LECTURE 31: Spin 1/2

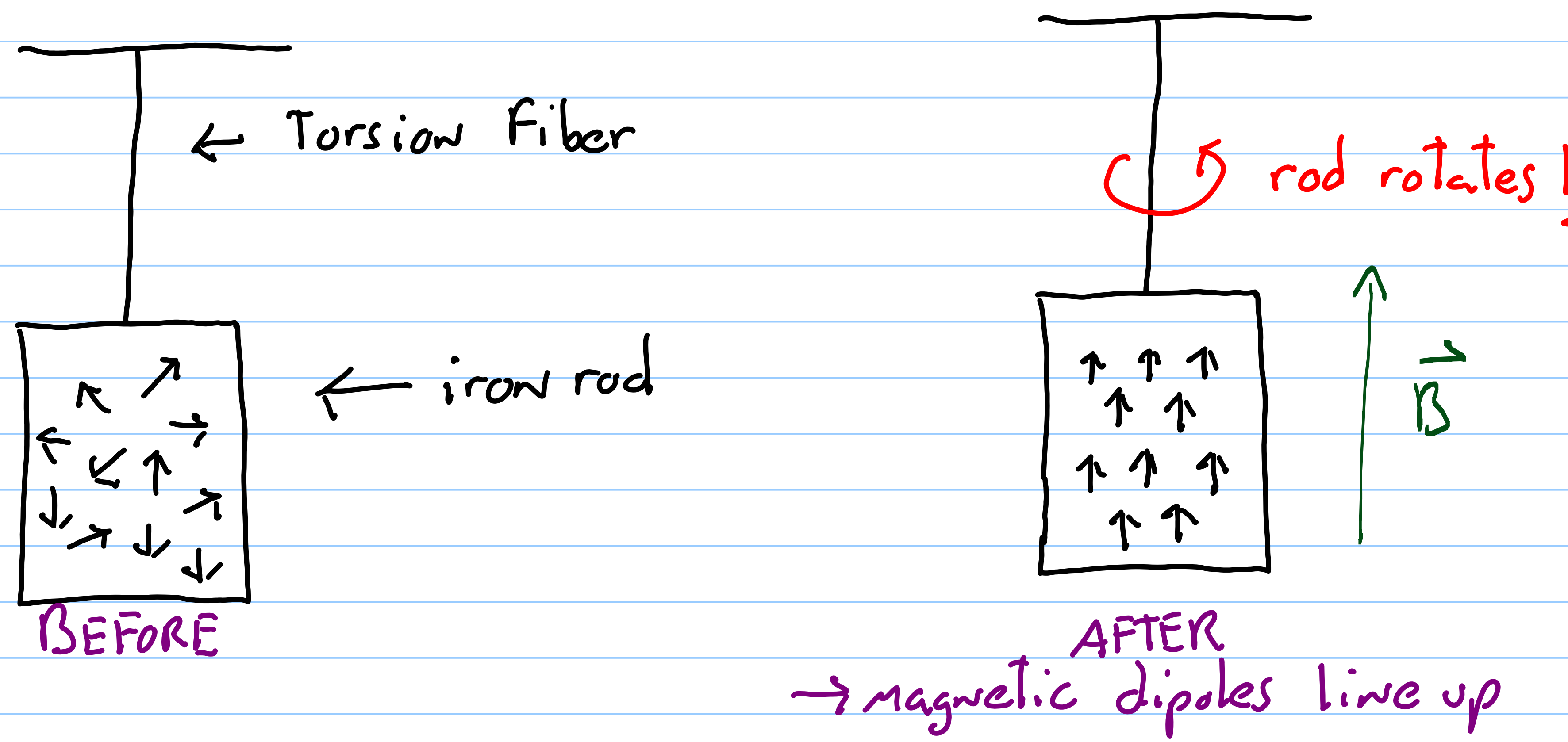
What I expect you to learn:

- What is "intrinsic spin"
- What is a gyromagnetic ratio
- How this relates to the Stern-Gerlach results
- What are the relevant operators for spin
- How to derive and work with the Pauli matrices (example problem)

(Roughly covers chapter 6.8 of the textbook)

