# LECTURE 26: The Harmonic Oscillator (Continued)



What I expect you to learn:

- How to obtain the energy eigenvalues of the harmonic oscillator using a method of Dirac

- How to contruct the wave eigenfunctions using the method above

- How to derive and work with the relevent operators in matrix representation

(Corresponds to chapter 5.6, 5.7 of the textbook)

(Problem set 4: see page 17 of lecture 24)

# THE HARMONIC OSCILLATOR

THE HAMILTONIAN:

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2}Kx^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$$

→ we introduced the following operators

$$\hat{a}_\pm = \frac{1}{\sqrt{2}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} \mp i\frac{\hat{p}}{\sqrt{m\hbar\omega}}\right]$$

aka "raising" and "lowering" operators

WE CAN REWRITE $\hat{H}$ IN TERMS OF $a_\pm$:

$$\hat{H} = \frac{\hbar\omega}{2}(a_-a_+ + a_+a_-)$$ → check that this True!

$$\rightarrow \quad a_-a_+ - a_+a_- = 1 \quad \rightarrow \quad a_+a_- = a_-a_+ - 1$$

$$\hat{H} = \hbar\omega(a_-a_+ - 1/2) = \hbar\omega(a_+a_- + 1/2)$$

# THE HARMONIC OSCILLATOR



$[\hat{H}, \hat{a}_\pm] = \pm \hbar\omega \hat{a}_\pm$ , $\qquad \hat{H}\hat{a}_\pm - \hat{a}_\pm \hat{H} = \pm\hbar\omega\hat{a}_\pm$

$\hat{H}|E_n\rangle = E_n|E_n\rangle$ , using $\qquad \hat{H}\hat{a}_\pm = \hat{a}_\pm\hat{H} \pm \hbar\omega\hat{a}_\pm$

$\hat{H}\hat{a}_\pm|E_n\rangle = (a_\pm \hat{H} \pm \hbar\omega a_\pm)|E_n\rangle$

$\hat{H}\,(a_\pm|E_n\rangle) = (E_n \pm \hbar\omega)(a_\pm|E_n\rangle)$

i.e. $\quad \hat{a}_+|E_n\rangle = |☺\rangle$ $\qquad$ i.e a new state vector

$\qquad \hat{a}_-|E\rangle = |☹\rangle$

$\hat{H}|☺\rangle = (E_n + \hbar\omega)|☺\rangle$

$$\boxed{\Rightarrow |☺\rangle \text{ is an eigenket of } \hat{H}}$$

$\hat{H}|☺\rangle = (E_n + \hbar\omega)|☺\rangle$

$\hat{H}|E\rangle = E_n|E\rangle$

$\hat{H}|☹\rangle = (E_n - \hbar\omega)|☹\rangle$

# HARMONIC OSCILLATOR (CONT.)

WITH $a_-|E_0\rangle = 0$ AND

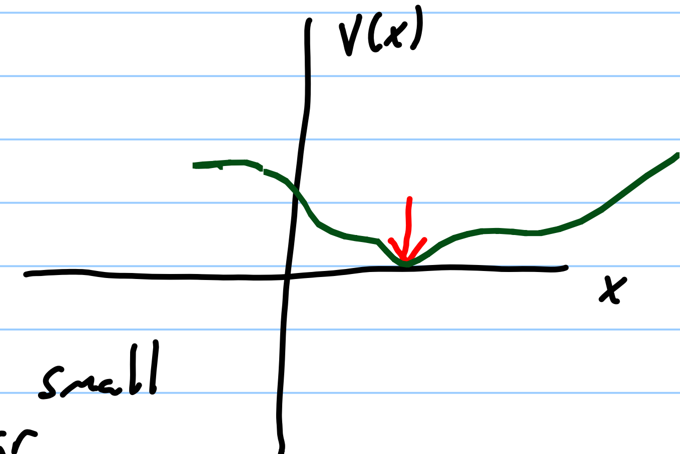
WITH $(\hat{H} - \frac{1}{2}\hbar\omega)|E_0\rangle = 0$

WE FOUND

$$E_0 = \frac{\hbar\omega}{2}, \quad E_1 = \frac{3}{2}\hbar\omega, \quad E_n = (n+\tfrac{1}{2})\hbar\omega$$

zero-point energy

As you cool particles in a given potential, they will eventually reach a minimum energy and the potential (with small displacement of particles) will appear locally as a harmonic oscillator potential



You may be confused at this point with "braket" or Dirac notation, operators, matrix representation but at least you can now understand what animated characters are saying in PG-rated movies...

