LECTURE 33: Spin Angular Momentum (Part II)

What I expect you to learn:

- Express eigenstates of spin operator in arbitrary direction

- Spin Precession

(Roughly covers chapter 6.8 of the textbook)

(PROBLEM SET 5 DEADLINE EXTENDED: NOW DUE DEC. 6th)

ANOTHER SPIN 1/2 PROBLEM (MORE GENERAL)

- FIND THE EIGENVALUES OF AND EIGENSTATES OF THE SPIN OPERATOR $\vec{S}$ OF AN ELECTRON IN THE DIRECTION OF A UNIT VECTOR N ($\vec{n}$ IS ARBITRARY)

$$\vec{n}\cdot\vec{S}\,|\varphi\rangle = \hbar/2\,\lambda\,|\varphi\rangle$$

$\hookrightarrow$ Eigenvector of $\hat{S}$

$$\vec{n} = (\sin\theta\cos\varphi)\,\vec{x} + (\sin\theta\sin\varphi)\,\vec{y} + \cos\theta\,\vec{z}$$

$\hookrightarrow$ unit vectors

$$\vec{n}\cdot\vec{S} = \hat{S}_x \sin\theta\cos\varphi + \hat{S}_y \sin\theta\sin\varphi + \hat{S}_z \cos\theta$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\varphi - i\sin\varphi) \\ \sin\theta(\cos\varphi + i\sin\varphi) & -\cos\theta \end{pmatrix}$$

$$= \hbar/2\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}$$

PROBLEM CONT.

$$\begin{vmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2} e^{-i\varphi}\sin\theta \\ \frac{\hbar}{2}\sin\theta\, e^{i\varphi} & -\frac{\hbar}{2}\cos\theta - \lambda \end{vmatrix}$$

$$-\frac{\hbar^2}{4}\cos^2\theta + \lambda^2 - \frac{\hbar^2}{4}\sin^2\theta = 0$$

$$\lambda^2 = \hbar^2/4 \Rightarrow \lambda = \pm \hbar/2$$

eigenvector for $\hbar/2$:

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \hbar/2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1\cos\theta + x_2 e^{-i\varphi}\sin\theta = x_1$$

$$x_1(1-\cos\theta) = x_2 e^{-i\varphi}\sin\theta \qquad (1)$$

PROBLEM CONT.

$$(1-\cos\theta) = 2\sin^2\theta/2$$

$$\sin\theta = 2\cos\theta/2 \, \sin\theta/2$$

(1) becomes: $\quad \chi_1 \cdot 2\sin^2\theta/2 = \chi_2 e^{-i\varphi} \cdot 2\cos\theta/2 \, \sin\theta/2$

$$\chi_2 = \chi_1 \tan\theta/2 \; e^{i\varphi}$$

$$\frac{\chi_2}{\chi_1} = \frac{\sin\theta/2 \, e^{i\varphi}}{\cos\theta/2}$$

$$|\varphi_1\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \, e^{i\varphi} \end{pmatrix} , \quad |\varphi_2\rangle = \begin{pmatrix} -\sin\theta/2 \\ e^{i\varphi}\cos\theta/2 \end{pmatrix}$$

Probability of measuring $\hat{S}_z = -\hbar/2$ if electron is in $|\varphi_2\rangle$ eigenstate?

PROBLEM CONT.

