LECTURE 30: Angular Momentum (part 2)

What I expect you to learn:

- What are the raising and lowering operators for $|l,m\rangle$
- How to work in the position representation in spherical coordinates
- How to solve problems in the Matrix representation

(Roughly covers chapter 6.1-6.6 of the textbook)

ANGULAR MOMENTUM (RECAP)

CM:

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

QM:

$$\hat{L} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_{op} & y_{op} & z_{op} \\ \frac{\hbar}{i}\frac{\partial}{\partial x} & \frac{\hbar}{i}\frac{\partial}{\partial y} & \frac{\hbar}{i}\frac{\partial}{\partial z} \end{vmatrix}$$

We focus on two operators:

$\hat{L}^2$, $\hat{L}^2 |\psi\rangle = \ell(\ell+1)\hbar^2 |\psi\rangle$

$\hat{L}_z$, $\hat{L}_z |\psi\rangle = m\hbar |\psi\rangle$

ANGULAR MOMENTUM (RECAP)

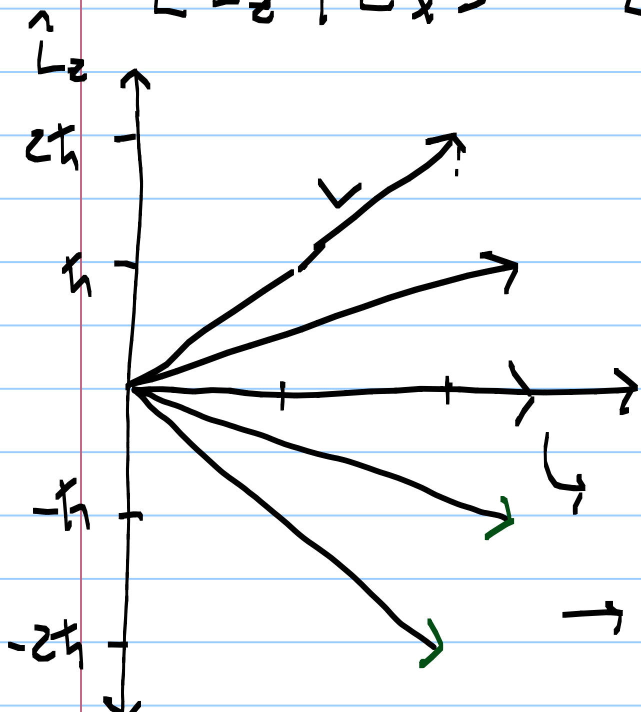
$$[\hat{L}^2, \hat{L}_z] = 0$$

→ $\hat{L}_z$, $\hat{L}^2$ share eigenstates

→ one can measure $\hat{L}_z$ and $\hat{L}^2$ at the same time

$$[\hat{L}_z, \hat{L}_x] = [\hat{L}_z, \hat{L}_y] \neq 0$$

→ cannot measure simultaneously two components of the angular momentum



↳ $\sqrt{l(l+1)}\,\hbar = \sqrt{6}\,\hbar$

→ sometimes called space quantisation (orientation of L in space)

ANGULAR MOMENTUM (RECAP)

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$$

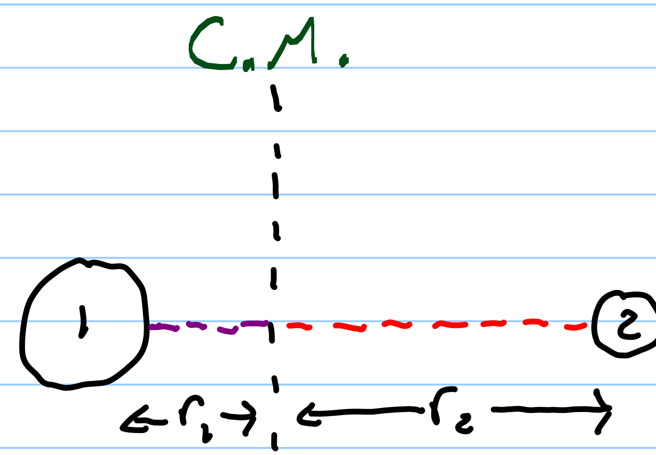
$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

FOR DIATOMIC MOLECULES

$$E = \frac{1}{2}m_1 v_{1,r}^2 + \frac{1}{2}m_2 v_{2,r}^2$$

$$E = \frac{m_1 + m_2}{2m_1 m_2} L^2 = \frac{L^2}{2\mu r_0^2}$$

$$\rightarrow E = \frac{\ell(\ell+1)\hbar^2}{2\mu r_0^2}$$



$r_0 = r_1 + r_2$

"rigid rotator"

