LECTURE 27: The Last Postulate

What I expect you to learn:

- -How quatum systems evolve with time
- -What is the Evolution Operator
- -What are the Heisenberg and Schrodinger "pictures"

(Corresponds to chapter 5.7 of the textbook)

(Problem set 4: due Wednesday Nov. 22nd)

GENERAL SOLUTION FOR A TIME INDEP. POTENTIAL

WE SAW THAT WE COULD EXPRESS $\psi(x,t) = \psi(x)F(t)$

WHERE $F(t) = C \exp\left(-iEt/\hbar\right)$ or $C(T_0) \exp\left[\frac{-iE}{\hbar}(t-T_0)\right]$

WE HAVE $\psi(x) = \sum_n A_n \psi_n(x)$

$$\Rightarrow \psi(x,t) = \sum_n A_n \psi_n(x)\, C(T_0) \exp\left[\frac{-iE_n}{\hbar}(t-T_0)\right]$$

we could rewrite $\psi(x,t) = \sum_n A_n \psi_n(x) \exp\left(-iEt/\hbar\right)$

with $A_n = C_n(T_0) \exp\left(iE_nT_0/\hbar\right)$

IF WE KNOW THE WAVE FUNCTION ψ AT A PARTICULAR TIME T_0, we can DETERMINE ψ FOR ALL VALUES OF T:

$$A_n = \exp\left(iET_0/\hbar\right) \int \psi_n^*(x)\, \psi(x,T_0)\, dx$$

$$\psi(x,t) = \sum_n \left[\int \psi_n^*(x')\, \psi(x',T_0)\, dx'\right] \psi_n(x) \exp\left[\frac{-iE}{\hbar}(t-T_0)\right]$$

GENERAL SOLUTION FOR A TIME INDEP. POTENTIAL

So we have: $\Psi(x,t) = \sum_n A_n \psi_n(x) \exp(-iEt/\hbar)$

$$P(x,t) = \Psi^*(x,t)\,\Psi(x,t)$$

$$= \sum_n \sum_m A_m^* A_n \exp\left[-i(E_n - E_m)t/\hbar\right] \psi_m^*(x)\,\psi_n(x)$$

$$= \sum_n |A_n|^2 |\psi_n|^2 + \sum_{\substack{n \\ n\neq m}} \sum_m A_m^* A_n \exp\left[\frac{-it}{\hbar}(E_n - E_m)\right] \psi_m^*(x)\,\psi_n(x)$$

→ Pure eigenstates have no time dep.

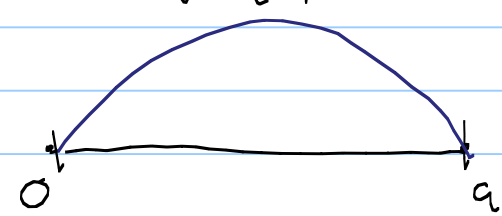
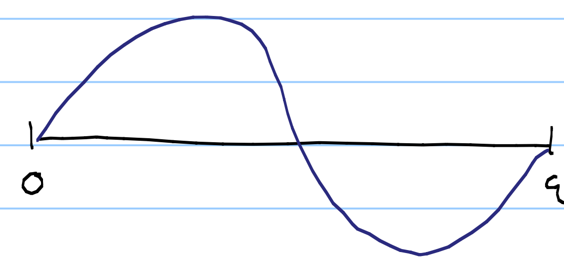
→ the larger the ΔE, the faster the probability evolves versus time

→ $P(x,t)$ is time dependent, physical quantities will in general depend on time. Show however that $\langle H \rangle$ does not depend on time (exercise).