$$|+\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

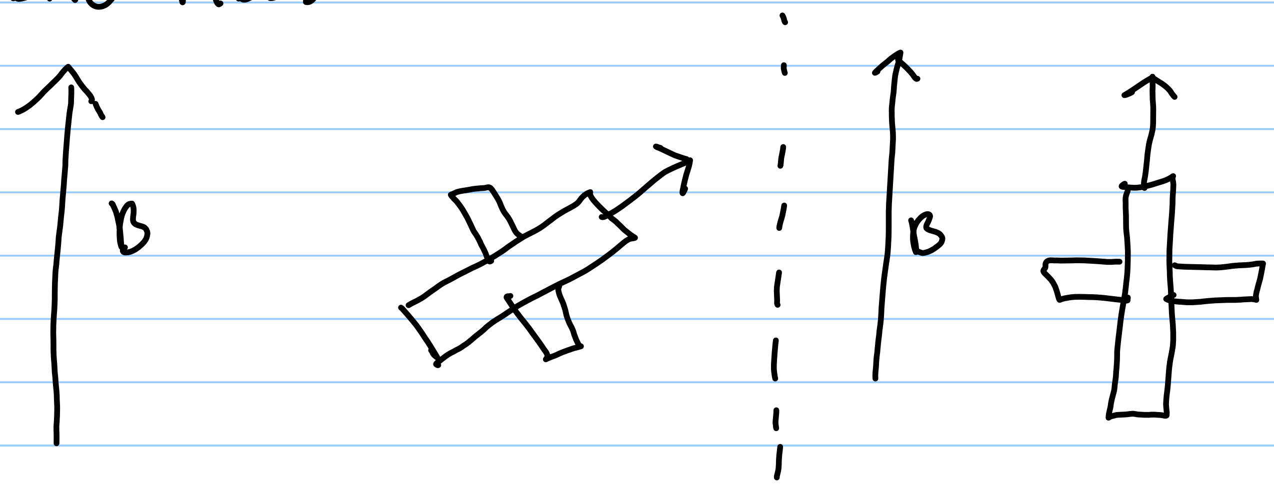
$$|\varphi_2\rangle = -\sin\theta/_2 \, |+\rangle + e^{i\varphi} \cos\theta/_2 \, |-\rangle$$

$$|\langle \varphi_2 | - \rangle|^2 = e^{i\varphi} \cos\theta/_2 \, e^{-i\varphi} \cos\theta/_2$$
$$= \cos^2 \theta/_2$$

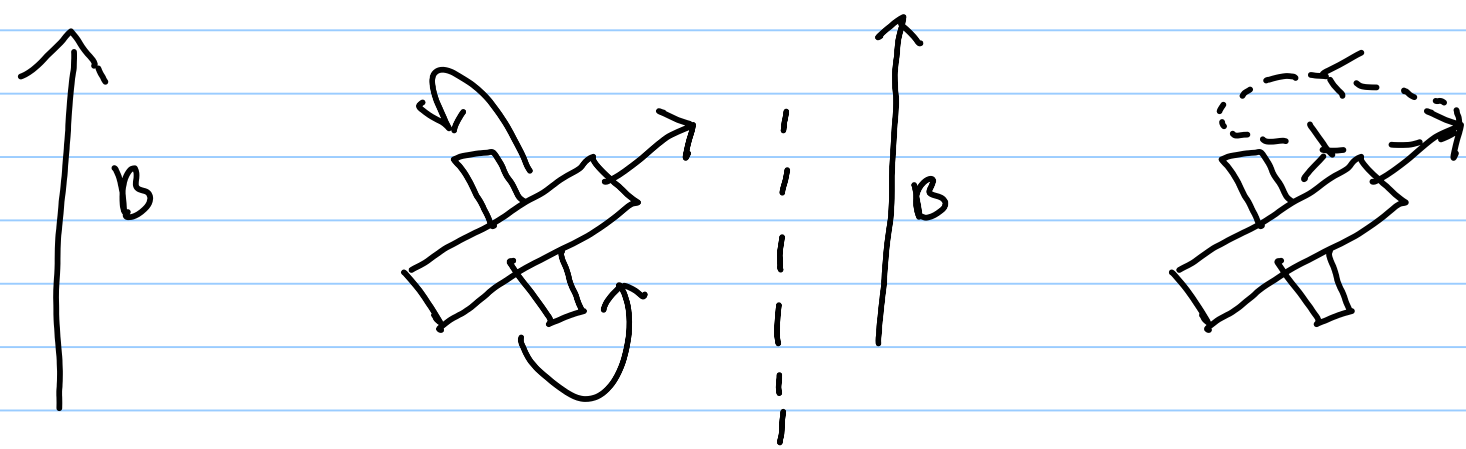
→ recall that the same problem for $\vec{n}$ in x-z plane gave the same answer

SPIN PRECESSION

CONSIDER A MAGNET IN AN EXTERNAL MAGNETIC FIELD:



NOW SUPPOSE THE MAGNET ROTATES WRT ITS MAGNETIC FIELD AXIS:



SPIN PRECESSION (CONT)