Derivation for Rigid Rotator:

$E = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2$, angular velocity: $\boxed{v_1 = r_1 \omega}$

$$E = \frac{1}{2} m_1 \omega^2 r_1^2 + \frac{1}{2} m_2 \omega^2 r_2^2 = \frac{1}{2} \underbrace{\left(m_1 r_1^2 + m_2 r_2^2\right)}_{\text{moment of inertia "I"}} \omega^2$$

$$E = \frac{1}{2} I \omega^2$$

In the centre of mass: $m_1 r_1 = m_2 r_2$

$$m_1 r_1^2 + m_2 r_2^2 = m_1 r_1 \frac{m_2}{m_1} r_2 + m_2 r_2 \frac{m_1}{m_2} r_1$$

$$= m_2 r_1 r_2 + m_1 r_2 r_1$$

$$= (m_2 + m_1) r_2 r_1 = \frac{(m_2 + m_1)^2}{(m_2 + m_1)} r_2 r_1$$

$$= \frac{m_2^2 r_2 r_1}{(m_2 + m_1)} + \frac{m_1^2 r_2 r_1}{(m_2 + m_1)} + \frac{2 m_2 m_1 r_2 r_1}{(m_2 + m_1)}$$

Derivation (cont.)

$$= \frac{m_2 m_1}{(m_2 + m_1)} \cdot \left(\frac{m_2}{m_1} r_2 r_1 + \frac{m_1}{m_2} r_2 r_1 + 2 r_2 r_1 \right)$$

$$= \mu \cdot \left(r_1^2 + r_2^2 + 2 r_2 r_1 \right)$$

$$= \mu \cdot (r_1 + r_2)^2 = \mu r_0^2 = I$$

$$L = m_1 w r_1^2 + m_2 w r_2^2 \qquad (m_1 v_1 r_1 + m_2 v_2 r_2)$$

$$= m_1 r_1 w \frac{m_2 r_2}{m_1} + m_2 r_2 w \frac{m_1}{m_2} r_1$$

$$= m_2 w r_1 r_2 + m_1 w r_2 r_1$$

$$= (m_2 + m_1) \cdot w r_1 r_2$$

$$= \frac{(m_2 + m_1)^2}{(m_1 + m_2)} \cdot w r_1 r_2$$

Derivation (cont'd)

$$= \frac{m_2^2}{(m_1+m_2)} w r_1 r_2 + \frac{m_1^2}{(m_1+m_2)} w r_1 r_2 + \frac{2 m_1 m_2 w r_1 r_2}{(m_1+m_2)}$$

$$= \frac{m_2 m_1 r_1^2 w}{(m_1+m_2)} + \frac{m_1 m_2 w r_2^2}{(m_1+m_2)} + \frac{2 m_1 m_2 w r_1 r_2}{(m_1+m_2)}$$

$$= \frac{m_2 m_1}{(m_1+m_2)} w \cdot (r_1^2 + r_2^2 + 2 r_1 r_2)$$

$$= \mu w r_0^2 \quad = I w \quad = L$$

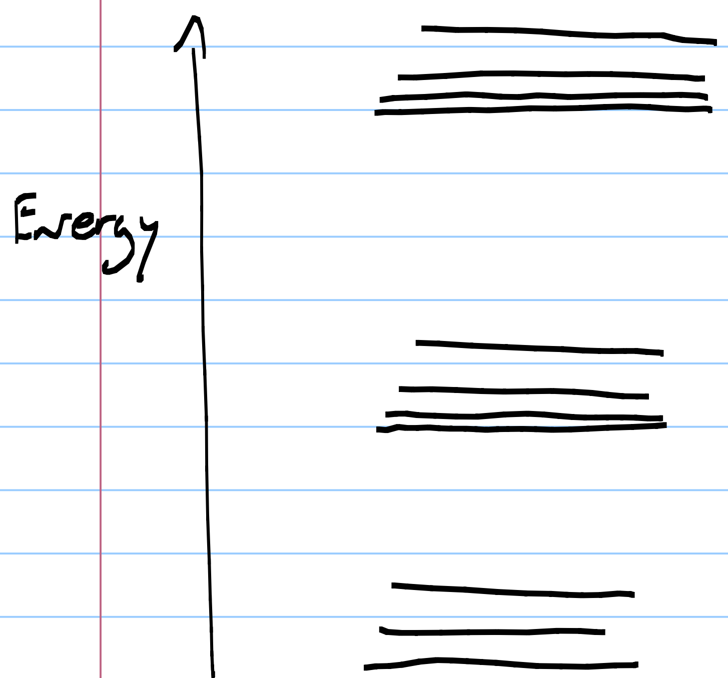
$$E = \frac{1}{2} I w^2 = \frac{1}{2} \frac{I^2 w^2}{I} = \boxed{\frac{L^2}{2I}}$$

