LECTURE 25: The Harmonic Oscillator (redux)

What I expect you to learn:

- How to obtain the energy eigenvalues of the harmonic oscillator using a method of Dirac

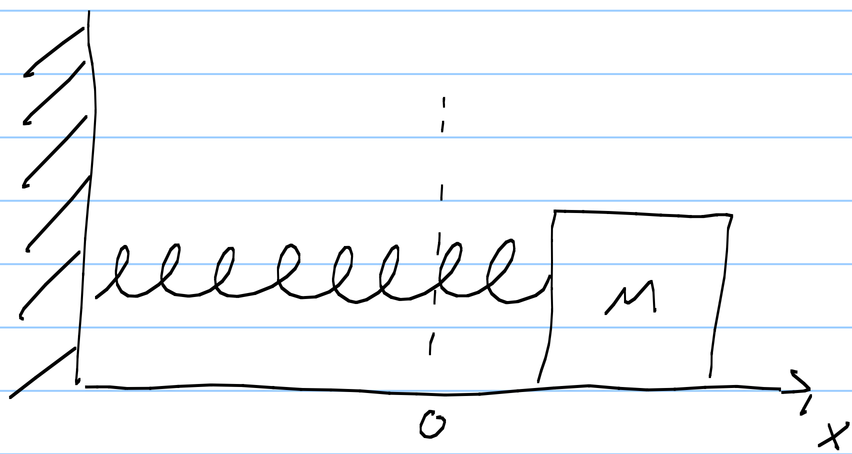
- How to contruct the wave eigenfunctions using the method above

(Roughly reviews chapter 5.7 of the textbook)

(Solutions to prob. set 3 will be posted Friday afternoon)

THE HARMONIC OSCILLATOR (REVISITED)

Consider the following setup:



mass "M" oscillating about x_0 due to restoring force exerted by a spring.
$F = -Kx$

$$F = ma \rightarrow -Kx = m\frac{d^2x}{dT^2} \rightarrow m\frac{d^2x}{dT^2} + Kx = 0 \quad (1)$$

the solution to (1) is?

$$F = Kx = \frac{dV}{dx} \rightarrow V = \frac{1}{2}Kx^2$$

This parabolic potential is of great importance in both classical and quantum physics

THE HARMONIC OSCILLATOR

Consider the following arbitrary potential

To estimate the motion of a particle at "a" subjected to $V(x)$, I could expand $V(x)$ using a Taylor series:

$$V(x) = V(a) + (x-a)V'(a) + \frac{1}{2!}(x-a)^2 V''(a) + \frac{1}{3!}(x-a)^3 V'''(a) + \dots$$

Since particle is at "a" → minimum, $V'(a) = 0$
We can choose "a" to be at the origin so we'll get

$$V(x) = \frac{1}{2} K x^2 + \dots$$, with $K = V''(a)$

THE HARMONIC OSCILLATOR

With $V(x) = \frac{1}{2} K x^2$, the Hamiltonian $= T + V$, will be:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} K x^2$$

the Schrödinger equation: $H\psi = E\psi$, will be:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2} K x^2 \psi(x) = E\psi(x) \qquad (2)$$

we can rewrite (2) in terms of dimensionless eigenvalues:

$\lambda = \frac{2E}{\hbar\omega}$, $\omega = \sqrt{\frac{K}{m}}$, and the dimensionless variable

$\xi = \alpha x$, $\alpha = \left(\frac{Km}{\hbar^2}\right)^{1/4} = \left(\frac{m\omega}{\hbar}\right)^{1/2}$, (2) becomes:

$$\frac{d^2\psi(\xi)}{d\xi^2} + (\lambda - \xi^2)\psi(\xi) = 0 \qquad (3)$$

THE HARMONIC OSCILLATOR

$$\frac{d^2\psi(\xi)}{d\xi^2} + (\lambda - \xi^2)\psi(\xi) = 0 \qquad (4)$$

The solutions to (4) involve Hermite polynomials. I encourage you to look at how the solutions are obtained in the textbook. You will learn about Hermite Polynomials in later courses.

To be physically valid, solutions to (4) require:

$$\lambda = 2n+1, \quad n = 0, 1, 2$$

They are given by:

$$\psi_n(\xi) = e^{-\xi^2/2} H_n(\xi)$$

$H_n(\xi) \longrightarrow$ Hermite polynomials

$$H_n(\xi) = e^{\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)^n e^{-\xi^2/2}$$

$$H_0(\xi) = 1, \quad H_1(\xi) = 2\xi, \quad H_2(\xi) = 4\xi^2 - 2$$

THE HARMONIC OSCILLATOR

→ FIRST SOLUTION OBTAINED USING POSITION REPRESENTATION

→ THE FOLLOWING SOLUTION COMES FROM DIRAC:

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2} K x^2 = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$\omega = \sqrt{\frac{K}{m}}$

→ we introduce the following operators

$$a_{\pm} = \frac{1}{\sqrt{2}} \left[\left(\frac{m\omega}{\hbar} \right)^{1/2} x \mp i \frac{p}{\sqrt{m\hbar\omega}} \right]$$

