LECTURE 24: The QM Postulates (part 3)

What I expect you to learn:

- What are the QM postulates
- How to work with the Dirac notation
- How to work with the matrix representation of operators

(Roughly corresponds to sections 5.6-5.7 of textbook)

(Problem set 4 assigned at the end of the notes)

MATRIX REPRESENTATION

THE ACTION OF A LINEAR HERMITIAN OPERATOR $\hat{A}$ ON A STATE VECTOR PRODUCES ANOTHER STATE VECTOR:

$$|\chi\rangle = \hat{A}|\psi\rangle$$

THE STATE VECTOR χ CAN BE EXPANDED IN TERMS OF THE EIGENVECTOR BASIS $|\psi_n\rangle$:

$$|\chi\rangle = \sum_m a_m|\psi_m\rangle$$

THE COEFFICIENTS a_m ARE GIVEN BY

$$a_m = \langle\psi_m|\chi\rangle$$

$$= \langle\psi_m|A|\psi\rangle \quad \longrightarrow \text{expand } |\psi\rangle = \sum_n b_n|\psi_n\rangle$$

$$= \sum_n \langle\psi_m|A|\psi_n\rangle b_n = \sum_n A_{mn} b_n \qquad (1)$$

(1) CAN BE WRITTEN AS A MATRIX EQUATION

MATRIX REPRESENTATION

With $a_n = \langle \psi_n | \psi \rangle$
$b_n^* = \langle \chi | \psi_n \rangle$

we can write $\langle \chi | \psi \rangle$ as: $\sum_n \langle \chi | \psi_n \rangle \langle \psi_n | \psi \rangle$

$= \sum_n b_n^* a_n$ or $\boxed{b^\dagger \cdot a}$

$$\rightarrow \langle \chi | \psi \rangle = (b_1^*, b_2^*, \ldots) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix}$$

DIAGONAL MATRICES:

THE MATRIX ELEMENTS OF A IN THE BASIS OF THE $|\psi_n\rangle$:

$$A_{mn} = \langle \psi_m | A | \psi_n \rangle = a_n \langle \psi_n | \psi_m \rangle = a_n \delta_{nm}$$

A in the basis of its orthonormal eigenvectors is diagonal

MATRIX REPRESENTATION

WE WILL WANT TO SOLVE MATRIX EIGENVALUE EQUATIONS:

$$A u_n = a_n u_n$$

u_n → eigenvector of A

A → matrix representing observable A in a given basis.

A is Hermitian → eigenvalues are real and they can be found with the equation:

$$\boxed{\det | A - a_n I | = 0}$$

I → identity matrix

Example:

$$\begin{pmatrix} A_{11}-a_1 & A_{12} & A_{13} \\ A_{21} & A_{22}-a_1 & A_{23} \\ A_{31} & A_{32} & A_{33}-a_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{vmatrix} A_{11}-a_1 & A_{12} & A_{13} \\ A_{21} & A_{22}-a_1 & A_{23} \\ A_{31} & A_{32} & A_{33}-a_1 \end{vmatrix} = 0$$

MATRIX REPRESENTATION

MATH REFRESHER EXAMPLES: Find eigenvalues and eigenvectors of:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - 1 = 0 \ , \quad \lambda^2 = 1$$

We have two eigenvalues: $\lambda = 1$, $\lambda = -1$

CASE 1: $\lambda = 1$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} x_2 = x_1 \\ x_1 = x_2 \end{matrix}$$

let's pick $x_1 = 1 \Rightarrow x_2 = 1$, eigenvector: $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_1$

CASE 2: $\lambda = -1$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -1 \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} x_2 = -x_1 \\ x_1 = -x_2 \end{matrix}$$

MATRIX REPRESENTATION

we pick $x_1 = 1 \Rightarrow x_2 = -1$, eigenvector: $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = v_2$

$$v_1 \cdot v_2 = (1, 1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 - 1 = 0$$

v_1, v_2 are orthogonal but they are not normalized.

EXAMPLE: NORMALIZE $\begin{pmatrix} 1 \\ 3 \\ 2i \end{pmatrix}$

WE WANT: $(x_1^*, x_2^*, x_3^*) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = |x_1|^2 + |x_2|^2 + |x_3|^2 = 1$

$$c^* (1, 3, -2i)\, c \begin{pmatrix} 1 \\ 3 \\ 2i \end{pmatrix} = |c|^2 (1 + 4 + 9) = 1$$

$$\Rightarrow c = \frac{1}{\sqrt{14}}$$

MATRIX REPRESENTATION

ANOTHER EXAMPLE: a system is in the following state:
$$|\psi\rangle = \sqrt{\tfrac{2}{7}}\,|\varphi_1\rangle + \sqrt{\tfrac{3}{7}}\,|\varphi_2\rangle + \sqrt{\tfrac{1}{7}}\,|\varphi_3\rangle + \sqrt{\tfrac{1}{7}}\,|\varphi_4\rangle$$
where $|\varphi_n\rangle$ are the eigenstates of the system's Hamiltonian such that:
$$\hat{H}|\varphi_n\rangle = n^2 \varepsilon_0 |\varphi_n\rangle$$