BACK TO THE INFINITE WELL

CONSIDER THE FOLLOWING WAVE FUNCTION (at $T=0$)

$$\psi(x) = A\sin\left(\frac{\pi x}{a}\right) + A\sin\left(\frac{2\pi x}{a}\right)$$

$\psi(x) =$  $+$ 

$n=1$ $n=2$

$$\psi(x,t) = \underbrace{A\sin\left(\frac{\pi x}{a}\right)}_{F_1} e^{-\frac{iE_1}{\hbar}T} + \underbrace{A\sin\left(\frac{2\pi x}{a}\right)}_{F_2} e^{-\frac{iE_2}{\hbar}T}$$

$$\psi(x,t) = F_1 e^{-iE_1T/\hbar} + F_2 e^{-iE_2t/\hbar}$$

$$P(x,t) = (F_1 e^{iE_1T/\hbar} + F_2 e^{iE_2T/\hbar})(F_1 e^{-iE_1T/\hbar} + F_2 e^{-iE_2T/\hbar})$$

$$= F_1^2 + F_2^2 + F_1F_2\left[e^{i(E_1-E_2)T/\hbar} + e^{-i(E_1-E_2)T/\hbar}\right]$$

$$= F_1^2 + F_2^2 + 2F_1F_2\cos\left(\frac{T}{\hbar}(E_1-E_2)\right)$$

THE INFINITE WELL (cont.)

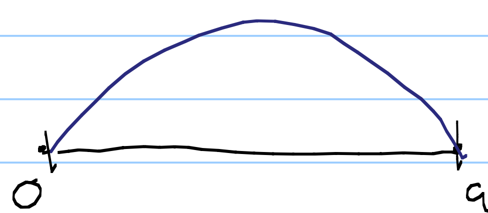
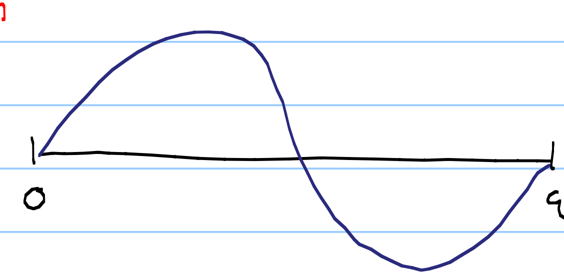
$$P(x,t) = F_1^2 + F_2^2 + 2F_1F_2 \cos\left(\frac{t}{\hbar}(E_1 - E_2)\right)$$

oscillates between $(F_1+F_2)^2$ and $(F_1-F_2)^2$

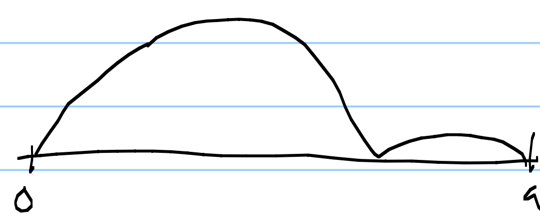
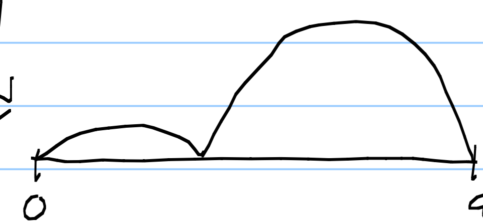
we have $E_1 = \frac{\hbar^2\pi^2}{2ma^2}$, $E_2 = \frac{4\hbar^2\pi^2}{2ma^2}$, $E_1 - E_2 = -3E_1$

$\Rightarrow \frac{3E_1}{\hbar} t = \pi, 3\pi, 5\pi, \ldots$

$\rightarrow$ obtain same config. when $\frac{3E_1}{\hbar} t = 2\pi, 4\pi, \ldots$

$\psi(x) =$  $+$ 

$t = \frac{\hbar\pi}{3E_1}$

$P(x,0) =$  $P(x,t)$ 

THE LAST POSTULATE

A SUMMARY OF THE POSTULATES WE'VE SEEN

1- INTRODUCED THE STATE VECTOR $|\psi\rangle$

*2- THE SUPERPOSITION PRINCIPLE: $|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$