MAGNETISM AND ANGULAR MOMENTUM

Einstein - de Haas EFFECT

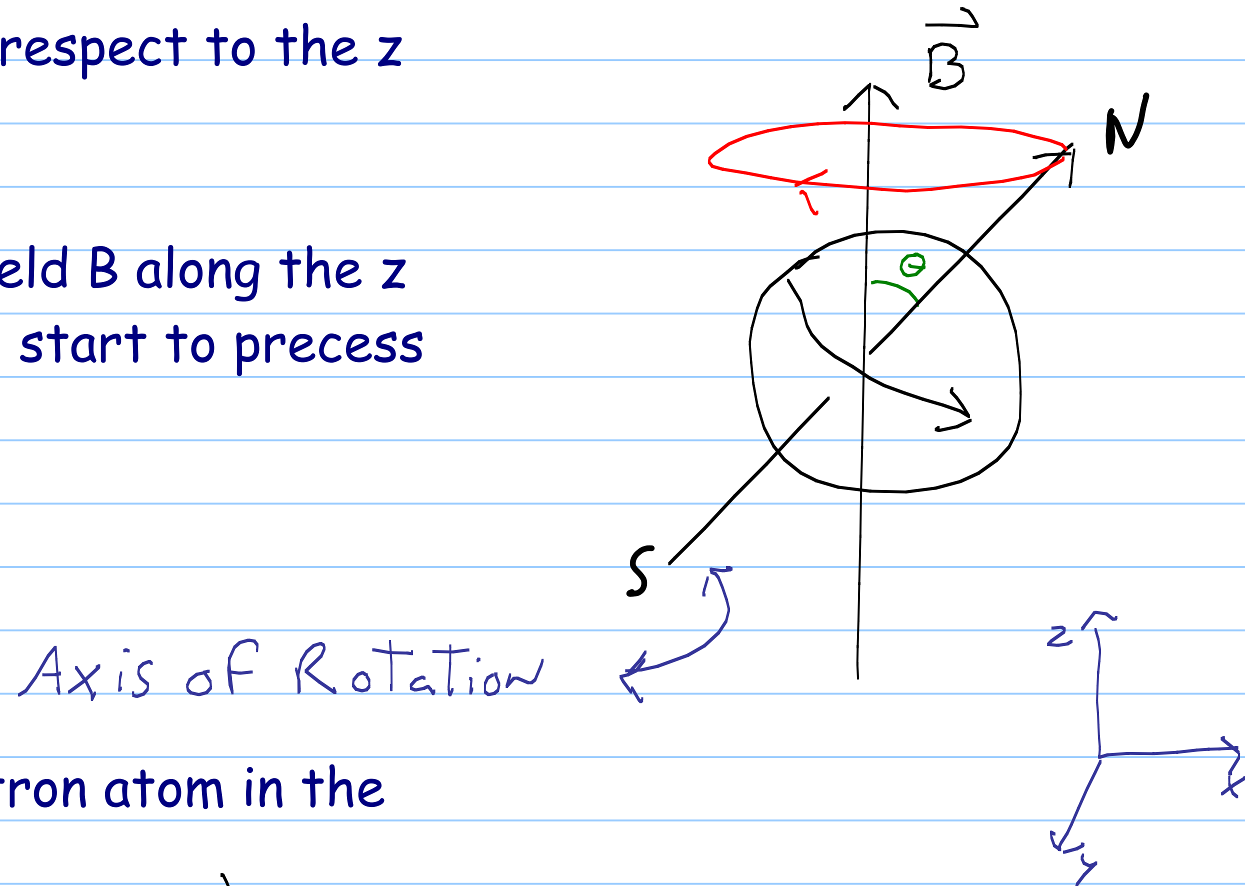


→ this effect links angular momentum and magnetism. IT does not demonstrate quantisation

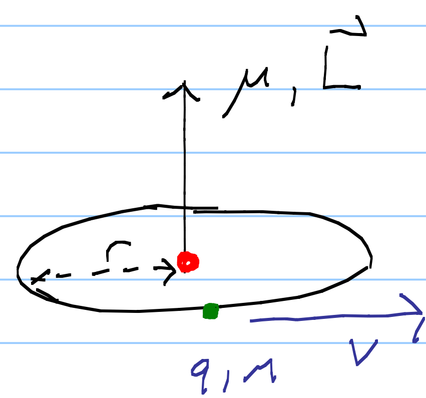
RECAP FROM LECTURE 7

Consider a charged sphere whose rotation axis is at an angle θ with respect to the z axis

By applying a magnetic field B along the z direction, the sphere will start to precess about the z axis