In the movie "The Incredibles", the vilain "Symdrome" is an expert on QM.

From the Movie Database Trivia webpage:

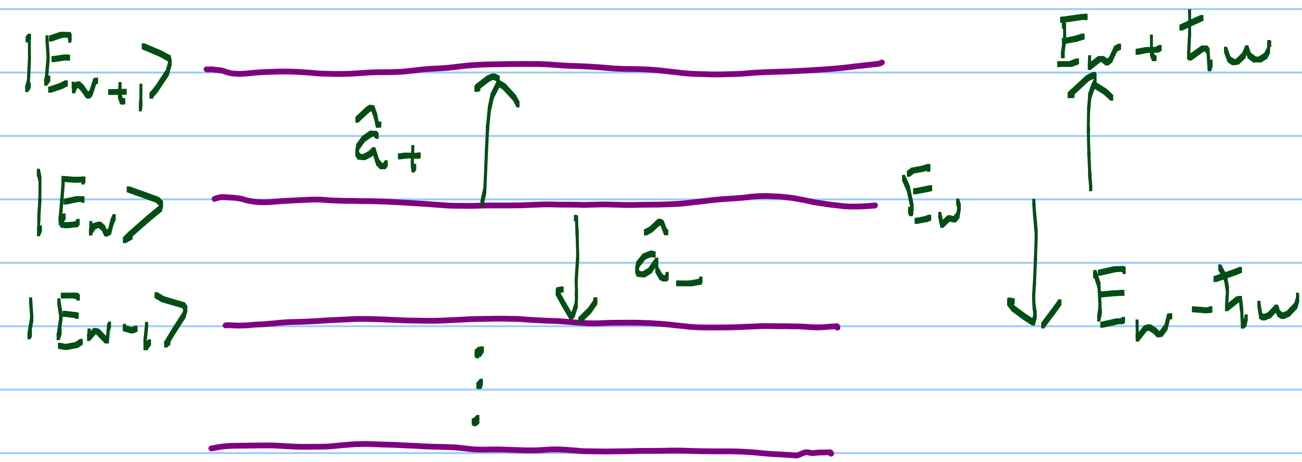
"Syndrome's "zero-point energy" beam is based on an actual physics concept, the zero-point field, demonstrated in 1948 via the Casimir Effect and essential to Stephen Hawking's theory that black holes eventually evaporate. Harnessing the zero-point field would be quite a feat, as it would yield a truly infinite source of energy."

NEXT STEP WAS TO FIND NORMALISATION CONSTANTS:

$$|E_n\rangle = \frac{1}{\sqrt{n!}}\, a_+^n |E_0\rangle$$

$$a_- |E_n\rangle = \sqrt{n}\, |E_{n-1}\rangle$$

→ we have recursive relations that allow us to calculate the normalisation of any eigenstate, starting from the ground state



# HARMONIC OSCILLATOR



WE NOW CALULATE THE WAVE FUNCTIONS IN THE POSITION REP. :

$$\langle x | E_0 \rangle = \psi_0(x)$$

$$\langle x | a_- | E_0 \rangle = 0 \quad \to \text{since} \quad \hat{a}_- | E_0 \rangle = 0$$

$$\hat{a}_- = \frac{1}{\sqrt{2}} \left[ \left( \frac{m\omega}{\hbar} \right)^{1/2} \hat{x} + \frac{i \hat{p}}{\sqrt{m\omega\hbar}} \right]$$

$$= \frac{1}{\sqrt{2}} \left[ \left( \frac{m\omega}{\hbar} \right)^{1/2} \hat{x} + \frac{i}{\sqrt{m\omega\hbar}} \cdot -i\hbar \frac{d}{dx} \right] .$$