→ if the energy is measured, what values will be obtained and with what probabilities?

possible values of E_n: ε_0, $4\varepsilon_0$, $9\varepsilon_0$, $16\varepsilon_0$

Probabilities: $P(E_n) = |\langle\varphi_n|\psi\rangle|^2$

$$P(E_1) = \left|\sqrt{\tfrac{2}{7}}\langle\varphi_1|\varphi_1\rangle\right|^2 = \frac{2}{7}\,, \quad P(E_2) = \left|\sqrt{\tfrac{3}{7}}\langle\varphi_2|\varphi_2\rangle\right|^2 = \frac{3}{7}$$

$$P(E_3) = P(E_4) = \frac{1}{7}$$

MATRIX REPRESENTATION

ANOTHER EXAMPLE: Consider a system whose Hamiltonian and state vector are given by:

$$H = \varepsilon \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad |\psi\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1-i \\ 1-i \\ 1 \end{pmatrix}$$

1 – WHAT ARE THE ENERGIES THAT CAN BE MEASURED AND WITH WHAT PROBABILITIES?

2 – CALCULATE $\langle \hat{H} \rangle$

1 – FIRST STEP, FIND EIGENVALUES:

$$\left| \begin{pmatrix} -\lambda & i\varepsilon & 0 \\ -i\varepsilon & -\lambda & 0 \\ 0 & 0 & -\varepsilon-\lambda \end{pmatrix} \right| = 0$$

$$-\lambda \cdot -\lambda \cdot -(\varepsilon+\lambda) + i\varepsilon \cdot i\varepsilon \cdot -(\varepsilon+\lambda)$$
$$= -\lambda^2\varepsilon - \lambda^3 + \varepsilon^3 + \varepsilon^2\lambda = 0$$

MATRIX REPRESENTATION

$$\varepsilon^2(\varepsilon+\lambda) - \lambda^2(\varepsilon+\lambda) = 0$$

$$\rightarrow \quad \lambda_1 = \varepsilon, \quad \lambda_2 = -\varepsilon, \quad \lambda_3 = -\varepsilon$$

Eigenvector 1 associated with λ_1:

$$\begin{pmatrix} 0 & i\varepsilon & 0 \\ -i\varepsilon & 0 & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \varepsilon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{aligned} i\varepsilon x_2 &= \varepsilon x_1 & &\rightarrow & i x_2 &= x_1 \\ -i\varepsilon x_1 &= \varepsilon x_2 & &\rightarrow & -i x_1 &= x_2 \\ -\varepsilon x_3 &= \varepsilon x_3 & &\rightarrow & x_3 &= 0 \end{aligned} \quad \rightarrow \quad \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = |\varphi_1\rangle$$

We normalize $\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \rightarrow c^2(1+1+0) = 1, \quad c = \frac{1}{\sqrt{2}}$

Eigenvector 2 associated with $\lambda_2 = -\varepsilon$:

MATRIX REPRESENTATION

$$\begin{pmatrix} 0 & i\varepsilon & 0 \\ -i\varepsilon & 0 & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\varepsilon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{aligned} i\varepsilon x_2 &= -\varepsilon x_1 & &\rightarrow & i x_2 &= -x_1 \\ -i\varepsilon x_1 &= -\varepsilon x_2 & &\rightarrow & i x_1 &= x_2 \quad \rightarrow \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = |\varphi_2\rangle \\ \varepsilon x_3 &= \varepsilon x_3 & &\rightarrow & x_3 &= 0 \end{aligned}$$

$x_3 = 1$

normalization : $\frac{1}{\sqrt{2}}$

third vector $\rightarrow$ pick orthonormal vector to the other two: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |\varphi_3\rangle$

We can write

$$|\psi\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1-i \\ 1-i \\ 1 \end{pmatrix} \quad \text{as: } \sqrt{\frac{2}{5}}\, |\varphi_1\rangle + \sqrt{\frac{2}{5}}\, |\varphi_2\rangle + \frac{1}{\sqrt{5}}\, |\varphi_3\rangle$$

MATRIX REPRESENTATION

Probability of measuring $E_1 = \varepsilon$:

$$P_1 = |\langle e_1|\psi\rangle|^2 = \left|\sqrt{\tfrac{2}{5}}\langle e_1|e_1\rangle\right|^2 = \frac{2}{5}$$