Now consider a one-electron atom in the Bohr Model:



q : charge of electron $= -e$

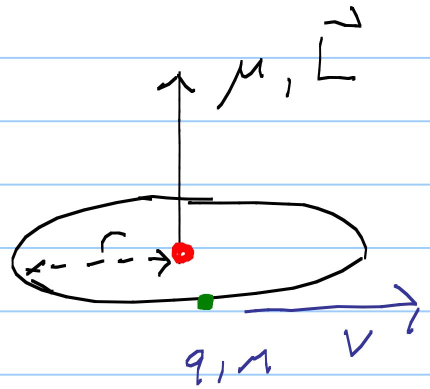
m : mass of electron $= m_e$

r : radius of orbit

v : velocity of electron

μ : magnetic moment of atom

L : angular momentum



How long does it take for the electron to orbit?

$$\rightarrow \frac{2\pi r}{v} = t_{orbit}$$

$$\Rightarrow \text{current } I = q/T_{orbit} = \frac{qv}{2\pi r}$$

You'll see in E and M that the magnetic moment for a dipole is

$$\mu = I \cdot \text{area} = \pi r^2 I$$

$$\rightarrow \mu = \pi r^2 \cdot \frac{qv}{2\pi r} = \boxed{\frac{qvr}{2}}$$

The orbital angular momentum: $L = mvr$

$\Rightarrow \mu = \frac{q}{2m} L$, with $\hbar$ natural unit of L,

we can write: $\boxed{\mu_B = \frac{e\hbar}{2m_e}}$ $\rightarrow \mu_B$ is "Bohr magneton"

Gyromagnetic Ratio

$\mu = \frac{q}{2m} L$ Tells us how the magnetic moment and the angular momentum are related

$\frac{q}{2m}$: gyromagnetic ratio

The electron's intrinsic angular momentum i.e "spin" is equal to $\frac{1}{2}\hbar$. Using this value, we should get

$$\mu = \frac{e}{2m_e}\frac{\hbar}{2}$$

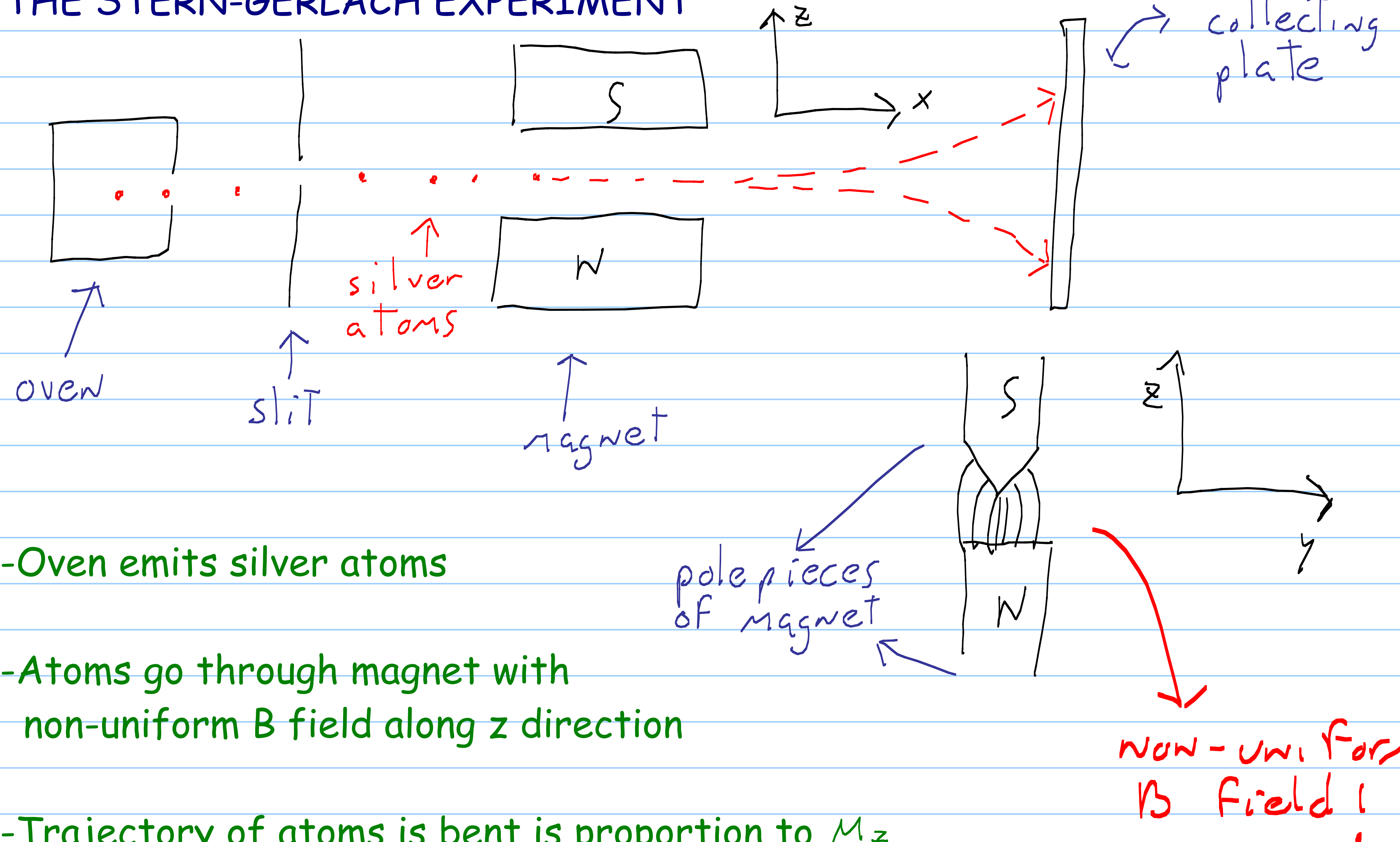
But we get $\mu = \frac{e\hbar}{2m_e}$ or $\mu = g \cdot \frac{e}{2m_e}\frac{\hbar}{2}$

