Introduction to Quantum Mechanics: Problem Set 1

1.6 Using Planck's spectral distribution law (1.13) for $\rho(\lambda, T)$, prove that the total energy density $\rho_{\text{Tot}} = aT^4$, where $a = 8\pi^5 k^4/15h^3c^3$.

Solution: From Planck's spectral distribution (1.13):

$$\rho(\lambda, T) = \frac{8\pi hc}{\lambda^5}\frac{1}{\exp(hc/\lambda kT) - 1} \tag{1}$$

We can obtain the total energy density as a function of temperature by integrating this expression over all possible wavelengths.

$$\begin{aligned}\rho_{\text{Tot}}(T) &= \int_0^\infty \rho(\lambda, T)\mathrm{d}\lambda \\ &= \int_0^\infty \frac{8\pi hc}{\lambda^5}\frac{1}{\exp(hc/\lambda kT) - 1}\,\mathrm{d}\lambda\end{aligned}$$

In order to evaluate this integral we need to make a change of variables. Define $x = hc/\lambda kT$. Now $\lambda = hc/xkT$ and

$$\mathrm{d}\lambda = \frac{-hc}{x^2kT}\mathrm{d}x$$

Substituting into the integral we obtain

$$\begin{aligned}\rho_{\text{Tot}}(T) &= 8\pi hc \int_\infty^0 (hc/xkT)^{-5}\,\frac{1}{e^x - 1}\left(\frac{-hc}{kTx^2}\right)\mathrm{d}x \\ &= -8\pi hc\left(\frac{kT}{hc}\right)^5\left(\frac{-hc}{kT}\right)\int_0^\infty \frac{x^3}{e^x - 1}\mathrm{d}x \\ &= 8\pi hc\left(\frac{k^4T^4}{h^3c^3}\right)\int_0^\infty \frac{x^3}{e^x - 1}\mathrm{d}x\end{aligned}$$

Note that the bounds of the integral are inverted by the change of variables since when λ goes to ∞, x goes to zero and when λ goes to zero x goes to ∞.

We are given in the problem that

$$\int_0^\infty \frac{x^3}{e^x - 1}\mathrm{d}x = \frac{\pi^4}{15}$$

So we have

$$\begin{aligned}\rho_{\text{Tot}}(T) &= 8\pi hc\left(\frac{k^4T^4}{h^3c^3}\right)\frac{\pi^4}{15}\\ &= \frac{8\pi^5k^4}{15h^3c^3}T^4\end{aligned}$$

As desired.

1.7 The photoelectric work function W for lithium is 2.3 eV.

(a) Find the threshold frequency ν_t and the corresponding threshold wavelength λ_t

(b) If ultraviolet light of wavelength $\lambda = 3000\mathring{A}$ is incident on a lithium surface calculate the maximum kinetic energy of the photoelectrons and the value of the stopping potential V_0.

Solution:

(a) Einstein's application of conservation of energy to the photoelectric effect appears as equation 1.28 in the text,

$$\frac{1}{2}mv_{\text{max}}^2 = h\nu - W$$

By definition the threshold frequency is frequency at which electrons begin to be emitted from the metal, but with zero kinetic energy. It is given by equation 1.29.

$$\begin{aligned}\nu_t &= \frac{W}{h}\\ &= \frac{(2.3eV)\left(1.602\times 10^{-19}J/eV\right)}{6.626^{-34}J\cdot s}\\ &= 5.6\times 10^{14}Hz\end{aligned}$$

Since this is light we're talking about $\lambda = c/\nu$ where c is the speed of light, so

$$\begin{aligned}\lambda_t &= \left(3\times 10^8 m/s\right)/\left(5.56\times 10^{14}Hz\right)\\ &= 5.4\times 10^{-7}m\end{aligned}$$

So $\lambda_t = 540nm$ and $\nu_t = 560THz$. This puts the threshold frequency in the green band of the visible spectrum which is a completely reasonable result.

(b) The kinetic energy at a given frequency is given by equation 1.28. We are given the photon wavelength and can convert to frequency with $\nu = c/\lambda$.

$$\begin{aligned}\nu &= \frac{3\times 10^{8} m/s}{3000\times 10^{-10} m}\\ &= 10^{15} Hz\end{aligned}$$

Plugging this into 1.28:

$$\begin{aligned}\frac{1}{2}mv_{\text{max}}^2 &= h\nu - W\\ &= \left(6.626\times 10^{-34} J\cdot s\right)\left(10^{15} Hz\right) - (2.3 eV)\left(1.602\times 10^{-19} J/eV\right)\\ &= 2.94\times 10^{-19} J\end{aligned}$$

$2.94\times 10^{-19} J/1.602\times 10^{-19} J/eV = 1.84 eV$, so the electrons have 1.84 eV of kinetic energy. Again, this is the right energy scale for interactions with photons in the visible spectrum. If each electron is acquiring 1.84 eV then the stopping potential is $V_0 = E_k/e = 1.84V$.