→ from problem set 4

Note that: $a_+ = a_-^{\dagger}$, $a_- = a_+^{\dagger}$

$[a_-, a_+] = 1$ (problem set #4) (5)

THE HARMONIC OSCILLATOR

WE CAN REWRITE $\hat{H}$ in TERMS OF $a_\pm$:

$$\hat{H} = \frac{\hbar\omega}{2}\left(a_- a_+ + a_+ a_-\right)$$

From commutator

$$\rightarrow \quad a_- a_+ - a_+ a_- = 1 \quad \rightarrow \quad a_+ a_- = a_- a_+ - 1$$

$$\Rightarrow \hat{H} = \hbar\omega\left(a_- a_+ - 1/2\right) = \hbar\omega\left(a_+ a_- + 1/2\right)$$

FROM YOUR PROBLEM SET YOU HAVE/WILL FOUND/FIND:

$$[H, a_\pm] = \pm\hbar\omega a_\pm \qquad (6)$$

$\hat{H}|E\rangle = E|E\rangle$, USING (6) WE HAVE:

$$\hat{H} a_\pm |E\rangle = \left(a_\pm \hat{H} \pm \hbar\omega a_\pm\right)|E\rangle$$

$$\hat{H}\left(a_\pm|E\rangle\right) = \left(E \pm \hbar\omega\right)\left(a_\pm|E\rangle\right)$$

$\rightarrow$ $a_\pm|E\rangle$ eigenstate with eigenvalue $(E \pm \hbar\omega)$

HARMONIC OSCILLATOR (CONT.)

IN PARTICULAR:

$$\hat{H}\left(a_-|E\rangle\right) = (E-\hbar\omega)\left(a_-|E\rangle\right)$$

NOTE THAT $\hat{H}$ CONTAINS THE SQUARES OF THE OPERATORS $\hat{x}$ AND $\hat{p}$ $\Rightarrow$ $\langle H\rangle$ MUST BE POSITIVE.

→ IF E_0 IS THE SMALLEST EIGENVALUE THEN

$$a_-|E_0\rangle = 0 \quad (7)$$

MULTIPLY (7) BY $\hbar\omega a_+$:

$$\hbar\omega\, a_+ a_- |E_0\rangle = \left(\hat{H} - \tfrac{1}{2}\hbar\omega\right)|E_0\rangle = 0$$

$$\Rightarrow \quad \hat{H}|E_0\rangle = E_0|E_0\rangle = \tfrac{1}{2}\hbar\omega|E_0\rangle$$

$$\boxed{E_0 = \frac{\hbar\omega}{2}}$$

HARMONIC OSCILLATOR (CONT.)

NOW WE START USING a_+ ON $|E_0\rangle$:

$$H(a_+|E_0\rangle) = (E_0 + \hbar\omega)\, a_+|E_0\rangle$$

$\Rightarrow$ THE EIGENSTATE $a_+|E_0\rangle$ HAS EIGENVALUE OF $E_0 + \hbar\omega = \frac{3}{2}\hbar\omega$

So we have: $E_0 = \frac{\hbar\omega}{2}$, $E_1 = \frac{3}{2}\hbar\omega$, $E_n = (n + \frac{1}{2})\hbar\omega$

WE NEED TO FIND THE NORMALISATION CONSTANTS:

→ ASSUME $|E_n\rangle$ IS NORMALISED, THEN

$$|E_{n+1}\rangle = C_{n+1}\, a_+|E_n\rangle$$
$$\langle E_{n+1}| = C_{n+1}\langle E_n|a_+^\dagger = C_{n+1}\langle E_n|a_-$$

$$\langle E_{n+1}|E_{n+1}\rangle = 1 = |C_{n+1}|^2 \langle E_n|a_- a_+|E_n\rangle$$

$$|C_{n+1}|^2 \langle E_n | a_- a_+ | E_n \rangle = 1 \quad (8)$$

$$\rightarrow \quad a_- a_+ | E_n \rangle = \left[\frac{\hat{H}}{\hbar\omega} + 1/2 \right] | E_n \rangle$$

$$\rightarrow \quad \hat{H} | E_n \rangle = E_n | E_n \rangle = (n + 1/2)\hbar\omega | E_n \rangle$$

$$(8) = |C_{n+1}|^2 \langle E_n | \left[\frac{E_n}{\hbar\omega} + 1/2 \right] | E_n \rangle$$

$$= |C_{n+1}|^2 \left[\frac{1}{2} \langle E_n | E_n \rangle + \left(\frac{n + 1/2}{\hbar\omega} \right) \hbar\omega \langle E_n | E_n \rangle \right]$$

$$= |C_{n+1}|^2 (n+1) = 1 \implies \boxed{C_{n+1} = \frac{1}{\sqrt{n+1}}}$$

HARMONIC OSCILLATOR

WE FOUND: $|E_{n+1}\rangle = \frac{1}{\sqrt{n+1}}\, a_+ |E_n\rangle$ (9)

