LECTURE 28: Recap of Recent lectures

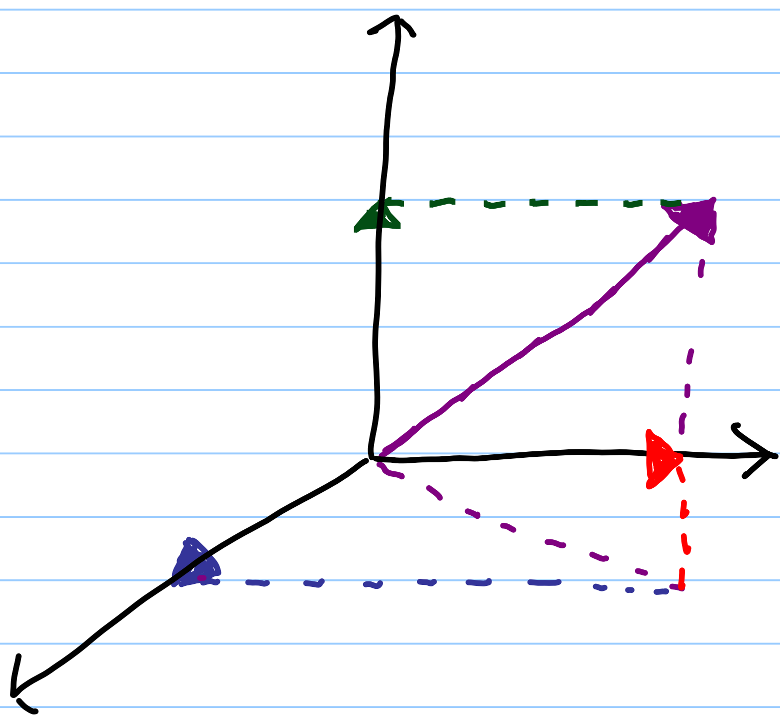
(Problem set 4 due Wednesday)

State Vectors and Inner products

Consider a particle in an infinite well.

Let's construct state vectors using the first three energy eigenvectors.

We can visualize our state vector in 3D:



$$|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle + \gamma|\psi_3\rangle$$

Prob of finding particle in ground state:

$$|\langle\psi_1|\psi\rangle|^2 =$$

$$|\alpha\langle\psi_1|\psi_1\rangle + \beta\langle\psi_1|\psi_2\rangle + \gamma\langle\psi_1|\psi_3\rangle|^2$$

$$|\alpha\cdot 1 \quad \beta\cdot 0 \quad \gamma\cdot 0|^2$$

$$= |\alpha|^2$$

3

We projected $|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle + \gamma|\psi_3\rangle$:

Projector: $|\psi_n\rangle\langle\psi_n|$

$$|\psi\rangle = \sum_{n=1}^{3} |\psi_n\rangle\langle\psi_n|\psi\rangle$$

$$|\psi\rangle = \sum_{n=1}^{3} a_n |\psi_n\rangle$$

$a_n = \langle\psi_n|\psi\rangle$

$a_1 = \alpha$
$a_2 = \beta$
$a_3 = \gamma$

onto the $|x\rangle$ basis (projection operator: $|x\rangle\langle x|$)

$|x\rangle\langle x|\psi\rangle$

$$|\psi\rangle = \int_{-\infty}^{\infty} dx \; |x\rangle\langle x|\psi\rangle$$

$\langle x|\psi\rangle \rightarrow c(x)$ or $\underline{\psi(x)}$

$$= \int_{-\infty}^{\infty} dx \; c(x)\,|x\rangle \quad \text{or} \quad \int_{-\infty}^{\infty} dx \; \psi(x)\,|x\rangle$$

$$|\psi\rangle = \sum_{n=1}^{3} a_n |\psi_n\rangle$$

$$\longrightarrow \langle x|\psi\rangle = \sum_{n=1}^{3} a_n \langle x|\psi_n\rangle$$

$$\updownarrow$$

$$\psi(x) = \sum_{n=1}^{3} a_n \psi_n(x)$$

we had:

$$|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle + \gamma|\psi_3\rangle$$