WE HAVE :

$$\langle x | \hat{x} | E_0 \rangle = x \langle x | E_0 \rangle = x \psi_0(x)$$

$$\langle x | \hat{p} | E_0 \rangle = \int_{-\infty}^{\infty} dp \, \langle x | \hat{p} | p \rangle \langle p | E_0 \rangle$$

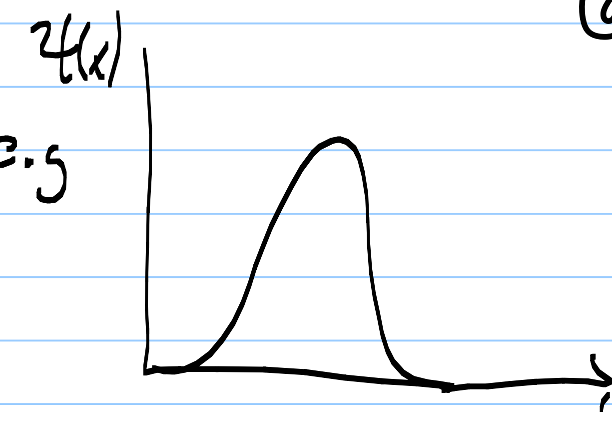
$$\hookrightarrow \quad \hat{p} | p \rangle = p | p \rangle$$

$$= \int_{-\infty}^{\infty} dp \; p \langle x | p \rangle \langle p | E_0 \rangle$$

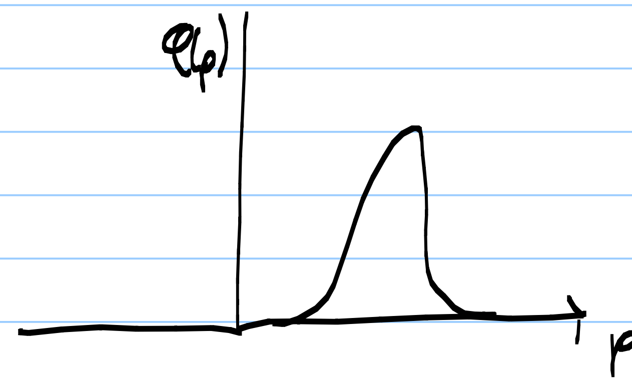
REMEMBER THAT:

$$\psi(x) = \langle x | \psi \rangle = \int_{-\infty}^{\infty} \langle x | p \rangle \langle p | \psi \rangle \, dp$$

e.g




$$\langle p | \psi \rangle = \varphi(p)$$




We saw before:

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp \, \varphi(p) \, e^{ipx/\hbar}$$

$$\Rightarrow \langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

$$= \int_{-\infty}^{\infty} dp \; p \langle x | p \rangle \langle p | E_0 \rangle$$

$$= \int_{-\infty}^{\infty} dp \; p \cdot \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \langle p | E_0 \rangle$$

$$= \frac{\hbar}{i} \frac{d}{dx} \int_{-\infty}^{\infty} dp \langle x | p \rangle \langle p | E_0 \rangle = \frac{\hbar}{i} \frac{d}{dx} \langle x | E_0 \rangle$$

$$\Rrightarrow \quad \langle x | \hat{p} | E_0 \rangle = \frac{\hbar}{i} \frac{d}{dx} \psi_0(x)$$

$$\Rightarrow \left( m\omega x + \hbar \frac{d}{dx} \right) \psi_0(x) = 0$$

$$\psi_0(x) = C e^{-m\omega x^2/2\hbar}$$

Normalising $\psi_0(x)$ yields $C = \left( \frac{m\omega}{\hbar\pi} \right)^{1/4}$

An aside:

We have: $\langle x|\hat{x}|E\rangle = x\langle x|E\rangle$

$$\langle x|\hat{p}|E\rangle = \frac{\hbar}{i}\frac{d}{dx}\langle x|E\rangle\,,\quad \langle x|\hat{p}^2|E\rangle = -\hbar^2\frac{d^2}{dx^2}\langle x|E\rangle$$

$$\langle x|V(\hat{x})|E\rangle = V(x)\langle x|E\rangle$$

e.g. $\frac{1}{2}m\omega^2x^2$

$$\langle x|\tfrac{1}{2}m\omega^2\hat{x}^2|E\rangle = \langle x|\tfrac{1}{2}m\omega^2 x\hat{x}|E\rangle$$
$$= \langle x|\tfrac{1}{2}m\omega^2x^2|E\rangle = \tfrac{1}{2}m\omega^2x^2\langle x|E\rangle$$

$$\langle x|\hat{H}|E\rangle = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\langle x|E\rangle + V(x)\langle x|E\rangle = E\langle x|E\rangle$$

!

$$\longrightarrow \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

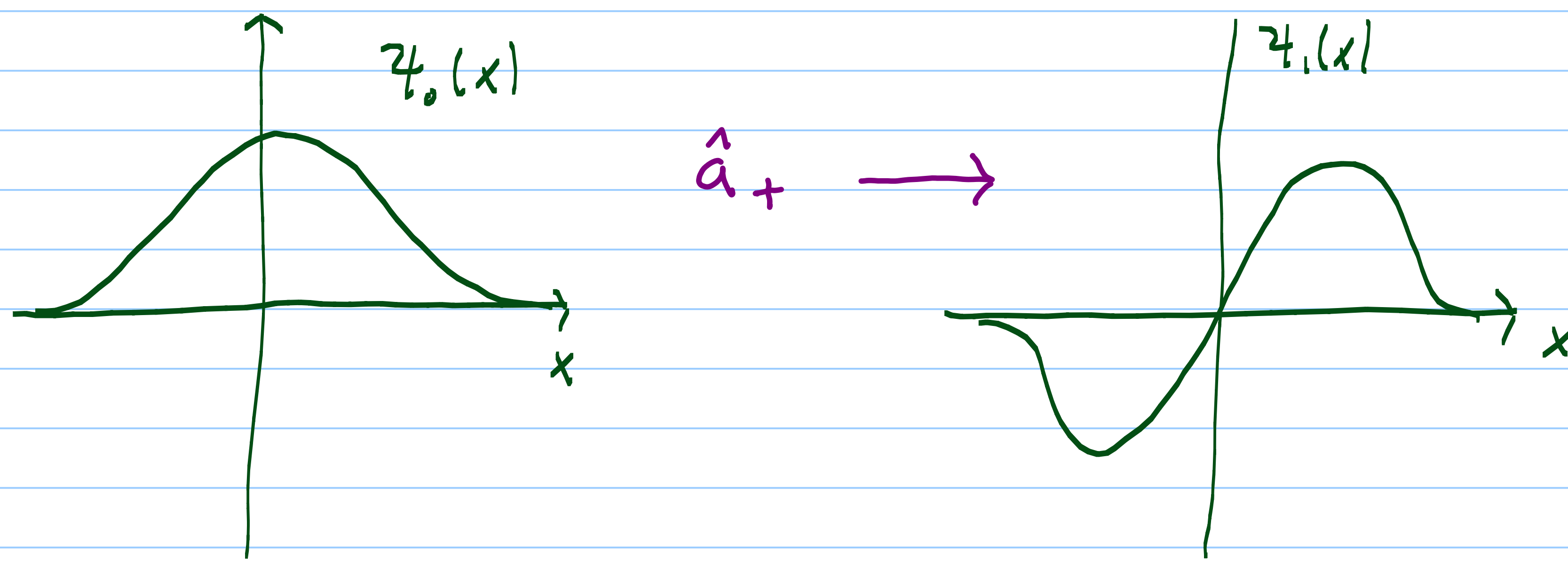
# Harmonic Oscillator



$$\langle x | E_n \rangle = \frac{1}{\sqrt{n!}} \langle x | a_+^n | E_0 \rangle$$