3- ASSOCIATES A LINEAR HERMITIAN OPERATOR TO EVERY OBSERVABLE

*4- A MEASUREMENT OF AN OBSERVABLE YIELDS AN EIGENVALUE OF THE OPERATOR AND LEAVES THE SYSTEM IN AN EIGENSTATE OF THE OPERATOR

5- $\langle A\rangle = \langle\psi|A|\psi\rangle$

6- $|\psi\rangle = \sum_n a_n|\psi_n\rangle$, $|\psi\rangle = \int a(x)|\psi_a\rangle\, da$

7- THE TIME EVOLUTION OF $|\psi\rangle$ IS DETERMINED BY THE TIME-DEPENDENT SCHRÖDINGER EQUATION:

$$\boxed{i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle}$$

FIRST-ORDER DIFF. LINEAR EQUATION IN TIME.
IF YOU KNOW $|\psi\rangle$ AT $T=0$, YOU KNOW $|\psi\rangle$ FOR ALL T

THE EVOLUTION OPERATOR

(7)

WE INTRODUCE AN EVOLUTION OPERATOR $U(T,T_0)$:

$|\psi(t)\rangle = U(T,T_0)\,|\psi(T_0)\rangle$

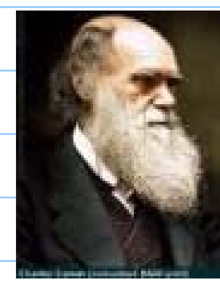
BANNED IN KANSAS ...

$$\hat{U}(T_0,T_0) = I \qquad (1)$$

$$\hat{U}(T,T_0) = \hat{U}(T,t')\,\hat{U}(t',T_0) \qquad (2)$$

$$\hat{U}^{-1}(T,T_0) = \hat{U}(T_0,T) \qquad (3)$$

$\hat{U}\,\Big|$ 

$\Big\rangle = \Big|$  $\Big\rangle$

THE EVOLUTION OPERATOR (cont.)

$$\rightarrow \quad i\hbar \frac{d}{dt}\left(U(T,T_0)\,|\psi(T_0)\rangle \right) = \hat{H}\left(U(T,T_0)\,|\psi(T_0)\rangle \right) \quad (4)$$

$$\rightarrow \quad i\hbar \frac{d}{dT} U(T,T_0) = \hat{H}\, U(T,T_0) \quad (5)$$

USING (1) AND (5)

$$\rightarrow \quad \hat{U}(T,T_0) = I - \frac{i}{\hbar}\int_{T_0}^{T} \hat{H}\,\hat{U}(t',T_0)\,dt' \quad (6)$$

$$\frac{d}{dt}\hat{U}(T,T_0) = \frac{1}{i\hbar}\hat{H}\,\hat{U}(T,T_0) = -\frac{i}{\hbar}\, U(T,T_0)$$

$$\Rightarrow \int_{T_0}^{T} \frac{d}{dt}\hat{U}(T,T_0)\,dT = \hat{U}(T,T_0) - \hat{U}(T_0,T_0)$$

$$= \hat{U}(T,T_0) - \underline{I}$$

EVOLUTION OPERATOR (CONT.)

CONSERVATION OF PROB. IMPLIES:

$$\langle \psi(T_0) | \psi(T_0) \rangle = \langle \psi(t) | \psi(t) \rangle$$

$$\langle \psi(t) | \psi(t) \rangle = \langle \hat{U}(T, T_0) \psi(T_0) | \hat{U}(T, T_0) \psi(T_0) \rangle$$

$$= \langle \psi(T_0) | \hat{U}^{\dagger}(T, T_0) \hat{U}(T, T_0) | \psi(T_0) \rangle$$

$$\Rightarrow \hat{U}^{\dagger} \hat{U} = I \quad \text{AND} \quad \hat{U} \hat{U}^{\dagger} = I$$

HOW DOES $\hat{U}$ CHANGE WHEN $T \to T + \delta T$?

$$i\hbar \left(\hat{U}(T_0 + \delta T, T_0) - \underbrace{\hat{U}(T_0, T_0)}_{= I} \right) = \hat{H} \hat{U}(T_0 + \delta T, T_0) \, \delta T$$

from 6

$$\hat{U}(T_0 + \delta T, T_0) = I - \frac{i}{\hbar} \hat{H} \, \delta T$$

$\rightarrow \hat{H}$ generates an infinitesimal Time Translation

EVOLUTION OPERATOR CONT.