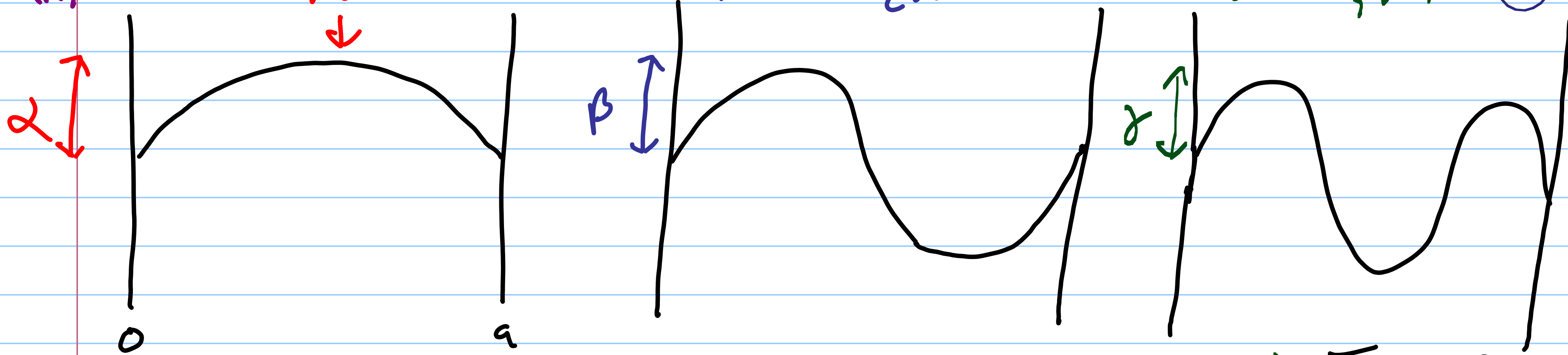
$$\Rightarrow \langle x|\psi\rangle = \alpha\langle x|\psi_1\rangle + \beta\langle x|\psi_2\rangle + \gamma\langle x|\psi_3\rangle$$

$$\psi(x) = \alpha\psi_1(x) + \beta\psi_2(x) + \gamma\psi_3(x)$$

$$\langle p|\psi\rangle = a\langle p|\psi_1\rangle + b\langle p|\psi_2\rangle + c\langle p|\psi_3\rangle$$

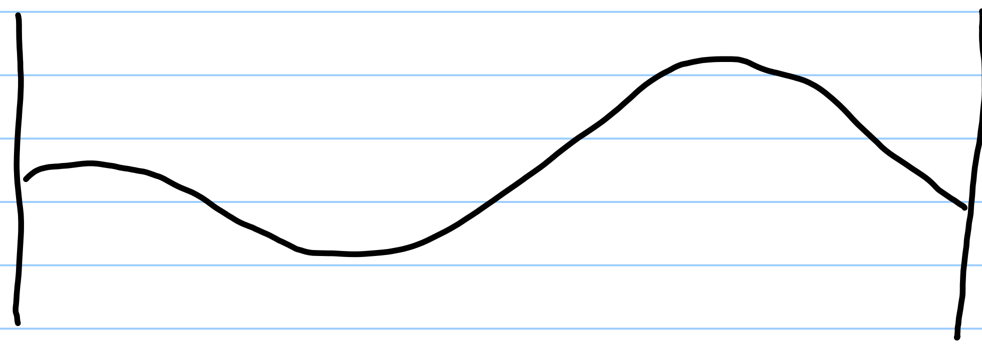
$$\psi(p) \text{ or } \varphi(p) = a\,\varphi_1(p) + b\,\varphi_2(p) + c\,\varphi_3(p)$$

$\psi(x) = \alpha\, \psi_1(x) + \beta\, \psi_2(x) + \gamma\, \psi_3(x)$ ⑤



$\psi_1(x) = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a}$ $\psi_2(x) = \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a}$ $\psi_3(x) = \sqrt{\frac{2}{a}} \sin \frac{3\pi x}{a}$

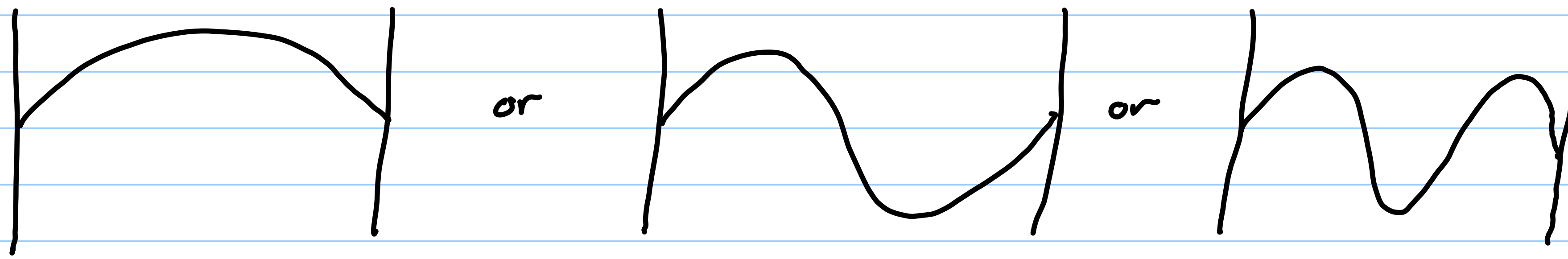
$\psi(x) =$



(for example)