1.16 Consider the Compton scattering of a photon of wavelength λ_0 by a free electron moving with a momentum of magnitude P in the same direction as that of the incident photon.

(a) Show that in this case the Compton equation (1.42) becomes

$$\Delta\lambda = 2\lambda_0 \frac{(p_0 + P)\,c}{E - Pc}\sin^2\left(\frac{\theta}{2}\right)$$

where $p_0 = h/\lambda_0$ is the magnitude of the incident photon momentum, θ is the photon scattering angle and $E = \left(m^2c^4 + P^2c^2\right)^{1/2}$.

(b) What is the maximum value of the electron momentum after the collision? Compare with the case P=0 considered in the text.

(c) Show that if the free electron initially moves with a momentum of magnitute P in a direction opposite to that of the incident photon, the Compton shift becomes

$$\Delta\lambda = 2\lambda_0 \frac{(p_0 - P)\,c}{E + Pc}\sin^2\left(\frac{\theta}{2}\right)$$

Solution

(a) The Compton equation is a consequence of conservation of momentum and energy in the collision between a photon and an electron.

The electron moves in the same direction as the photon. Define this direction to be the x direction. Before the collision the photon momentum is given by the de Broglie relation $p_0 = h/\lambda_0$ and the electron momentum by P. We'll define the magnitude of the photon momentum after the collision to be h/λ_f and the magnitude of electron momentum after the collision to be P_f. Conservation of momentum in the x direction gives us

$$p_0 + P = \frac{h}{\lambda_f}\cos\theta + P_f\cos\phi$$

Where θ is the direction of the photon relative to the x-axis after the collision and ϕ is the direction of the electron.

In the y direction conservation of momentum gives us

$$0 = \frac{h}{\lambda_f}\sin\theta + P_f\sin\phi$$

Energy is also conserved. Planck's formula tells us that the photon energy is $E_\gamma = hf$, and since we're in free space where the speed of light is $c = \lambda f$, $E_\gamma = hc/\lambda$. The electron's energy is given by the relativistic energy formula $E = (m^2c^4 + P^2c^2)^{1/2}$. Thus

$$\frac{hc}{\lambda_0} + E = \frac{hc}{\lambda_f} + \sqrt{m^2c^4 + P_f^2c^2}$$

I've been careful to leave in those terms that appear in the desired expression like E for the initial electron energy. The goal now is to eliminate those terms that don't appear in that expression, namely ϕ and P_f. I have three equations, so by clever substitution I should be able to eliminate these two variables and be left with an equation in the remaining variables.

I'll start by eliminating ϕ. I notice that ϕ only appears in the momentum equations and that I have one for $\cos\phi$ and one for $\sin\phi$. If I square these and add them together I'll get 1, independent of what ϕ is. Let's try.

$$P_f\cos\phi = (p_0 + P) - \frac{h}{\lambda_f}\cos\theta$$

$$P_f\sin\phi = -\frac{h}{\lambda_f}\sin\theta$$

Squaring, I get

$$
\begin{aligned}
P_f^2 \cos^2 \phi &= \left(p_0 + P - \frac{h}{\lambda_f}\cos\theta\right)^2 \\
&= p_0^2 + P^2 + \frac{h^2}{\lambda_f^2}\cos^2\theta + 2Pp_0 - \frac{2Ph}{\lambda_f}\cos\theta - \frac{2p_0 h}{\lambda_f}\cos\theta
\end{aligned}
$$

$$
P_f^2 \sin^2\theta = \frac{h^2}{\lambda_f^2}\sin^2\theta
$$

And adding them together,

$$
\begin{aligned}
P_f^2 &= p_0^2 + P^2 + \frac{h^2}{\lambda_f^2}\cos^2\theta + 2Pp_0 - \frac{2p_0 h}{\lambda_f}\cos\theta - \frac{2Ph}{\lambda_f}\cos\theta + \frac{h^2}{\lambda_f^2}\sin^2\theta \\
&= p_0^2 + P^2 + \frac{h^2}{\lambda_f^2} + 2Pp_0 - \frac{2\left(p_0 + P\right)h}{\lambda_f}\cos\theta
\end{aligned}
$$