Probability of measuring $-\varepsilon$:

$$|\langle e_2|\psi\rangle|^2 + |\langle e_3|\psi\rangle|^2 = \frac{3}{5}$$

$$2-\quad \langle\hat{H}\rangle = \frac{3}{5}\cdot -\varepsilon + \frac{2}{5}\cdot\varepsilon = \frac{-\varepsilon}{5}$$

or :

$$\langle\hat{H}\rangle = \langle\psi|\hat{H}|\psi\rangle = \frac{\varepsilon}{5}(1+i, 1+i, 1)\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1-i \\ 1-i \\ 1 \end{pmatrix}$$

$$= \frac{\varepsilon}{5}(1+i, 1+i, 1)\begin{pmatrix} i+1 \\ -i-1 \\ -1 \end{pmatrix} = \frac{\varepsilon}{5}\cdot[i+1-1+i-i-1+1-i-1] = \frac{-\varepsilon}{5}$$

MATRIX REPRESENTATION

One more example: consider a system whose HAMILTONIAN is GIVEN BY:

$$\hat{H} = \begin{pmatrix} \varepsilon & -\varepsilon & 0 \\ -\varepsilon & \varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix}$$

AND AN OPERATOR $\hat{A}$:

$$\hat{A} = \begin{pmatrix} 0 & 4a & 0 \\ 4a & 0 & 1 \\ 0 & a & 0 \end{pmatrix}$$

1 - If we measure the energy, what values can we obtain?

2 - If we measure the energy and obtain $-\varepsilon$ and then measure A, what values of A can we obtain and with what probabilities?

3 - Calculate the uncertainty ΔA

MATRIX REPRESENTATION

$$1-\quad \left|\begin{pmatrix} \varepsilon-\lambda & -\varepsilon & 0 \\ -\varepsilon & \varepsilon-\lambda & 0 \\ 0 & 0 & -\varepsilon-\lambda \end{pmatrix}\right| = 0$$

$$(\varepsilon-\lambda)\cdot(\varepsilon-\lambda)\cdot-(\varepsilon+\lambda) + -1\cdot-\varepsilon\cdot-\varepsilon\cdot-(\varepsilon+\lambda)$$

$$= (\varepsilon-\lambda)^2\cdot-(\varepsilon+\lambda) + \varepsilon^2(\varepsilon+\lambda)$$

$$\lambda_1 = 0, \quad \lambda_2 = -\varepsilon, \quad \lambda_3 = 2\varepsilon$$

$\lambda_1 = 0$

$$\begin{pmatrix} \varepsilon & -\varepsilon & 0 \\ -\varepsilon & \varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\varepsilon x_1 - \varepsilon x_2 = 0 \qquad x_1 = x_2 \qquad \rightarrow |\varphi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$-\varepsilon x_3 = 0 \qquad x_3 = 0$$

MATRIX REPRESENTATION

$\lambda_2 = -\varepsilon$

$$\begin{pmatrix} \varepsilon & -\varepsilon & 0 \\ -\varepsilon & \varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\varepsilon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \begin{aligned} \varepsilon x_1 - \varepsilon x_2 &= -\varepsilon x_1 \\ -\varepsilon x_1 + \varepsilon x_2 &= -\varepsilon x_2 \\ -\varepsilon x_3 &= -\varepsilon x_3 \end{aligned}$$

$\hookrightarrow$ use $x_3 = 1$

$\rightarrow x_1 = 0 = x_2$

$$|e_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

2-

Eigenvalues of $\hat{A}$: $\quad a_1 = -\sqrt{17}\,a, \quad a_2 = 0, \quad a_3 = \sqrt{17}\,a$

Eigenvectors of $\hat{A}$: $\quad \dfrac{1}{\sqrt{34}} \begin{pmatrix} 4 \\ -\sqrt{17} \\ 1 \end{pmatrix}, \quad \dfrac{1}{\sqrt{17}} \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \quad \dfrac{1}{\sqrt{34}} \begin{pmatrix} 4 \\ \sqrt{17} \\ 1 \end{pmatrix}$

$\boxed{\text{MATRIX REPRESENTATION}}$

After obtaining $-\varepsilon$, system is in state $|\varphi_2\rangle$

Prob of measuring a_2: $|\langle\varphi_2|\varphi_2\rangle|^2$

$$= \left| \frac{1}{\sqrt{17}} (1 \quad 0 \quad -4) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{16}{17} = 94\%$$

etc.

3- Since the system is in state $|\varphi_2\rangle$ before we measure $\hat{A}$ we have:

$$\Delta A = \sqrt{\langle\varphi_2|A^2|\varphi_2\rangle - \langle\varphi_2|A|\varphi_2\rangle^2}$$

First Term: $a^2(0,0,1) \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = a^2$

MATRIX REPRESENTATION

second Term : $\langle \varphi_2 | A | \varphi_2 \rangle =$

$$a\,(0,0,1)\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$\Rightarrow \Delta A = a$

→ Will the probability of measuring E_2 first and then a_2 be the same as measuring a_2 first and then E_2? How could you check this quickly?

PROBLEM SET #4:

1 -Using the derivation in lecture 22, pages 17-18 show that the wave packet that minimizes $\Delta x \Delta p$ is the Gaussian wave packet (see page 215 of textbook for hints)

2-Textbook problem 5.11

3-Textbook problem 5.12