$$\varphi(p) = \alpha \underbrace{\frac{\pi\hbar}{a}}_{p_1} + \beta \underbrace{\frac{2\pi\hbar}{a}}_{p_2} + \gamma \underbrace{\frac{3\pi\hbar}{a}}_{p_3}$$

MEASUREMENT OF ENERGY OR MOMENTUM WILL YIELD
EIGENVECTOR $|\psi\rangle$ $\rightarrow$ EIGENFUNCTION $\psi(x)$



$\rightarrow$ MEASURING MOMENTUM AGAIN WILL LEAVE
$|\psi\rangle$ UNCHANGED :

$$|\psi\rangle = |\psi_1\rangle \quad \text{or} \quad |\psi_2\rangle \quad \text{or} \quad |\psi_3\rangle$$

with $|\psi\rangle = |\psi_2\rangle$, $\langle\psi_1|\psi\rangle = \langle\psi_1|\psi_2\rangle = 0$

$$\langle\psi_2|\psi\rangle = \langle\psi_2|\psi_2\rangle = 1$$

$$\langle\psi_3|\psi\rangle = \langle\psi_3|\psi_3\rangle = 0$$

$$|\psi\rangle = \int |x\rangle\langle x|\psi_2\rangle \, dx = \int \psi_2(x) \, |x\rangle \, dx$$

EIGENFUNCTIONS OF MOMENTUM IN POSITION REP. :

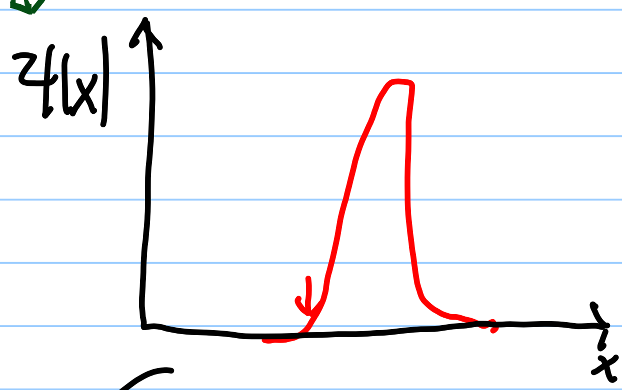
PROBABILITIES :

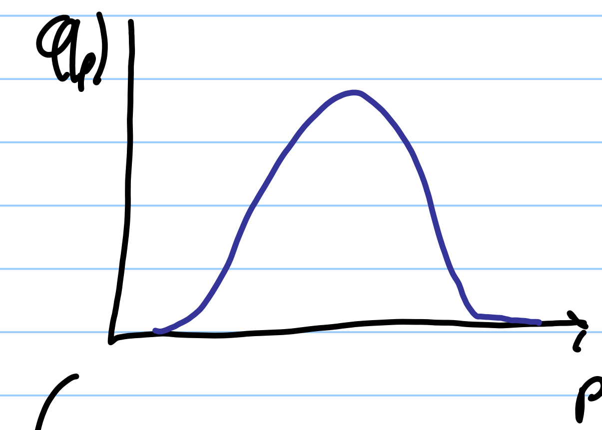
MEASURING POSITION WILL YIELD EIGENVECTOR, EIGENFUNCTION

$$\psi(x) = \delta(x - x')$$

REALISTICALLY, $\psi(x)$ would not be so "spiked"