ANGULAR MOMENTUM (DIATOMIC MOLECULES)

In reality the diatomic molecule has both rotational and vibrational energy levels.

To first approximation one can write:

$$E_{n,\ell} = (n + 1/2)\hbar\omega_0 + \frac{\ell(\ell+1)\hbar^2}{2\mu r_0^2}$$



ORBITAL MOMENTUM (RECAP)

$$\hat{L}_z F(\theta,\varphi) = m\hbar F(\theta,\varphi), \quad \hat{L}^2 F(\theta,\varphi) = \ell(\ell+1)\hbar^2 F(\theta,\varphi)$$

$$F = P_{\ell,m}(\theta)\, e^{im\varphi} \equiv Y_{\ell,m}(\theta,\varphi)$$

ℓ	m	$Y_{\ell m}(\theta,\varphi)$
0	0	$\frac{1}{\sqrt{4\pi}}$
1	0	$\sqrt{\frac{3}{4\pi}} \cos\theta$
	± 1	$\mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\varphi}$
2	0	$\sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$
	± 1	$\sqrt{\frac{15}{8\pi}} \cos\theta \sin\theta\, e^{\pm i\varphi}$
	± 2	$\sqrt{\frac{15}{32}} \sin^2\theta\, e^{\pm 2i\varphi}$

ANGULAR MOMENTUM (cont.)

WE INTRODUCE THE FOLLOWING OPERATORS:

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y \quad , \quad \hat{L}_- = \hat{L}_x - i\hat{L}_y$$

VERIFY THE FOLLOWING:

$$[\hat{L}^2, L_\pm] = 0 \qquad (1)$$

$$[\hat{L}_z, L_\pm] = \pm\hbar L_\pm \qquad (2)$$

LET'S PLAY WITH (2): $\hat{L}_z L_+ - \hat{L}_+ \hat{L}_z = \hbar L_+$

$$\hat{L}_z [\hat{L}_+ Y_{\ell m}] = (\hbar L_+ + \hat{L}_+ \hat{L}_z) Y_{\ell m}$$

$$= (\hbar \hat{L}_+ + \hat{L}_+ m\hbar) Y_{\ell m}$$

$$= \hbar(m+1) [\hat{L}_+ Y_{\ell m}]$$

Also $\hat{L}_z \hat{L}_- Y_{\ell m} = \hbar(m-1) \hat{L}_- Y_{\ell m}$

ANGULAR MOMENTUM (cont.)

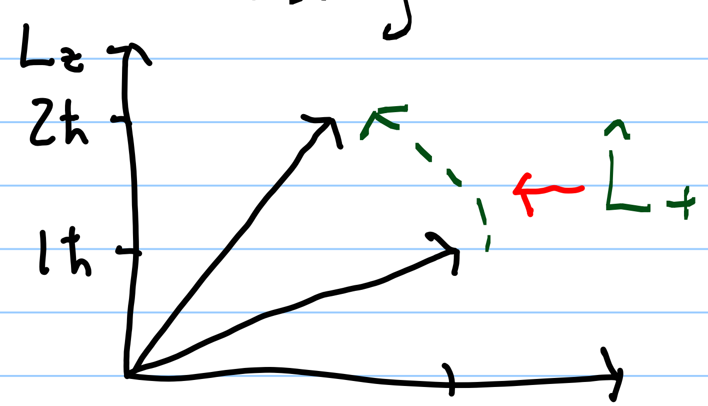
$$\hat{L}^2(\hat{L}_+ Y_{\ell m}) = \hat{L}_+ \hat{L}^2 Y_{\ell m} = \ell(\ell+1)\hbar^2 (L_+ Y_{\ell m})$$

→ so $\hat{L}_\pm$ generates a new eigenstate of both $\hat{L}^2$ and $\hat{L}_z$.

→ eigenvalue of $\hat{L}^2$ remains the same

→ eigenvalue of $\hat{L}_z$ changes by one unit of $\hbar$.