A SOLUTION TO: $i\hbar \frac{d}{dT} U(T,T_0) = \hat{H}\, U(T,T_0)$

is: $$U(T,T_0) = \exp\left[-i\hbar \hat{H}\,(T-T_0)\right]$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar}\right)^n \hat{H}^n (T-T_0)^n$$

FIRST TWO TERMS:
$$I - \frac{i}{\hbar} \hat{H}\,(T-T_0)$$

$$\Rightarrow \quad |\psi(t)\rangle = \exp\left[\frac{-i}{\hbar} \hat{H}\,(T-T_0)\right] |\psi(T_0)\rangle$$

$\hat{H} \rightarrow$ Time indep. potential

AN EXAMPLE:

$$|\psi(0)\rangle = \frac{1}{5}\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \quad \hat{H} = \varepsilon \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{pmatrix}$$

1- FIND WHAT VALUES CAN BE MEASURED FOR THE ENERGY AND WITH WHAT PROBABILITIES.

2- FIND $|\psi(t)\rangle$ AND EXPAND IN TERMS OF THE EIGENSTATES OF $\hat{H}$.

3- FIND THE EXPECTATION VALUE OF $\hat{H}$ AT TIMES 0 AND T.

(SOLUTION ON BLACKBOARD, PARTS OF THE SOLUTIONS FOLLOW)

FIRST STEP, DIAGONALISE THE MATRIX AND FIND THE EIGENVALUES:

$$E_1 = -5\varepsilon \quad , \quad E_2 = 3\varepsilon , \quad E_3 = 5\varepsilon$$

STEP TWO, FIND THE EIGENVECTORS:

$$|\varphi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad , \quad |\varphi_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad |\varphi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

DETERMINE THE PROBABILITIES FOR OBTAINAINING THE E_n:

$$P(E_1) = |\langle \varphi_1 | \psi(0)\rangle|^2 = \left| \frac{1}{5\sqrt{2}} (0, -1, 1) \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right|^2 = \frac{8}{25}$$

$$P(E_2) = 9/25 \quad , \quad P(E_3) = 8/25$$

EXPAND $|\psi_0\rangle$ IN TERMS OF EIGENVECTORS:

$$|\psi_0\rangle = \frac{1}{5}\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \frac{\sqrt{8}}{5} |\varphi_1\rangle + \frac{3}{5} |\varphi_2\rangle + \frac{\sqrt{8}}{5} |\varphi_3\rangle$$

APPLY TIME EVOLUTION OPERATOR

$$|\psi(t)\rangle = \exp\left[\frac{-i}{\hbar}\hat{H}\,(t) \quad |\psi(0)\rangle\right]$$

EXPAND $|\psi(t)\rangle$ IN TERMS OF EIGENVECTORS:

$$|\psi(t)\rangle = \frac{\sqrt{8}}{5}\, e^{-iE_1 T/\hbar}|\varphi_1\rangle + \frac{3}{5}\, e^{-iE_2 T/\hbar}|\varphi_2\rangle + \frac{\sqrt{8}}{5}\, e^{-iE_3 T/\hbar}|\varphi_3\rangle$$