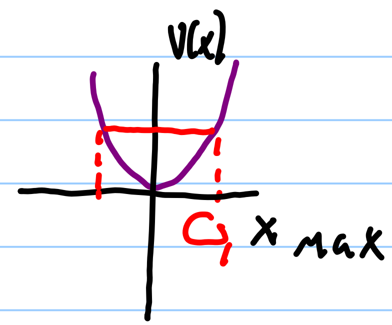
$$\psi_n(x) = \frac{1}{\sqrt{n!}} (a_+)^n \psi_0(x)$$

$$= \frac{1}{\sqrt{n!}} \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \left[\sqrt{\frac{m\omega}{2\hbar}}\, x - \sqrt{\frac{\hbar}{2m\omega}} \frac{d}{dx}\right]^n e^{-m\omega x^2/2\hbar}$$

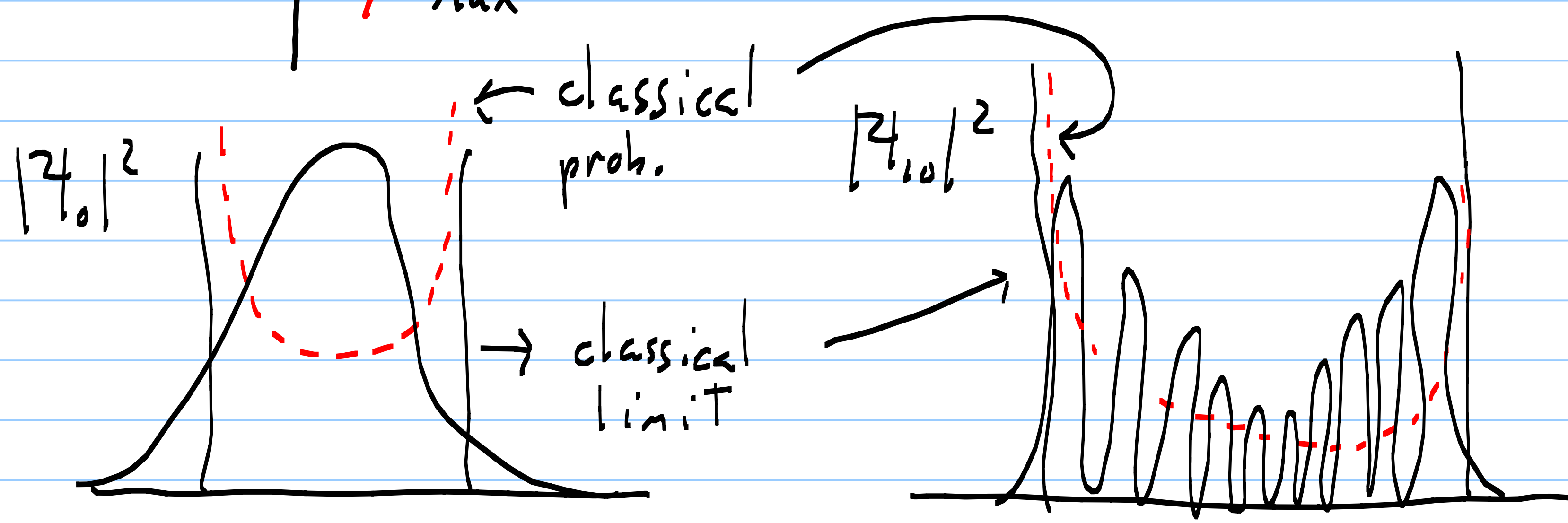


In classical physics:

- The particle will more likely be found on the edge of the potential
- The particle does not go beyond $V(x) = \max\left(\frac{p^2}{2m}\right)$




