LECTURE 35: Bell's Theorem

What I expect you to learn:

-How to compose systems of two spin 1/2 particles

-What is Bell's Theorem

(Roughly covers sections 6.10 and 17.3 of the textbook)

Notes:

- Problem set 5 due today
- I will post solutions to problem set 5 Friday afternoon. You need to hand-in your problem set at the latest Friday in class.
- I will post a practice final exam with solutions next weekend
- Friday we will discuss finish today's material and discuss the exam

(FINAL EXAM: Dec 19th, 9-11am, BN2S)

ADDITION OF TWO SPINS

$$\vec{J} = \vec{L} + \vec{S}$$

WE'LL STUDY THE SIMPLE CASE OF SYSTEM OF TWO ELECTRONS:

$\hat{S}_1$ operates on electron 1

$\hat{S}_2$ operates on electron 2

$$[\hat{S}_{1x}, \hat{S}_{1y}] = i\hbar \hat{S}_{1z}$$

$$[\hat{S}_{2x}, \hat{S}_{2y}]\ i\hbar \hat{S}_{2z}$$

$$[\hat{S}_1, \hat{S}_2] = 0$$

Define $\hat{S} = \hat{S}_1 + \hat{S}_2$

ADDING SPINS (CONT.)

$$
\begin{aligned}
[\hat{S}_x, \hat{S}_y] &= [(\hat{S}_{1x} + \hat{S}_{2x}), (\hat{S}_{1y} + \hat{S}_{2y})] \\
&= [\hat{S}_{1x}, \hat{S}_{1y}] + [\hat{S}_{2x}, \hat{S}_{2y}] \\
&= i\hbar \hat{S}_{1z} + i\hbar \hat{S}_{2z} \\
&= i\hbar \hat{S}_z
\end{aligned}
$$

- TWO - SPIN SYSTEM HAS 4 STATES:

$$\chi^1_+ \chi^2_+ , \chi^1_+ \chi^2_- , \chi^1_- \chi^2_+ , \chi^1_- \chi^2_-$$

WE HAVE:

$$\hat{S}_1^2 \chi^1_\pm = \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2 \chi^1_\pm$$

$$\hat{S}_{1z} \chi^1_\pm = \pm\hbar \chi^1_\pm$$

ADDING SPINS (cont)

$$\hat{S}_z \chi^1_\pm \chi^2_\pm = (\hat{S}_{1z} + \hat{S}_{2z})\, \chi^1_\pm \chi^2_\pm$$

$$= (\hat{S}_{1z} \chi^1_\pm)\, \chi^2_\pm + \chi^1_\pm (\hat{S}_{2z} \chi^2_\pm)$$

$$\rightarrow \quad \hat{S}_z \chi^1_+ \chi^2_+ = \hbar\, \chi^1_+ \chi^2_+$$

$$\hat{S}_z \chi^1_+ \chi^2_- = \hat{S}_z \chi^1_- \chi^2_+ = 0$$

$$\hat{S}_z \chi^1_- \chi^2_- = -\hbar\, \chi^1_- \chi^2_-$$

4 states $\rightarrow$ expect with $l=1$ to get 3 combinations : $m=-1,0,1$
+ 1 state with $l=0$, $m=0$

ADDING SPINS

DEFINE : $\hat{S}_- = S_{1-} + S_{2-}$

WE WILL APPLY THIS TO THE $m=1$ STATE:

REMEMBER: $S_-^1 \chi_+^1 = \hbar \chi_-^1$

$$\hat{S}_- \chi_+^1 \chi_+^2 = (\hat{S}_-^1 \chi_+^1)\chi_+^2 + \chi_+^1(\hat{S}_-^2 \chi_+^2)$$

$$= \hbar \chi_-^1 \chi_+^2 + \hbar \chi_+^1 \chi_-^2$$

$$= \sqrt{2}\hbar \cdot \frac{\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2}{\sqrt{2}}$$

$$\hat{S}_- \left(\frac{\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2}{\sqrt{2}} \right)$$

$$= (\hat{S}_{1-} + \hat{S}_{2-})(\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2)\frac{1}{\sqrt{2}}$$