TO DETERMINE WHAT HAPPENS FOR A QUANTUM SYSTEM, WE'LL NEED TO SOLVE SCHRÖDINGER'S EQUATION.

$$\hat{H} = -\vec{\mu} \cdot \vec{B}$$

$$\mu = \frac{-eg}{2m_e} \hat{S}$$

gyromagnetic moment of the electron:

2.0023193043718

$\pm$ 0.0000000000076 (2004)

we'll use $g = 2$, $\mu_B = \frac{e\hbar}{2m_e}$

Let's chose $\vec{z}$ so that it points in the B field direction.

$$\rightarrow \hat{H} = \frac{2\mu_B}{\hbar} B \cdot \hat{S} = \frac{2\mu_B}{\hbar} \hat{S}_z B = \frac{2\mu_B}{\hbar} \cdot \frac{\hbar}{2} \sigma_z B = \mu_B \sigma_z B$$

SPIN PRECESSION (CONT)

$$\hat{H}|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$$

$$\rightarrow \mu_B B \sigma_z |\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$$

SINCE σ_z IS 2x2 MATRIX, $|\psi\rangle$ WILL BE EXPRESSED AS A 2-COMPONENT VECTOR:

$$|\psi\rangle : \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$ aka "spinor"

$$\mu_B B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

SPIN PRECESSION (cont.)

$$\mu_B B \psi_+ = i\hbar \frac{\partial \psi_+}{\partial T}$$

$$-\mu_B B \psi_- = i\hbar \frac{\partial \psi_-}{\partial T}$$

Solutions:

$$\psi_+ = A_+ e^{-i(\mu_B B/\hbar)T}$$
$$\psi_- = A_- e^{i(\mu_B B/\hbar)T}$$

IN MATRIX FORM:

$$|\psi(t)\rangle = \begin{pmatrix} A_+ e^{-i(\mu_B B/\hbar)T} \\ A_- e^{i(\mu_B B/\hbar)T} \end{pmatrix} \qquad \boxed{|\psi(0)\rangle = \begin{pmatrix} A_+ \\ A_- \end{pmatrix}}$$

SPIN PRECESSION (cont.)

IF ELECTRON HAS SPIN "UP" IN THE z DIRECTION AT $T=0$:

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\psi(t)\rangle = \begin{pmatrix} e^{-i(\mu_B B/\hbar)T} \\ 0 \end{pmatrix}$$

PROBABILITY OF MEASURING SPIN UP AT A LATER TIME:

$$P = |\langle + | \psi(t)\rangle|^2 = \left| (1, 0) \begin{pmatrix} e^{-i(\mu_B B/\hbar)T} \\ 0 \end{pmatrix} \right|^2$$

$$= \underline{\underline{e^{+i(\mu_B B/\hbar)T} \, e^{-i(\mu_B B/\hbar)T} = 1}}$$

SPIN PRECESSION (cont.)

WHAT IF $|\psi(0)\rangle$ IS IN THE $|+\rangle_x$ EIGENSTATE?

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow \quad A_+ = \frac{1}{\sqrt{2}} = A_-$$

$$|\psi\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\, e^{-i(\mu_B B/\hbar)t} \\ \frac{1}{\sqrt{2}}\, e^{i(\mu_B B/\hbar)t}\end{pmatrix}$$

define $\boxed{\omega = 2\mu_B B/\hbar}$ → Larmor frequency

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i(\omega/2)t} \\ e^{i(\omega/2)t}\end{pmatrix}$$

SPIN PRECESSION (cont.)

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$|\psi(2\pi/\omega)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\pi}\\e^{i\pi}\end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

remember that $e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$

what about $T = \pi/\omega$?

$$|\psi(\pi/\omega)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\pi/2}\\e^{i\pi/2}\end{pmatrix} = \frac{-i}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