$$\sqrt{n+1}\, |E_{n+1}\rangle = a_+ |E_n\rangle$$

$$\Rightarrow \quad a_+ |E_0\rangle = \sqrt{1}\, |E_1\rangle$$

$$a_+ |E_1\rangle = \sqrt{2}\, |E_2\rangle$$

$$a_+ |E_2\rangle = \sqrt{3}\, |E_3\rangle$$

$$\Rightarrow \quad |E_3\rangle = \frac{a_+}{\sqrt{3}} |E_2\rangle = \frac{a_+}{\sqrt{3}} \frac{a_+}{\sqrt{2}} |E_1\rangle = \frac{a_+}{\sqrt{3}} \frac{a_+}{\sqrt{2}} \frac{a_+}{\sqrt{1}} |E_0\rangle$$

$$\rightarrow \quad \boxed{|E_n\rangle = \frac{1}{\sqrt{n!}}\, a_+^n |E_0\rangle}$$

HARMONIC OSCILLATOR

WE FOUND: $|E_{n+1}\rangle = \frac{1}{\sqrt{n+1}}\, a_+ |E_n\rangle$ (9)

USE THIS TO FIGURE OUT NORMALISATION CONSTANT FOR $|E_{n-1}\rangle$:

$$|E_n\rangle = \frac{1}{\sqrt{(n-1)+1}}\, a_+ |E_{n-1}\rangle = \frac{1}{\sqrt{n}}\, a_+ |E_{n-1}\rangle$$

Operate on both sides with a_-

$$\boxed{a_- |E_n\rangle} = \frac{1}{\sqrt{n}}\, a_- a_+ |E_{n-1}\rangle = \frac{1}{\sqrt{n}} \left(\frac{\hat{H}}{\hbar\omega} + \frac{1}{2} \right) |E_{n-1}\rangle$$

$$= \frac{1}{\sqrt{n}} \left[\frac{(E_{n-1})}{\hbar\omega} |E_{n-1}\rangle + \frac{1}{2} |E_{n-1}\rangle \right]$$

$$= \frac{1}{\sqrt{n}} \left[\frac{(n - 1/2)\hbar\omega}{\hbar\omega} |E_{n-1}\rangle + \frac{1}{2} |E_{n-1}\rangle \right] = \boxed{\sqrt{n}\, |E_{n-1}\rangle}$$

HARMONIC OSCILLATOR

WE NOW CALULATE THE WAVE FUNCTIONS IN THE POSITION REP. :

$$\langle x | E_0 \rangle = \psi_0(x)$$

$$\langle x | a_- | E_0 \rangle = 0$$

$$a_- = \frac{1}{\sqrt{2}} \left[\left(\frac{m\omega}{\hbar} \right)^{1/2} x_{op} + \frac{i\, p_{op}}{\sqrt{m\omega\hbar}} \right]$$

$$= \frac{1}{\sqrt{2}} \left[\left(\frac{m\omega}{\hbar} \right)^{1/2} x_{op} + \frac{i}{\sqrt{m\omega\hbar}} \cdot -i\hbar \frac{d}{dx} \right] .$$

WE HAVE :

$$\langle x | x_{op} | E_0 \rangle = x \langle x | E_0 \rangle = x\, \psi_0(x)$$

$$\langle x | p_{op} | E_0 \rangle = \int_{-\infty}^{\infty} dp\, \langle x | p_{op} | p \rangle \langle p | E_0 \rangle$$

$$= \int_{-\infty}^{\infty} dp\; p \langle x | p \rangle \langle p | E_0 \rangle$$

HARMONIC OSCILLATOR

$$= \int_{-\infty}^{\infty} dp \; p \langle x | p \rangle \langle p | E_0 \rangle$$

$$= \int_{-\infty}^{\infty} dp \; p \cdot \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \langle p | E_0 \rangle$$

$$= \frac{\hbar}{i} \frac{d}{dx} \int_{-\infty}^{\infty} dp \langle x | p \rangle \langle p | E_0 \rangle = \frac{\hbar}{i} \frac{d}{dx} \langle x | 0 \rangle$$

$$= \frac{\hbar}{i} \frac{d}{dx} \psi_0(x)$$

$$\Rightarrow \left(m\omega x + \hbar \frac{d}{dx} \right) \psi_0(x) = 0$$

$$\psi_0(x) = C e^{-m\omega x^2/2\hbar}$$

Normalising $\psi_0(x)$ yields $C = \left(\frac{m\omega}{\hbar\pi} \right)^{1/4}$

HARMONIC OSCILLATOR

$$\langle x | E_n \rangle = \frac{1}{\sqrt{n!}} \langle x | a_+^n | E_0 \rangle$$

$$\psi_n(x) = \frac{1}{\sqrt{n!}} (a_+)^n \psi_0(x)$$

$$= \frac{1}{\sqrt{n!}} \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \left[\sqrt{\frac{m\omega}{2\hbar}}\, x - \sqrt{\frac{\hbar}{2m\omega}}\, \frac{d}{dx}\right]^n e^{-m\omega x^2/2\hbar}$$