ADDING SPINS

FIRST TERM

$$= (\hat{S}_{1-} + \hat{S}_{2-})(\chi^1_- \chi^2_+) \cdot \frac{1}{\sqrt{2}}$$

$$= 0 + \frac{\hbar}{\sqrt{2}} \chi^1_- \chi^2_-$$

2nd TERM:

$$= (\hat{S}_{1-} + \hat{S}_{2-})(\chi^1_+ \chi^2_-) \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{\hbar}{\sqrt{2}} \chi^1_- \chi^2_- + 0$$

$$= \sqrt{2}\,\hbar\, \chi^1_- \chi^2_-$$

REMAINING STATE WOULD BE (?)

$$\frac{1}{\sqrt{2}} (\chi^1_+ \chi^2_- - \chi^1_- \chi^2_+)$$

DOES $S = 0$ FOR THIS STATE?

ADDING SPINS

SET $\quad Y_\pm = \frac{1}{\sqrt{2}}\left(\chi_+^1 \chi_-^2 \pm \chi_-^1 \chi_+^2\right)$

$$\hat{S}^2 = (\hat{S}_1 + \hat{S}_2)^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_1 \cdot \hat{S}_2$$

$$= \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_{1z}\hat{S}_{2z} + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}$$

$$\rightarrow \hat{S}_1^2 Y_\pm = \frac{1}{\sqrt{2}}\left(\hat{S}_1^2 \chi_+^1 \chi_-^2 \pm \hat{S}_1^2 \chi_-^1 \chi_+^2\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{3}{4}\hbar^2 \chi_+^1 \chi_-^2 \pm \frac{3}{4}\hbar^2 \chi_-^1 \chi_+^2\right)$$

$$= \frac{3}{4}\hbar^2 Y_\pm$$

$$\rightarrow \hat{S}_2^2 = \frac{3}{4}\hbar^2 Y_\pm$$

ADDING SPINS (CONT.)

$$\rightarrow 2\hat{S}_{1z}\hat{S}_{2z}\chi_{\pm} = 2\cdot\frac{\hbar}{2}\cdot -\frac{\hbar}{2} = -\frac{\hbar^2}{2}\chi_{\pm}$$

$$\rightarrow \left(\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}\right)\chi_{+} = \frac{1}{\sqrt{2}}\left(\hat{S}_{1+}\chi^1_+\hat{S}_{2-}\chi^2_- + \hat{S}_{1+}\chi^1_-\hat{S}_{2-}\chi^2_+ + \hat{S}_{1-}\chi^1_+\hat{S}_{2+}\chi^2_- + \hat{S}_{1-}\chi^1_-\hat{S}_{2+}\chi^2_+\right)$$

$$= \frac{1}{\sqrt{2}}\left(0 + \hbar^2\chi^1_+\chi^2_- + \hbar^2\chi^1_-\chi^2_+ + 0\right)$$

$$= \hbar^2\chi_+$$

$$\left(\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}\right)\chi_- = -\hbar^2\chi_-$$

$$S^2\chi_{\pm} = \hbar^2\left(3/4 + 3/4 - 1/2 \pm 1\right)\chi_{\pm}$$

$$= 0 \quad \text{for } \chi_-$$

BELL'S THEOREM

$$\frac{1}{\sqrt{2}}\left(\chi_+^1\chi_-^2 - \chi_-^1\chi_+^2\right) = |\chi\rangle$$

measure electron 1 along $\hat{S}_a \cdot \vec{a}$, 2 along $S_b \cdot \vec{b}$

$$E(\hat{a},\hat{b}) = \langle \chi | \hat{S}_a \cdot \vec{a}\; \hat{S}_b \cdot \vec{b} | \chi \rangle$$

FOR EXAMPLE WE SAW:

$$\vec{n}\cdot\hat{S} = (\sin\theta\, \vec{x} + \cos\theta\, \vec{z}) \cdot (\hat{S}_x \vec{x} + S_y \vec{y} + \hat{S}_z \vec{z})$$

unit vectors

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{n}\cdot\hat{S} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

BELL (cont)