$g = 2.00231930437 66\ (87)$ → measured

→ see electron spin resonance in PHY 225

(See lecture 7) ↓

THE STERN-GERLACH EXPERIMENT



- Oven emits silver atoms

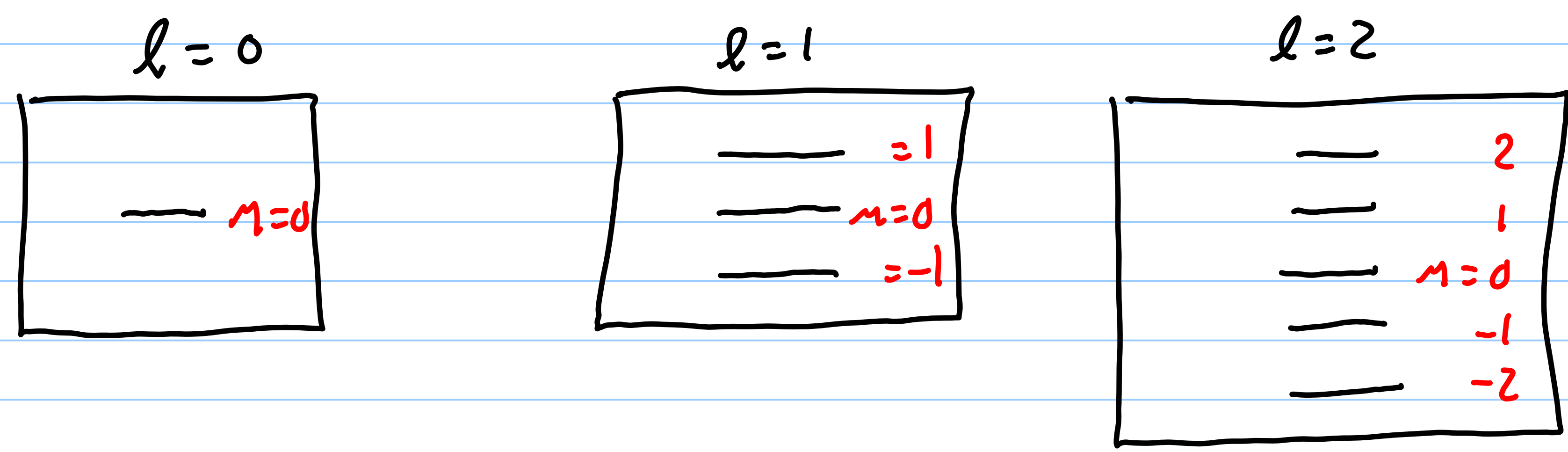
- Atoms go through magnet with non-uniform B field along z direction

- Trajectory of atoms is bent is proportion to μ_z

- Observe pattern of silver atoms on collecting plate

SPIN ANGULAR MOMENTUM

FROM WHAT WE HAVE SEEN UP UNTIL NOW WITH ORBITAL ANGULAR MOMENTUM, WE SHOULD OBSERVE:



WHAT WE ARE MISSING IS "SPIN". IT HAS NO CLASSICAL ANALOG

WE HAVE FOUND "ELEMENTARY" PARTICLES THAT HAVE SPIN $1/2$ AND SPIN 1.