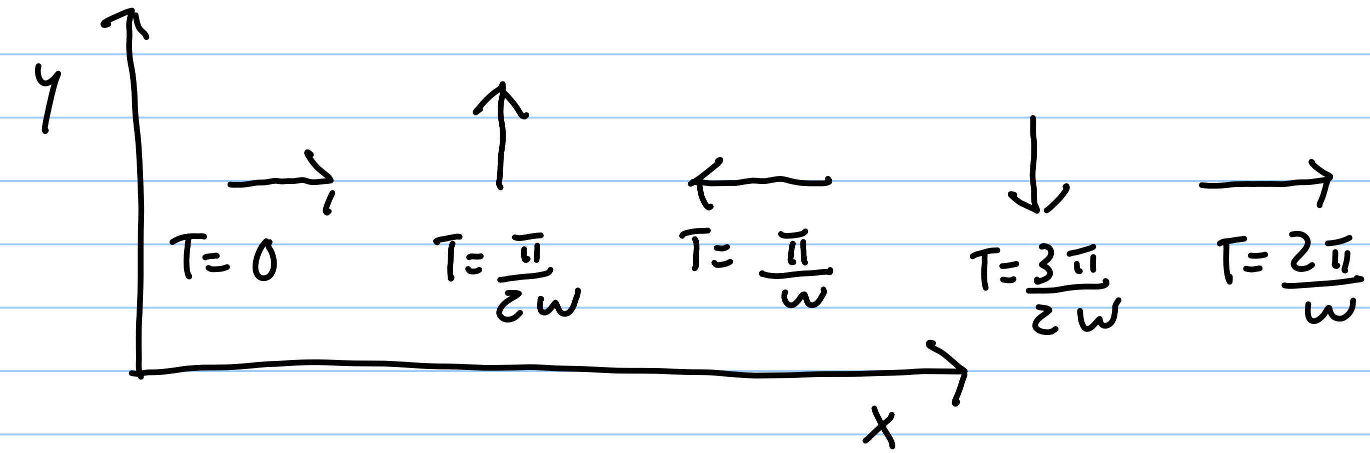
electron is now in $\boxed{|-\rangle_x \text{ state}}$

$$|\psi(\pi/2\omega)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\pi/4}\\e^{i\pi/4}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2} + i/\sqrt{2}\\1/\sqrt{2} - i/\sqrt{2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+i\\1-i\end{pmatrix}$$

$$= \frac{1}{2}(1+i)\begin{pmatrix}1\\i\end{pmatrix}$$

$\boxed{|+\rangle_y \text{ eigenstate}}$

SPIN PRECESSION (cont)



Electron is precessing, period $= \frac{2\pi}{\omega}$

$\omega = 2\mu_B B/\hbar$

BASIS OF MAGNETIC RESONANCE IMAGING

NMR uses nucleus

↳ nuclear magnetic resonance

MRI

Powerful magnet produces $\vec{B}$ field ($\sim 1.5T$) along $\hat{z}$.

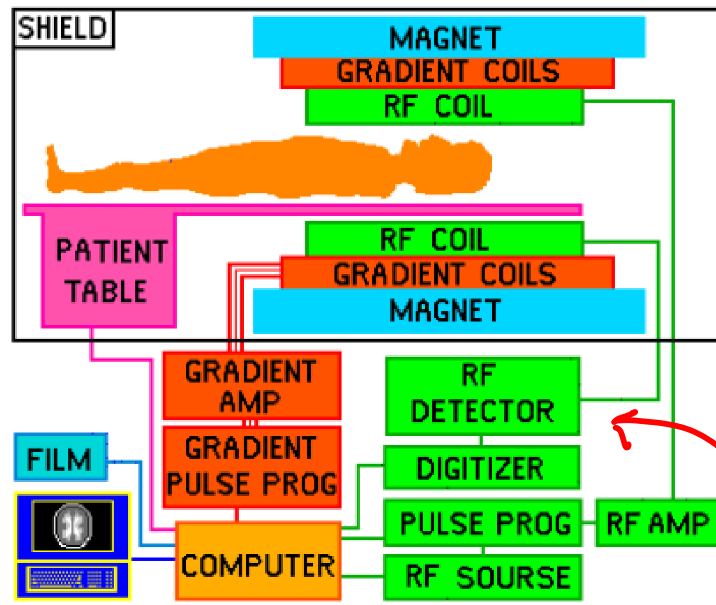
Designed to detect photons from spin flip of protons in hydrogen!

→ B field aligns spins along $\hat{z}$

→ secondary field (say along $\hat{x}$) induces spin flip

→ as proton flips back to lower energy state i.e. "relaxes", they precess about $B_{\hat{z}}$ field

→ protons emit radiation at precession frequency



→ detect radiation