$$= \frac{\sqrt{8}}{5}\begin{pmatrix} 0 \\ -1/\sqrt{2}\; e^{+i5\varepsilon T/\hbar} \\ 1/\sqrt{2}\cdot e^{+i5\varepsilon T/\hbar} \end{pmatrix} + \frac{3}{5}\begin{pmatrix} e^{-i3\varepsilon T/\hbar} \\ 0 \\ 0 \end{pmatrix} + \frac{\sqrt{8}}{5}\begin{pmatrix} 0 \\ 1/\sqrt{2}\; e^{-i5\varepsilon t/\hbar} \\ 1/\sqrt{2}\; e^{-i5\varepsilon T/\hbar} \end{pmatrix}$$

$$= \begin{pmatrix} 3/5\, e^{-i3\varepsilon T/\hbar} \\ -\frac{2}{5}\, e^{i5\varepsilon t/\hbar} + \frac{2}{5}\, e^{-i5\varepsilon T/\hbar} \\ 2/5\, e^{i5\varepsilon T/\hbar} + 2/5\, e^{-i5\varepsilon T/\hbar} \end{pmatrix} = \begin{pmatrix} 3/5\; e^{-i3\varepsilon T/\hbar} \\ 4/5i\; \sin(5\varepsilon T/\hbar) \\ 4/5\; \cos(5\varepsilon t/\hbar) \end{pmatrix}$$

Is $|\psi(t)\rangle$ normalised? i.e. $\langle\psi(t)|\psi(t)\rangle = 1$?

$$\left(\frac{3}{5}e^{i3\varepsilon t/\hbar},\ \frac{4}{5i}\sin(5\varepsilon t/\hbar),\ \frac{4}{5}\cos 5\varepsilon t/\hbar\right)\begin{pmatrix}\frac{3}{5}e^{-i3\varepsilon t/\hbar}\\ \frac{-4}{5i}\sin(5\varepsilon t/\hbar)\\ \frac{4}{5}\cos(5\varepsilon t/\hbar)\end{pmatrix}$$

$$= \frac{9}{25}e^{i3\varepsilon t/\hbar}\cdot e^{-i3\varepsilon t/\hbar}$$

$$+ \frac{16}{25}\cdot\sin^2(5\varepsilon t/\hbar) + \frac{16}{25}\cos^2(5\varepsilon t/\hbar)$$

$$= \frac{9}{25} + \frac{16}{25}\left(\sin^2(5\varepsilon t/\hbar) + \cos^2(5\varepsilon t/\hbar)\right)$$

$$= \frac{9}{25} + \frac{16}{25} = \frac{25}{25}$$

Question 3

$$\langle \psi_0 | \hat{H} | \psi_0 \rangle = \left(3/5,\ 0,\ 4/5\right) \begin{pmatrix} 3\varepsilon & 0 & 0 \\ 0 & 0 & 5\varepsilon \\ 0 & 5\varepsilon & 0 \end{pmatrix} \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix}$$

$$\boxed{= 27/25\ \varepsilon}$$

or

$$\langle H \rangle = \sum_n P_n E_n = \frac{8}{25} \cdot -5\varepsilon + \frac{9}{25} \cdot 3\varepsilon + \frac{8}{25} \cdot 5\varepsilon = \boxed{\frac{27}{25}\varepsilon}$$

or

$$\frac{8}{25} \langle \varphi_1 | \hat{H} | \varphi_1 \rangle + \frac{9}{25} \langle \varphi_2 | \hat{H} | \varphi_2 \rangle + \frac{8}{25} \langle \varphi_3 | \hat{H} | \varphi_3 \rangle$$

$$= \frac{8}{25} \cdot -5\varepsilon + 9/25 \cdot 3\varepsilon + \frac{8}{25} \cdot 5\varepsilon = \boxed{27/25\ \varepsilon}$$

$$\langle \psi(t) | \hat{H} | \psi(t) \rangle =$$

$$\left(3/5\, e^{+i3\varepsilon t/\hbar},\ \frac{4}{5i} \sin(5\varepsilon t/\hbar),\ \frac{4}{5} \cos(5\varepsilon t/\hbar)\right) \begin{pmatrix} 3\varepsilon & 0 & 0 \\ 0 & 0 & 5\varepsilon \\ 0 & 5\varepsilon & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} e^{-i3\varepsilon t/\hbar} \\ \frac{-4}{5i} \sin(5\varepsilon t/\hbar) \\ \frac{4}{5} \cos(5\varepsilon t/\hbar) \end{pmatrix}$$