TAKE $\vec{b} = \hat{z}$, $\vec{a} = \sin\theta\,\vec{x} + \cos\theta\,\vec{z}$

We have

$$\langle\psi|\left(\sin\theta\,\hat{S}_{1x} + \cos\theta\,\hat{S}_{1z}\right)\hat{S}_{2z}|\psi\rangle$$

$$\hat{S}_{2z}\ \frac{1}{\sqrt{2}}\left(\psi_1^+\psi_2^- - \psi_1^-\psi_2^+\right)$$

$$= \frac{1}{\sqrt{2}}\left(-\frac{\hbar}{2}\cdot\psi_1^+\psi_2^- - \frac{\hbar}{2}\psi_1^-\psi_2^+\right)$$

$$\hat{S}_{1z}\cos\theta = \frac{1}{\sqrt{2}}\left(-\frac{\hbar^2}{4}\psi_1^+\psi_2^- + \frac{\hbar^2}{4}\psi_1^-\psi_2^+\right)\cdot\cos\theta$$

$$= -\cos\theta\ \langle\psi|\psi\rangle\cdot\frac{\hbar^2}{4} \qquad (1)$$

BELL (CONT.)

SUPPOSE WE HAVE A HIDDEN VARIABLE λ

$$\int p(\lambda)\, d\lambda = 1 \qquad p(\lambda) \geq 0 \qquad (2)$$

result of messurement of electron 1 along

$\vec{a}$: $A(\vec{a}, \lambda)$

result of messurement of electron 1 along

$\vec{b}$: $B(\vec{b}, \lambda)$

OVERALL SPIN $= 0$ $\Rightarrow$ $A(\vec{a}, \lambda) = -B(\vec{a}, \lambda)$

BELL (cont.)

* FOR A "COMPLETE" THEORY A, B ARE SPECIFIED BY λ

* in A LOCAL THEORY WITH MEASUREMENTS WELL SEPARATED (OUTSIDE LIGHT CONE), A depends on λ and $\vec{a}$ but -NOT- $\vec{b}$

* JOINT MEASUREMENT OF S_a AND S_b IS UNCORRELATED PRODUCT OF A AND B

AVERAGE IS GIVEN BY

$$\mathcal{E}(\vec{a},\vec{b}) = \int p(\lambda)\, A(\vec{a},\lambda)\, B(\vec{b},\lambda)\, d\lambda$$

IF $\vec{a} = \vec{b}$, $\mathcal{E}(\hat{a},\hat{a}) = E(\hat{a},\hat{a}) = -1$

BELL (CONT)

FOR A MEASUREMENT ALONG $\vec{c}$ WE WOULD HAVE:

$$\mathcal{E}(\vec{a},\vec{c}) = \int p(\lambda)\, A(\vec{a},\lambda)\, B(\vec{c},\lambda)\, d\lambda$$

$$\mathcal{E}(\vec{a},\vec{b}) - \mathcal{E}(\vec{a},\vec{c}) = \int d\lambda\, p(\lambda) \left[A(\vec{a},\lambda) B(\vec{b},\lambda) - A(\vec{a},\lambda) B(\vec{c},\lambda) \right]$$

Use $A(\vec{a},\lambda) = -B(\vec{a},\lambda)$, $(A(\vec{b},\lambda))^2 = 1$

$$= -\int d\lambda\, p(\lambda)\, A(\vec{a},\lambda)\, A(\vec{b},\lambda) \left(1 + A(\vec{b},\lambda) B(\vec{c},\lambda) \right)$$

$$\rightarrow \left| \mathcal{E}(\vec{a},\vec{b}) - \mathcal{E}(\vec{a},\vec{c}) \right| < \int d\lambda\, p(\lambda) \left[1 + A(\vec{b},\lambda) B(\vec{c},\lambda) \right]$$

or $\left| \mathcal{E}(\vec{a},\vec{b}) - \mathcal{E}(\vec{a},\vec{c}) \right| < 1 + \mathcal{E}(\vec{b},\vec{c})$

$$|\varepsilon(\vec{a},\vec{b}) - \varepsilon(\vec{a},\vec{c})| < 1 + \varepsilon(\vec{b},\vec{c})$$

EXAMPLE OF VIOLATION OF THIS INEQUALITY :

ANGLE BETWEEN $\vec{a}$ AND $\vec{c}$ = $2\pi/3$

$\vec{b}$ IN SAME PLANE AT $\pi/3$ OF BOTH

$$|E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})| = |-\cos(2\pi/3) + \cos(\pi/3)| = 1$$

$$1 + E(\vec{b},\vec{c}) = [1 - \cos \pi/3] = 1/2$$

WE HAVE : $|E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})| > 1 + E(\vec{b},\vec{c})$

EXPERIMENTS HAVE SHOWN THAT THIS IS CORRECT.