Practice Midterm Solutions

Q Using the Bohr Model, calculate energy levels of system of two quark bound state with same mass and potential $V(r) = -kr$.

A Force is actually constant $F = -\frac{\partial V}{\partial r} = k$.

Bohr model: orbits of one particle around another, but ang. momentum must be multiple of $\hbar$.

$$k = \frac{mv^2}{r} \qquad\qquad L = mvr = n\hbar$$

centripetal force — quantization condition

We get $v = \sqrt[3]{\frac{n\hbar k}{m^2}}$ $\quad r = \sqrt[3]{\frac{n^2\hbar^2}{km}}$

(Here m is the reduced mass of system. If quark mass is μ then $m = \frac{\mu_1\mu_2}{\mu_1+\mu_2} = \frac{\mu^2}{2\mu}$)

Kinetic energy $T = \frac{1}{2}mv^2 = \frac{1}{2}m\sqrt[3]{\frac{n^2\hbar^2k^2}{m^4}} = \frac{1}{2}\sqrt[3]{\frac{n^2\hbar^2k^2}{m}}$

Potential energy $V = -kr = -k\sqrt[3]{\frac{n^2\hbar^2}{km}} = -\sqrt[3]{\frac{n^2\hbar^2k^2}{m}}$

$$E_n = T + V = -\frac{1}{2}\sqrt[3]{\frac{\hbar^2k^2}{m}}\, n^{2/3}$$

Q Wavefunction $\Psi(x) = Ae^{-\mu|x|}$. Find A and $\phi(p)$

A Normalize $\Psi(x)$ for A. $\int_{-\infty}^{\infty}|\Psi(x)|^2dx = 1$ (Assume $A \geq 0$)

$$\int_{-\infty}^{\infty}|\Psi(x)|^2dx = A^2\int_{-\infty}^{\infty}e^{-2\mu|x|}dx = A^2\left[\int_{-\infty}^{0}e^{2\mu x}dx + \int_0^{\infty}e^{-2\mu x}dx\right] = \frac{A^2}{2\mu}\left[\left(e^{2\mu x}\right)_{-\infty}^{0} - \left(e^{-2\mu x}\right)_0^{\infty}\right]$$

$$= \frac{A^2}{\mu} = 1. \qquad \text{Thus } A = \sqrt{\mu}.$$

To find $\phi(p)$ use Fourier Transform.

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\psi(x)\,dx = \sqrt{\frac{\mu}{2\pi\hbar}}\left[\int_{-\infty}^{0} e^{(-\frac{ip}{\hbar}+\mu)x}\,dx + \int_{0}^{\infty} e^{(-\frac{ip}{\hbar}-\mu)x}\,dx\right]$$

$$= \sqrt{\frac{\mu}{2\pi\hbar}}\left\{\frac{1}{\mu-\frac{ip}{\hbar}}\left(e^{\mu x}e^{-ipx/\hbar}\right)_{-\infty}^{0} + \frac{-1}{\mu+\frac{ip}{\hbar}}\left(e^{-\mu x}e^{-ipx/\hbar}\right)_{0}^{\infty}\right]$$

$$\phi(p) = \sqrt{\frac{\mu}{2\pi\hbar}}\left[\frac{1}{\mu-\frac{ip}{\hbar}} + \frac{1}{\mu+\frac{ip}{\hbar}}\right] = \sqrt{\frac{\mu}{2\pi\hbar}}\;\frac{2\mu}{\mu^2+p^2/\hbar^2}$$

Q Using $\langle x\rangle = \int_{-\infty}^{\infty}\psi^*(x,t)\,x\,\psi(x,t)\,dx$, show $\langle p\rangle = -i\hbar\int_{-\infty}^{\infty}\psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t)\,dx$

A Ehrenfest's Th'm: $\langle p\rangle = m\dfrac{d\langle x\rangle}{dt}$

Now, $$\frac{d\langle x\rangle}{dt} = \frac{d}{dt}\left[\int_{-\infty}^{\infty}\psi^*(x,t)\,x\,\psi(x,t)\,dx\right] = \int_{-\infty}^{\infty}\psi^*(x,t)\,x\,\frac{\partial\psi(x,t)}{\partial t}\,dx + \int_{-\infty}^{\infty}\frac{\partial\psi^*(x,t)}{\partial t}\,x\,\psi(x,t)\,dx$$

But $\dfrac{\partial\psi(x,t)}{\partial t} = \dfrac{1}{i\hbar}H\psi(x,t)$ where $H = \left[-\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2} + V(x)\right]$

So $$\frac{d\langle x\rangle}{dt} = \frac{1}{i\hbar}\left[\int_{-\infty}^{\infty}\psi^* x(H\psi)\,dx - \int_{-\infty}^{\infty}(H\psi)^* x\,\psi\,dx\right]$$

$$= \frac{1}{i\hbar}\left[\int_{-\infty}^{\infty}\psi^* x\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi\right)dx - \int_{-\infty}^{\infty}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + V\psi^*\right)x\,\psi\,dx\right]$$

$$= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\psi^* x\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi^*}{\partial x^2}x\,\psi\right)dx$$

Now $$\int_{-\infty}^{\infty}\frac{\partial^2\psi^*}{\partial x^2}x\,\psi\,dx = \left[\cancel{\psi x\frac{\partial\psi^*}{\partial x}}\right]_{-\infty}^{\infty} \xrightarrow{} 0 \;-\int_{-\infty}^{\infty}\frac{\partial\psi^*}{\partial x}\frac{\partial}{\partial x}(x\psi)\,dx$$

$$= \left[\cancel{-\psi^*\frac{\partial}{\partial x}(x\psi)}\right]_{-\infty}^{\infty} \xrightarrow{} 0 \;+\int_{-\infty}^{\infty}\psi^*\frac{\partial^2}{\partial x^2}(x\psi)\,dx$$

Thus, $$\frac{d\langle x\rangle}{dt} = \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\psi^* x\frac{\partial^2\psi}{\partial x^2} - \psi^* x\frac{\partial^2\psi}{\partial x^2} - 2\psi^*\frac{\partial\psi}{\partial x}\right)dx$$

$$= -\frac{i\hbar}{m}\int_{-\infty}^{\infty}\psi^*\frac{\partial\psi}{\partial x}\,dx$$

Therefore $$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = -i\hbar\int_{-\infty}^{\infty}\psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t)\,dx$$