FERMIONS (half integer) BOSONS (INTEGER)

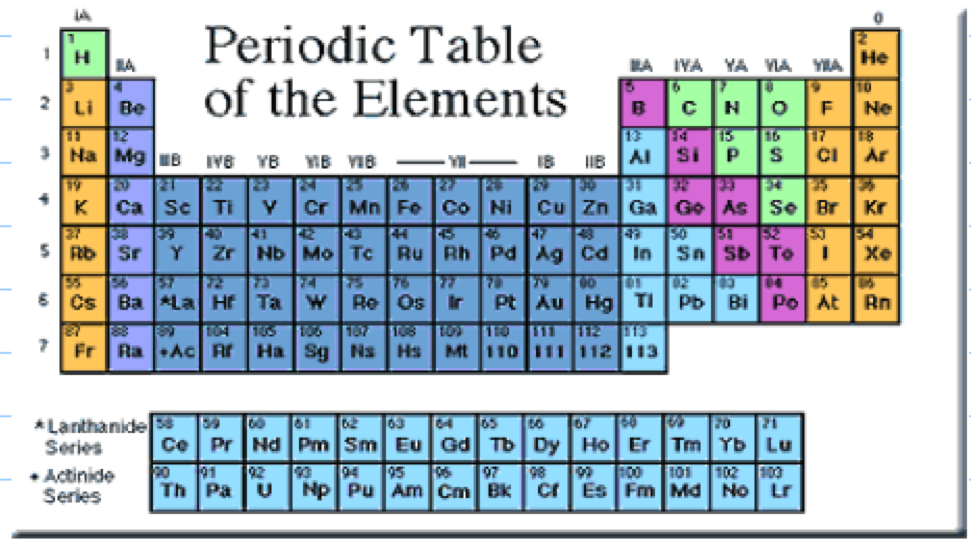
Gravitons have in theory spin 2. We are looking for a spin 0 particle (Higgs Boson).

But we were using silver !?

Silver(A_g) has 47 electrons

47^{th} electron occupies 5s orbital.

All other electrons are paired



SPLITTING OF THE BEAM INTO TWO LINES IS DUE TO THIS UNPAIRED ELECTRON

NOTE : - SPACE QUANTISATION (ORBITAL MOMENTUM)
EXPERIMENT: ZEEMAN EFFECT

SPIN 1/2 EXPERIMENT: ELECTRON SPIN RESONANCE

SPIN ANGULAR MOMENTUM

WE REPEAT WHAT WE DID IN LECTURES 29 AND 30

→ WE SAW THAT ORBITAL ANGULAR MOMENTUM WAS CHARACTERIZED BY TWO QUANTUM NUMBERS : ℓ, m

→ FOR SPIN WE WILL WORK WITH TWO NUMBERS ALSO : s , m_s and TWO OPERATORS $\hat{S}^2$, $\hat{S}_z$

→ since m_s takes on two values,
$s = 1/2$, $m_s \pm 1/2$

With $|\psi\rangle$ being eigenstate of $\hat{S}^2$, $\hat{S}_z$

we have:

$$\hat{S}^2 |\psi\rangle = \hbar^2 s(s+1) |\psi\rangle$$

$$\hat{S}_z |\psi\rangle = m_s \hbar |\psi\rangle$$

$$\Rightarrow [\hat{S}^2, \hat{S}_z] = 0 \quad , \quad [\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$$

SPIN (CONT.)

