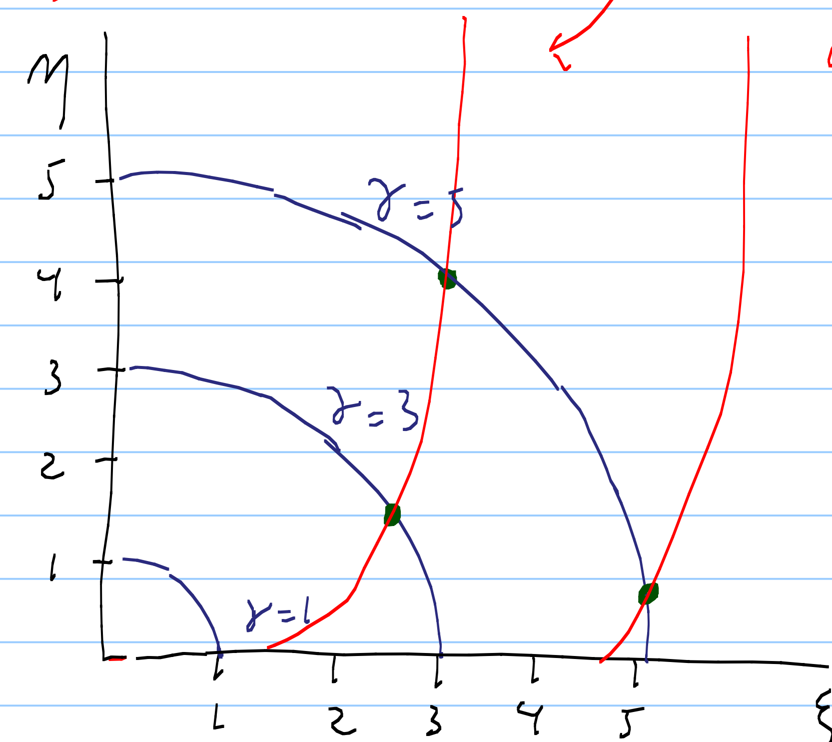
$$\eta^2 = \beta^2 a^2 = \frac{2m|E|a^2}{\hbar^2}$$

$$\xi^2 + \eta^2 = \gamma^2 = \frac{2mV_0 a^2}{\hbar^2}$$

even: $\eta = \xi \tan \xi$



odd: $\eta = -\xi \cot \xi$



THE FINITE WELL (cont.)

NOTE THAT FOR $\gamma = 1$ we have 1 bound state
$\gamma = 3$ " " 2 bound states
$\gamma = 5$ " " 4 " "

Remember that $\gamma = \sqrt{\dfrac{2mV_0a^2}{\hbar^2}}$

Example: Find the energy of the bound state when $\gamma = 1$

$$\xi \tan \xi = \sqrt{1-\xi^2}$$

$$\xi^2 \tan^2 \xi = 1 - \xi^2$$

$$\xi^2 (1 + \tan^2 \xi) = 1 = \xi^2 \left[\frac{\cos^2 \xi}{\cos^2 \xi} + \frac{\sin^2 \xi}{\cos^2 \xi} \right] = \frac{\xi^2}{\cos^2 \xi}$$

$$\Rightarrow \cos^2 \xi = \xi^2 \rightarrow \xi \simeq 0.74$$

$$(0.74 \tan 0.74)^2 = \frac{2mE a^2}{\hbar^2}, \qquad E = \frac{\hbar^2}{2ma^2}.$$

EXERCISE: WHAT IS THE NORMALISED EIGENFUNCTION ?

PROBLEM SET #3 DUE NOV. 10^{th}

1- Derive the expressions for T and R on page 16 of lecture 18

2- A particle in an infinite well is in the ground state of the system. The well has walls at $x = a$ and $x = -a$. If we move the walls instantaneously to $x = 2a$ and $x = -2a$, what is the probability of finding the particle in the ground state of the new system?

3- A particle is in an infinite well with walls at $x = 0$, $x = a$. Calculate:

- $\langle x_n \rangle$, $\langle p_n \rangle$, $\langle x_n^2 \rangle$, $\langle p_n^2 \rangle$
- calculate $\Delta x_n \Delta p_n$ ($\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$
- Estimate the ground state energy using the result above. Compare to the result we obtained,