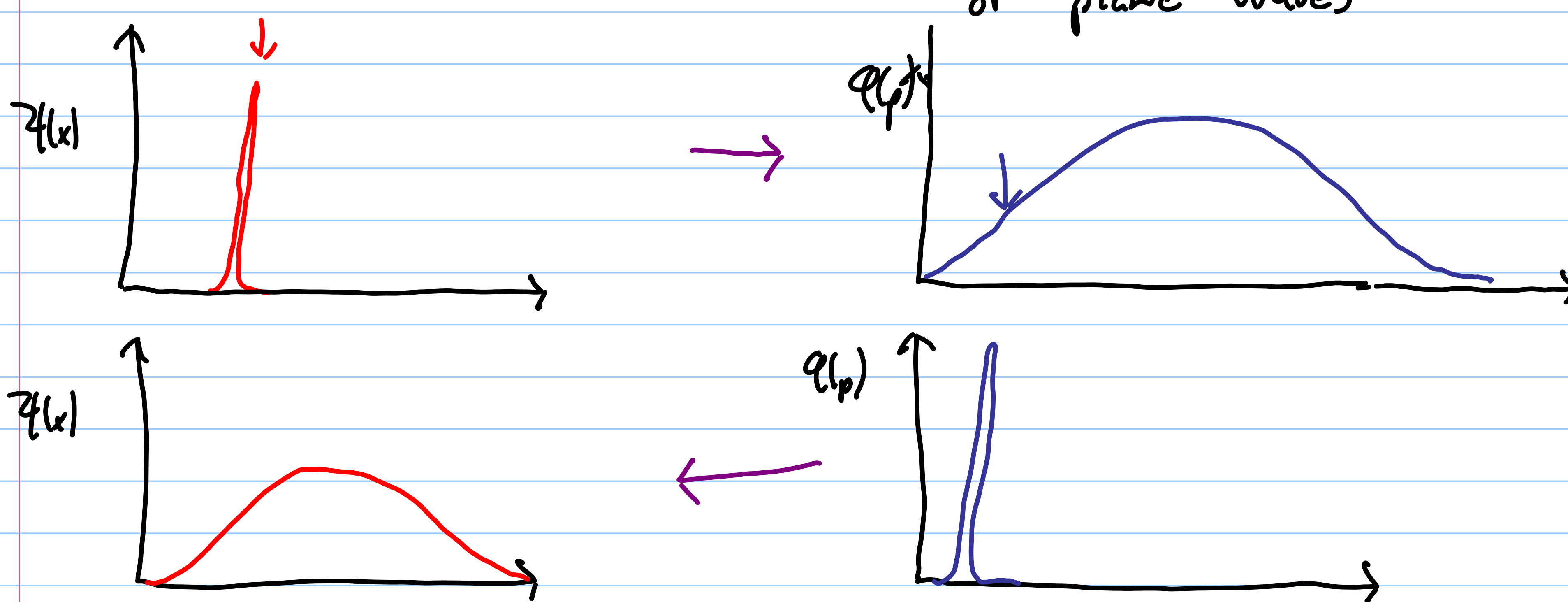
$\langle x|\psi\rangle$ $\langle p|\psi\rangle$





sum of plane waves

amplitude distribution of plane waves



OPERATORS

Quantities we can measure: (observables)

momentum
position
energy
angular momentum
spin
...

are associated with operators: $\hat{p}$ $\hat{x}$ $\hat{H}$ $\hat{L}$ $\hat{S}$

Value that are observed are eigenvalues of their operators.

$$\hat{O}\,|o\rangle = o\,|o\rangle$$

OPERATOR

EIGENVECTOR OF $\hat{O}$

eigenvalue

Back To our example: Expectation Values

Basis of $\hat{H}$:

$$|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle + \gamma|\psi_3\rangle$$

In our example, we associated the eigenvectors with unit vectors in 3D:

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |\psi_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad , \quad \hat{H} = \begin{pmatrix} \frac{\pi^2\hbar^2}{2ma^2} & 0 & 0 \\ 0 & \frac{4\pi^2\hbar^2}{2ma^2} & 0 \\ 0 & 0 & \frac{9\pi^2\hbar^2}{2ma^2} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix}}_{\hat{H}} \underbrace{\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}}_{|\psi\rangle} = \begin{pmatrix} E_1\alpha \\ E_2\beta \\ E_3\gamma \end{pmatrix}$$

$$\hat{H}|\psi\rangle = \alpha \hat{H}|\psi_1\rangle + \beta \hat{H}|\psi_2\rangle + \gamma \hat{H}|\psi_3\rangle$$
$$= \alpha E_1|\psi_1\rangle + \beta E_2|\psi_2\rangle + \gamma E_3|\psi_3\rangle$$

$$\begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} E_1\alpha \\ E_2\beta \\ E_3\gamma \end{pmatrix}$$

Expectation value (average) for $\hat{H}$:

$$\langle\psi|\hat{H}|\psi\rangle = (\alpha^*, \beta^*, \gamma^*) \begin{pmatrix} E_1\alpha \\ E_2\beta \\ E_3\gamma \end{pmatrix} =$$

$$= |\alpha|^2 E_1 + |\beta|^2 E_2 + |\gamma^3| E_3$$

prob. of obtaining E_1