WE RE-INTRODUCE NEW RAISING AND LOWERING OPERATORS:

$$\boxed{\begin{aligned}\hat{S}_+ &= \hat{S}_x + i\hat{S}_y \\ \hat{S}_- &= \hat{S}_x - i\hat{S}_y\end{aligned}} \rightarrow \boxed{\begin{aligned}\hat{S}_+^T &= \hat{S}_- \\ \hat{S}_-^T &= \hat{S}_+\end{aligned}}$$

WE SKIP THE DERIVATION (see lecture 30)

$$\left.\begin{aligned}\hat{S}_- |s, m_s\rangle &\propto |s, m_s - 1\rangle \\ \hat{S}_+ |s, m_s\rangle &\propto |s, m_s + 1\rangle\end{aligned}\right]$$

s stays the same

m_s is raised or lowered by 1

$$\hat{S}_- |s, m_s\rangle = c_- |s, m_s - 1\rangle$$

$$\langle s, m_s | \hat{S}_-^T = \langle s, m_s - 1 | c_-^* = \langle s, m_s | \hat{S}_+$$

$$\Rightarrow \langle s, m_s | \hat{S}_+ \hat{S}_- | s, m_s\rangle = |c_-|^2$$

SPIN (cont.)

$$\hat{S}_+\hat{S}_- = (\hat{S}_x + i\hat{S}_y)(\hat{S}_x - i\hat{S}_y)$$
$$= \hat{S}_x^2 + \hat{S}_y^2 - \hat{S}_x i\hat{S}_y + i\hat{S}_y\hat{S}_x$$
$$= \hat{S}_x^2 + \hat{S}_y^2 - i[\hat{S}_x, \hat{S}_y]$$
$$= \hat{S}_x^2 + \hat{S}_y^2 - i \cdot i\hbar\hat{S}_z$$
$$= \hat{S}^2 - \hat{S}_z^2 + \hbar\hat{S}_z$$

$$\langle s, m_s | \hat{S}^2 - S_z^2 + \hbar\hat{S}_z | s, m_s\rangle = |c_-|^2$$

$$\langle s, m_s | s, m_s\rangle \cdot \left[s(s+1)\hbar^2 - m_s^2\hbar^2 + m_s\hbar^2\right] = |c_-|^2$$

$$= \hbar^2\left[s(s+1) - m_s^2 + m_s\right] = |c_-|^2$$

$$c_- = \hbar\sqrt{s(s+1) - m_s(m_s-1)}$$

$$\Rightarrow \boxed{\hat{S}_-|s, m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s-1)}\,|s, m_s-1\rangle} \quad (1)$$

Spin (cont.)

$$\hat{S}_+ |s, m_s\rangle = \hbar \sqrt{s(s+1) - m_s(m_s+1)} \; |s, m_s+1\rangle \qquad (2)$$

$$\Rightarrow \sqrt{1/2(1/2+1) - -1/2(-1/2+1)} = \sqrt{3/4 + 1/2} = 1$$

same result for c_-

In the $\hat{S}_z$ basis:

$\hookrightarrow$ $\hat{S}_z$ eigenstate

$$\hat{S}_z |+\rangle = \frac{\hbar}{2} |+\rangle$$

$$\hat{S}_z |-\rangle = -\frac{\hbar}{2} |-\rangle$$

$\hat{S}_z$: $\begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$, eigenvectors : $|+\rangle : \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|-\rangle : \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

From (1),(2) : $\hat{S}_+ |-\rangle = \hbar |+\rangle$

$\hat{S}_- |+\rangle = \hbar |-\rangle$

SPIN (CONT.)

MATRIX ELEMENTS OF $\hat{S}_+$:

$$\langle +|\hat{S}_+|+\rangle = 0 \quad , \quad \langle +|\hat{S}_+|-\rangle = \hbar$$

$$\langle -|\hat{S}_+|+\rangle = 0 \quad , \quad \langle -|\hat{S}_+|-\rangle = 0$$

$$: \quad \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \hat{S}_- : \quad \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

BACK TO THE DEFINITION OF $\hat{S}_+$ AND $\hat{S}_-$:

$$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) \quad , \quad \hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-)$$

$$\hat{S}_x : \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \hat{S}_y : \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

PAULI MARTICES: $\sigma_x : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y : \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Spin (cont.)

WHAT ARE THE EIGENVECTORS OF $\hat{S}_x$?