Prob. of finding particle beyond $x_{max} = 0$



## – MATRIX REPRESENTATION OF $\hat{H}$, $\hat{a}_+$, $\hat{a}_-$, etc. IN THE $|E_n\rangle$ BASIS

$E_n = (n + 1/2)\hbar\omega$ , $\hat{H} = (a^+ a^- + 1/2)\hbar\omega$

$\Rightarrow$ $\hat{a}_+\hat{a}_-$ gives $n$ as eigenvalue : $\boxed{\hat{a}_+\hat{a}_- \equiv \hat{N}}$

$$\hat{H} = \begin{pmatrix} \hbar\omega/2 & 0 & 0 & \cdots \\ 0 & \frac{3}{2}\hbar\omega & 0 & \cdots \\ 0 & 0 & 5/2\,\hbar\omega & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ & & & & \cdots \end{pmatrix}$$

What is $\hat{a}_-$ in matrix rep.?

$\hat{a}_-|E_n\rangle = \sqrt{n}\,|E_{n-1}\rangle$

$\langle E_k|\hat{a}_-|E_n\rangle = \sqrt{n}\,\langle E_k|E_{n-1}\rangle = \sqrt{n}\,\delta_{k,n-1}$

$(\hat{a}_-)_{kn} = \delta_{k,n-1}\sqrt{n}$

$$\Rightarrow \quad a_- : \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \ddots \\ \cdot & \cdots & & & \end{pmatrix} \qquad a_+ : \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & \ddots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & & \end{pmatrix}$$

Matrix rep. For $\hat{x}$ :

with $a_\pm = \frac{1}{\sqrt{2}} \left[ \left( \frac{m\omega}{\hbar} \right)^{1/2} \hat{x} \mp i \frac{\hat{p}}{(m\omega\hbar)^{1/2}} \right]$

$$\Rightarrow \quad \hat{x} = \left[ \sqrt{2}\, a_+ + \frac{\hat{p}}{(m\omega\hbar)^{1/2}} \right] \cdot \sqrt{\frac{\hbar}{m\omega}} \qquad (1)$$

$$\hat{x} = \left[ \sqrt{2}\, a_- - \frac{\hat{p}}{(m\omega\hbar)^{1/2}} \right] \cdot \sqrt{\frac{\hbar}{m\omega}} \qquad (2)$$

$$(1)+(2) = \quad 2\hat{x} = \left[ \sqrt{2}\, a_+ + \sqrt{2}\, a_- \right] \sqrt{\frac{\hbar}{m\omega}}$$

$$\Rightarrow \quad \hat{x} = \sqrt{\frac{\hbar}{m\omega}} \left( a_+ + a_- \right)$$

SAME PROCEDURE FOR $\hat{p}$ YIELDS:

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\,(a_+ - a_-)$$

We can use the previous matrices for $a_+$ and $a_-$ To express $\hat{x}$ and $\hat{p}$ in matrix rep.:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & \cdots \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

CHECK THAT: $[\hat{x}, \hat{p}] = i\hbar$

$$\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} = \hat{H}$$

# Expectation Values of $\hat{x}^2$ and $\hat{p}^2$ 

$\hat{x} \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_+ + \hat{a}_-)$ $\qquad$ $\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}} (\hat{a}_+ - \hat{a}_-)$

$$\hat{x}^2 = \frac{\hbar}{2m\omega} (\hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_+\hat{a}_- + \hat{a}_- a_+)$$

let's use: $\hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ = 1$ $\quad$ i.e. $[\hat{a}_+, \hat{a}_-] = 1$

$$\Rightarrow \quad \hat{a}_+\hat{a}_- = 1 + \hat{a}_-\hat{a}_+$$

$$\hat{x}^2 = \frac{\hbar}{2m\omega} (\hat{a}_+^2 + \hat{a}_-^2 + 2\hat{a}_-\hat{a}_+ + 1)$$

$$\hat{p}^2 = -\frac{m\omega\hbar}{2} (\hat{a}_+^2 + \hat{a}_-^2 - 2\hat{a}_-\hat{a}_+ + 1)$$