→ $\hat{L}_\pm$ are "raising" and "lowering" operators.



ANGULAR MOMENTUM (cont.)

NOTE THAT:

$$\hat{L}_+\hat{L}_- = (\hat{L}_x + i\hat{L}_y)(\hat{L}_x - i\hat{L}_y) = \hat{L}_x^2 - \hat{L}_x i\hat{L}_y + i\hat{L}_y\hat{L}_x + \hat{L}_y^2$$

$$= \hat{L}_x^2 + \hat{L}_y^2 - i[\hat{L}_x, \hat{L}_y]$$

$$= \hat{L}_x^2 + \hat{L}_y^2 + \hbar\hat{L}_z$$

$$= \hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z \qquad (3)$$

$$\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z \qquad (4)$$

Normalizations:

$$L_+|l,m\rangle = C_+(l,m)\,|l,m+1\rangle \qquad (5)$$

$$L_-|l,m\rangle = C_-(l,m)\,|l,m-1\rangle$$

$$\langle l,m|\hat{L}_+^\dagger = \langle l,m|\hat{L}_- = \langle l,m+1|\,C_+^*(l,m) \qquad (6)$$

ANGULAR MOMENTUM (cont.)

5 · 6 :

$$|C_+(l,m)|^2 \langle l, m+1 | l, m+1 \rangle$$

$$= \langle l,m | \hat{L}_- \hat{L}_+ | l,m \rangle = \langle l,m | \hat{L}^2 - \hat{L}_z^2 - \hbar L_z | l,m \rangle$$

$$= \hbar^2 [\, l(l+1) - m^2 - m \,] \qquad = \hbar^2 [\, l(l+1) - m(m+1) \,]$$

$$= \hbar^2 [(l-m)(l+m+1)]$$

$$\Rightarrow \quad C_+(l,m) = \hbar \sqrt{(l-m)(l+m+1)}$$

$$C_-(l,m) = \hbar \sqrt{(l+m)(l-m+1)}$$

$$\langle l, m' | \hat{L}_z | l,m \rangle = \hbar m \, \delta_{mm'}$$

$$\langle l, m' | L_\pm | l,m \rangle = \hbar \sqrt{l(l+1) - m(m\pm 1)} \;\; \delta_{m', m\pm 1}$$

ANGULAR MOMENTUM (MATRIX REP.)

$$\langle l, m' | \hat{L}_z | l, m \rangle = \hbar m \, \delta_{m m'}$$

$$\langle l, m' | L_\pm | l, m \rangle = \hbar \sqrt{l(l+1) - m(m \pm 1)} \; \delta_{m', m \pm 1}$$

MATRIX REP. FOR $l = 1$, $m = -1, 0, 1$

$$\hat{L}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \hat{L}_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{L}_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \qquad \begin{aligned} \hat{L}_x &= \frac{1}{2} (\hat{L}_+ + \hat{L}_-) \\ \hat{L}_y &= \frac{-i}{2} (\hat{L}_+ - \hat{L}_-) \end{aligned}$$

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_y = \hbar/\sqrt{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

POSITION REPRESENTATION

Remember $\psi(x) = \langle x | \psi \rangle$

Now we have: $Y_{\ell,m}(\theta, \varphi) = \langle \theta, \varphi | \ell, m \rangle$

$\rightarrow$ ANGLES θ, φ ARE COORDINATES ON SURFACE OF UNIT SPHERE

WE HAD: $\displaystyle\int_{-\infty}^{\infty} dx \, |x\rangle\langle x| = 1$

WE NOW HAVE

$$\int d\Omega \, |\theta, \varphi\rangle\langle \varphi | \theta| \equiv \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \, |\theta, \varphi\rangle\langle \varphi, \theta| = 1$$

Position Representation

$$\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \varphi}$$

$$\hat{L}_{\pm} = \hbar e^{\pm i\varphi} \left(\pm \frac{\partial}{\partial \theta} + i \cot \frac{\partial}{\partial \varphi} \right)$$