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = 0 \qquad \lambda^2 - \frac{\hbar^2}{4} = 0 \quad \Rightarrow \quad \lambda = \pm \hbar/2$$

- with $\lambda = +\hbar/2$:

$$\begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \hbar/2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \boxed{\begin{aligned} \hbar/2\, x_2 &= \hbar/2\, x_1 \\ \hbar/2\, x_1 &= \hbar/2\, x_2 \end{aligned}}$$

$$\Rightarrow \quad |x\rangle_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- with $\lambda = -\hbar/2$

$$\begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\hbar/2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{aligned} \hbar/2\, x_2 &= -\hbar/2\, x_1 \\ \hbar/2\, x_1 &= -\hbar/2\, x_2 \end{aligned}$$

$\Rightarrow \; |-\rangle_{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$|+\rangle_x \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \quad |-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

NOTE THAT WITH : $\quad |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

WE CAN WRITE :

$$|+\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle \right)$$
$$|-\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle \right)$$

i.e. $|+\rangle_x$ eigenstate even mixture of $|+\rangle$ and $|-\rangle$

$\boxed{\text{EXERCISE : FIND EIGENVALUES AND EIGENVECTORS OF } \hat{S}_y}$

EXAMPLE PROBLEM:

- FIND THE EIGENVALUES AND EIGENSTATES OF THE SPIN OPERATOR $\hat{S}$ OF AN ELECTRON IN THE DIRECTION OF A UNIT VECTOR $\vec{n}$, WHERE $\vec{n}$ LIES IN THE XZ PLANE.

WE WANT TO SOLVE:

$$\vec{n} \cdot \hat{S} |\varphi\rangle = \frac{\hbar}{2} \lambda |\varphi\rangle$$

$$\vec{n} \cdot \hat{S} = (\sin\theta\, \vec{x} + \cos\theta\, \vec{z}) \cdot (\hat{S}_x \vec{x} + \hat{S}_y \vec{y} + \hat{S}_z \vec{z})$$

unit vectors

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{n} \cdot \hat{S} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$\begin{vmatrix} \hbar/2 \cos\theta - \lambda & \frac{\hbar}{2}\sin\theta \\ \frac{\hbar}{2}\sin\theta & -\hbar/2\cos\theta - \lambda \end{vmatrix} = 0$$

$$-\frac{\hbar^2}{4}\cos^2\theta - \lambda\frac{\hbar}{2}\cos\theta + \lambda\frac{\hbar}{2}\cos\theta + \lambda^2 - \frac{\hbar^2}{4}\sin^2\theta = 0$$

$$= -\frac{\hbar^2}{4} + \lambda^2 = 0 \quad , \quad \lambda = \pm\frac{\hbar}{2}$$

Eigenvectors:

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\cos\theta\, x_1 + \sin\theta\, x_2 = x_1 \qquad (4)$$
$$\sin\theta\, x_1 - \cos\theta\, x_2 = x_2 \qquad (5)$$

PROBLEM (CONT.)

$$x_2 \sin\theta = x_1 (1 - \cos\theta) \quad (6)$$
$$x_1 \sin\theta = x_2 (1 + \cos\theta) \quad (7)$$

$$(6) \rightarrow \frac{x_2}{x_1} = \frac{(1-\cos\theta)}{\sin\theta} \quad , \quad \operatorname{tg}\frac{\theta}{2} = \frac{(1-\cos\theta)}{\sin\theta}$$

$$\frac{x_2}{x_1} = \operatorname{Tg}\frac{\theta}{2} = \frac{\sin\theta/2}{\cos\theta/2} \quad , \quad \begin{matrix} x_1 = \cos\theta/2 \\ x_2 = \sin\theta/2 \end{matrix}$$

$$|\varphi_1\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \rightarrow \text{normalised}$$

PROCEEDING THE SAME WAY WE WOULD GET:

$$|\varphi_2\rangle = \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \quad , \quad \langle\varphi_1|\varphi_2\rangle = 0 \rightarrow \text{orthogonal}$$

- WHAT IS THE PROBABILITY OF MEASURING $\hat{S}_z = +\hbar/2$ THE SYSTEM IS IN THE $|\varphi_1\rangle$ EIGENSTATE?

PROBLEM CONT.

LET'S WRITE $|\varphi_1\rangle$ IN TERMS OF THE $\hat{S}_z$ EIGENSTATES:

$$|\varphi_1\rangle = \cos\theta/2 \underbrace{|+\rangle}_{\begin{pmatrix}1\\0\end{pmatrix}} + \sin\frac{\theta}{2} \underbrace{|-\rangle}_{\begin{pmatrix}0\\1\end{pmatrix}}$$

$$P = |\langle + | \varphi_1 \rangle|^2$$

$$= \left| \langle + | \cos\frac{\theta}{2} |+\rangle + \langle + | \sin\frac{\theta}{2} |-\rangle \right|^2$$

$$= \cos^2\frac{\theta}{2}$$