$$\langle E_n | \hat{x}^2 | E_n \rangle = \frac{\hbar}{2m\omega} \Big( \underbrace{\langle E_n | \hat{a}_+^2 | E_n \rangle}_{=0} + \underbrace{\langle E_n | \hat{a}_-^2 | E_n \rangle}_{=0}$$

$$+ 2\langle E_n | \hat{a}_-\hat{a}_+ | E_n \rangle + 1 \Big)$$

with $\hat{a}_- \hat{a}_+ = \hat{n}$

$$\langle E_n | \hat{X}^2 | E_n \rangle = \frac{\hbar}{2m\omega}\left[ 2\langle E_n | \hat{n} | E_n \rangle + 1 \right]$$

$$= \frac{\hbar}{2m\omega}(2n+1)$$

$$\langle E_n | \hat{P}^2 | E_n \rangle = \frac{m\hbar\omega}{2}(2n+1)$$

$$\langle E_n | \hat{H} | E_n \rangle = \frac{\hbar\omega}{2}(2n+1)$$

$$\Rightarrow \frac{m\omega^2}{2}\langle E_n | \hat{X}^2 | E_n \rangle = \frac{1}{2m}\langle E_n | \hat{P}^2 | E_n \rangle = \frac{1}{2}\langle E_n | \hat{H} | E_n \rangle$$

$\langle$Potential energy$\rangle$ = $\langle$Kinetic energy$\rangle$ = $\left\langle \frac{\text{Total energy}}{2} \right\rangle$

$\rightarrow$ virial theorem

$$2\langle T \rangle = n \langle U \rangle$$

$n = -1$ for $\frac{1}{r}$ potentials

# HEISENBERG UNCERTAINTY PRINCIPLE



For harmonic oscillator: $\langle \hat{x} \rangle = 0$

$$\langle \hat{p} \rangle = 0$$

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} = \sqrt{\langle \hat{x}^2 \rangle} = \sqrt{\frac{\hbar}{2m\omega}(2n+1)}$$

$$\Delta p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2} = \sqrt{\langle \hat{p}^2 \rangle} = \sqrt{\frac{m\hbar\omega}{2}(2n+1)}$$

$$\Delta x \Delta p = \frac{\hbar}{2}(2n+1) \quad , \quad n=0 \Rightarrow \Delta x \Delta p = \hbar/2$$

note that the previous results could have been derived with matrices.

Example problem:

A particle is placed in a harmonic oscillator potential. We add a constant electric field $\mathcal{E}$.
What is the wave function in the ground state?

New force due to $\mathcal{E}$: $-q\mathcal{E}$

Potential energy associated with force: $F = \frac{dV}{dx}$

$\Rightarrow V = -q\mathcal{E}x$

The Hamiltonian is then:

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 - q\mathcal{E}\hat{x}$$

$\rightarrow$ change of variables: $y = \hat{x} - q\mathcal{E}/m\omega^2$

$$y^2 = \hat{x}^2 + q^2\mathcal{E}^2/m^2\omega^4 - 2\hat{x}\frac{q\mathcal{E}}{m\omega^2}$$

Problem continued

$$\frac{1}{2} m\omega^2 \hat{x}^2 = \frac{1}{2} m\omega^2 y^2 - \frac{q^2\mathcal{E}^2}{2m\omega^2} - \frac{\cancel{2}}{\cancel{2}} \cancel{m\omega^2} \frac{\hat{x} q\mathcal{E}}{\cancel{m\omega^2}}$$

$$\Rightarrow \hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2m}\omega^2 y^2 - \frac{q^2\mathcal{E}^2}{2m\omega^2}$$

same as before but we've added a constant

before: $E_n = \langle E_n | \hat{H} | E_n \rangle = \hbar\omega(n + 1/2)$

now: $E_n = \hbar\omega(n + 1/2) - \frac{q^2\mathcal{E}^2}{2m\omega^2}$

before: $\psi_0(x) = C \exp\left(-m\omega x^2/2\hbar\right)$

now:

$$\psi_0(y) = C \exp\left(-m\omega y^2/2\hbar\right)$$