Example:

$$Y_{22}(\theta, \varphi) = A \sin^2\theta \, e^{2i\varphi}$$

$$Y_{21}(\theta, \varphi) \propto \hat{L}_- Y_{22}(\theta, \varphi) = \hbar e^{-i\varphi} \left(-\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \varphi} \right) Y_{22}(\theta, \varphi)$$

$$= \hbar e^{-i\varphi} \left[A \sin^2\theta \cdot i \cot\theta \cdot 2i \, e^{2i\varphi} - A e^{2i\varphi} \cdot 2 \sin\theta \cdot \cos\theta \right]$$

$$= \hbar e^{i\varphi} A \left[\sin\theta \cos\theta - 2 \sin\theta \cdot \cos\theta \right]$$

$$= B e^{i\varphi} \sin\theta \cos\theta$$

EXAMPLE PROBLEM:

A SYSTEM WITH ORBITAL MOMENTUM $\ell = 1$ is in THE FOLLOWING INITIAL STATE

$$|\psi\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{2} \\ \sqrt{3} \end{pmatrix}$$

1 - WHAT VALUES CAN WE OBTAIN IF WE MEASURE $\hat{L}_x$, WHAT ARE THE EIGENVECTORS?

2 - CALCULATE $\langle \hat{L}_z \rangle$ IF THE SYSTEM IS IN $\ell_x = -\hbar$

3 - " $\langle \hat{L}_z^2 \rangle$ " "

4 - ΔL_z " "

5 - repeat 2, 3, 4, 6 for $\hat{L}_y$

6 - if we measure $\hat{L}_x$ with the system in the initial state above, what values will we obtain? with what probabilities?

PROBLEM (CONT)

$$\hat{L}_x : \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{vmatrix} -\lambda & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & -\lambda & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & -\lambda \end{vmatrix}$$

$$-\lambda\left(\lambda^2 - \frac{\hbar^2}{2}\right) - \hbar/\sqrt{2}\left(\frac{\hbar}{\sqrt{2}} \cdot -\lambda\right) = 0$$

$$= -\lambda\left(\lambda^2 - \frac{\hbar^2}{2}\right) + \frac{\hbar^2}{2}\lambda = \lambda\left(-\lambda^2 + \hbar^2\right) = 0$$

$$\lambda = 0, -\hbar, \hbar$$

$$\begin{pmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\hbar\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{aligned} &\frac{\hbar}{\sqrt{2}} x_2 = -\hbar x_1 \\ &\frac{\hbar}{\sqrt{2}} x_1 + \frac{\hbar}{\sqrt{2}} x_3 = -\hbar x_2 \\ &\frac{\hbar}{\sqrt{2}} x_2 = -\hbar x_3 \end{aligned}$$

PROBLEM CONT.

$$\Rightarrow x_1 = x_3$$

$$\sqrt{2}\, x_1 = -x_2$$

PICK : $\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$

NORM : $\frac{1}{2}$

$$- \langle -1 | \hat{L}_2 | -1 \rangle = \frac{\hbar}{4} \begin{pmatrix} -1 & \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} = 0$$

$$- \langle -1 | \hat{L}_2^2 | -1 \rangle = \frac{\hbar^2}{4} (-1, \sqrt{2}, -1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix} \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} = \frac{\hbar^2}{2}$$

$$- \Delta \hat{L}_2 = \sqrt{\langle -1 | \hat{L}_2^2 | -1 \rangle - \langle -1 | L_2 | -1 \rangle^2} = \sqrt{\frac{\hbar^2}{2}} = \hbar / \sqrt{2}$$

PROBLEM (CONT)

NEED OTHER EIGENVECTORS:

$$|-1\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}, \quad |0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$|4\rangle = \begin{pmatrix} -\sqrt{3/14} \\ \sqrt{8/14} \\ \sqrt{3/14} \end{pmatrix}$$

$$\rightarrow \quad \sqrt{3/7}\ |0\rangle, \quad \sqrt{\frac{2}{7}}\ |1\rangle, \quad \sqrt{\frac{2}{7}}\ |-1\rangle$$

$$P(-1) = |\langle -1|4\rangle|^2 = \left|\sqrt{\frac{2}{7}}\right|^2 = \frac{2}{7}$$

PROBLEM SET 5

DUE: Dec 4th

1- PROBLEM 6.13 OF TEXTBOOK

2- A PARTICLE IN A SPHERICALLY SYMMETRIC POTENTIAL IS IN THE FOLLOWING STATE:

$$\psi(x, y, z) = C\,(xy + yz + zx)\,e^{-\alpha r^2}$$

a- what is the probability that the measurement of L^2 will give: i) 0 ii) $6\hbar^2$

b- if we find $\ell = 2$, what are the probabilities of finding $m = 2, 1, 0, -1, 2$?

3 - PART 5 OF PROBLEM ON PAGE 14, Lecture 30