$$\boxed{= 27/25\ \varepsilon}$$

TIME VARIATION OF EXPECTATION VALUES

$$\frac{d}{dt}\langle\hat{A}\rangle = \frac{d}{dt}\langle\psi|\hat{A}|\psi\rangle$$

$$= \left\langle\frac{d\psi}{dt}\right|\hat{A}|\psi\rangle + \left\langle\psi\left|\frac{d\hat{A}}{dt}\right|\psi\right\rangle + \left\langle\psi\right|\hat{A}\left|\frac{d\psi}{dt}\right\rangle$$

$$i\hbar\frac{\partial}{\partial t} = \hat{H} \rightarrow \frac{\partial}{\partial t} = \frac{\hat{H}}{i\hbar}$$

$$= \frac{-1}{i\hbar}\langle\hat{H}\psi|\hat{A}|\psi\rangle + \left\langle\psi\left|\frac{d\hat{A}}{dt}\right|\psi\right\rangle + \frac{1}{i\hbar}\langle\psi|\hat{A}|\hat{H}\psi\rangle$$

$$= \frac{-1}{i\hbar}\langle\psi|\hat{H}\hat{A}|\psi\rangle + \frac{1}{i\hbar}\langle\psi|\hat{A}\hat{H}|\psi\rangle + \left\langle\psi\left|\frac{d\hat{A}}{dt}\right|\psi\right\rangle$$

$$= \frac{1}{i\hbar}\langle\psi|\hat{A}\hat{H} - \hat{H}\hat{A}|\psi\rangle + \left\langle\psi\left|\frac{d\hat{A}}{dt}\right|\psi\right\rangle$$

$$= \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle + \left\langle\frac{\partial A}{\partial t}\right\rangle$$

TIME VARIATION OF EXPECTATION VALUES (V2)

$$\langle \hat{A} \rangle (t) = \langle \psi(t) | \hat{A} | \psi(t) \rangle$$

$$= \langle e^{-i\hat{H}t/\hbar} \psi(0) | \hat{A} | e^{-i\hat{H}t/\hbar} \psi(0) \rangle$$

$$= \langle \psi(0) | e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar} | \psi(0) \rangle$$

$$= \langle \psi(0) | \hat{A}(t) | \psi(0) \rangle$$

With no explicit time dependence of $\hat{A}$, we have

$$\rightarrow \frac{d}{dt} \hat{A}(t) = \frac{i}{\hbar} \hat{H} e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar} - \frac{i}{\hbar} e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar} \hat{H}$$

$$= \frac{i\hat{H}}{\hbar} \hat{A}(t) - \frac{i}{\hbar} \hat{A}(t) \hat{H} = \frac{i}{\hbar} [\hat{H}, \hat{A}(t)]$$

SCHRÖDINGER AND HEISENBERG PICTURES

SCHRÖDINGER PICTURE:

TIME-DEP WAVE FUNCTION THAT SATISFIES SCHRÖDINGER'S EQUATION

GET $\psi(t)$ FROM $\psi(0)$ USING: $\psi(t) = U(t, t_0)\, \psi(t_0)$

HEISENBERG PICTURE:

$$\psi_H \equiv \psi_0$$

$$= U^\dagger(t, t_0)\, \psi(t) = U(t_0, t)\, \psi(t) = \psi(0)$$

WE PUT THE TIME EVOLUTION IN THE OPERATOR:

$$A_H(t) = U^\dagger(t, t_0)\, \hat{A}\, U(t, t_0)$$

$$= U(t_0, t)\, \hat{A}\, U(t_0, t)$$