So we've eliminated ϕ; let's try to get rid of P_f. We'll use the conservation of energy equation for this. P_f is under the square-root, so we'll isolate the square root and square both sides of the equation:

$$
\begin{aligned}
\sqrt{m^2c^4 + P_f^2c^2} &= hc\left(\frac{1}{\lambda_0} - \frac{1}{\lambda_f}\right) + E \\
m^2c^4 + P_f^2c^2 &= h^2c^2\left(\frac{1}{\lambda_0} - \frac{1}{\lambda_f}\right)^2 + E^2 + 2Ehc\left(\frac{1}{\lambda_0} - \frac{1}{\lambda_f}\right) \\
P_f^2 &= \frac{h^2}{\lambda_0^2} + \frac{h^2}{\lambda_f^2} - 2\frac{h^2}{\lambda_0\lambda_f} + \frac{E^2}{c^2} + 2\frac{Eh}{\lambda_0 c} - 2\frac{Eh}{\lambda_f c} - m^2c^2
\end{aligned}
$$

Now we can expand out the E^2/c^2 term. $E^2 = m^2c^4 + P^2c^2$, so $E^2/c^2 = m^2c^2 + P^2$. This cancels the $-m^2c^2$.

$$
\begin{aligned}
P_f^2 &= \frac{h^2}{\lambda_0^2} + \frac{h^2}{\lambda_f^2} - 2\frac{h^2}{\lambda_0\lambda_f} + P^2 + 2\frac{Eh}{\lambda_0 c} - 2\frac{Eh}{\lambda_f c} \\
&= \frac{h^2}{\lambda_0^2} + \frac{h^2}{\lambda_f^2} - 2\frac{h^2}{\lambda_0\lambda_f} + P^2 + 2\frac{Eh}{c}\frac{\lambda_f - \lambda_0}{\lambda_f\lambda_0} \\
&= \frac{h^2}{\lambda_0^2} + \frac{h^2}{\lambda_f^2} - 2\frac{h^2}{\lambda_0\lambda_f} + P^2 + 2\frac{Eh}{c}\frac{\Delta\lambda}{\lambda_f\lambda_0}
\end{aligned}
$$

Combining this with the conservation of momentum result gives

$$
p_0^2 + P^2 + \frac{h^2}{\lambda_f^2} + 2Pp_0 - \frac{2\left(p_0 + P\right)h}{\lambda_f}\cos\theta = \frac{h^2}{\lambda_0^2} + \frac{h^2}{\lambda_f^2} - 2\frac{h^2}{\lambda_0\lambda_f} + P^2 + 2\frac{Eh}{c}\frac{\Delta\lambda}{\lambda_f\lambda_0}
$$

$$
p_0^2 + P^2 + \frac{h^2}{\lambda_f^2} + 2Pp_0 - \frac{2\left(p_0 + P\right)h}{\lambda_f}\cos\theta = p_0^2 + \frac{h^2}{\lambda_f^2} - 2\frac{h^2}{\lambda_0\lambda_f} + P^2 + 2\frac{Eh}{c}\frac{\Delta\lambda}{\lambda_f\lambda_0}
$$

Canceling terms gives

$$2Pp_0 - \frac{2(p_0+P)h}{\lambda_f}\cos\theta = -2\frac{h^2}{\lambda_0\lambda_f} + 2\frac{Eh}{c}\frac{\Delta\lambda}{\lambda_f\lambda_0}$$

If we now multiply both sides by $\lambda_f\lambda_0/2h$ we obtain

$$P\lambda_f - (p_0+P)\lambda_0\cos\theta = -h + \frac{E}{c}\Delta\lambda$$

We can now apply the trig identity $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$ to obtain

$$\begin{aligned}
P\lambda_f - (p_0+P)\lambda_0\left(1 - 2\sin^2\frac{\theta}{2}\right) &= -h + \frac{E}{c}\Delta\lambda \\
P(\lambda_f - \lambda_0) - \frac{h}{\lambda_0}\lambda_0 + 2\lambda_0(p_0+P)\sin^2\frac{\theta}{2} &= -h + \frac{E}{c}\Delta\lambda \\
P\Delta\lambda - h + 2\lambda_0(p_0+P)\sin^2\frac{\theta}{2} &= -h + \frac{E}{c}\Delta\lambda \\
\Delta\lambda(E/c - P) &= 2\lambda_0(p_0+P)\sin^2\frac{\theta}{2} \\
\Delta\lambda &= \frac{2\lambda_0(p_0+P)\sin^2\frac{\theta}{2}}{E/c - P} \\
\Delta\lambda &= \frac{2\lambda_0(p_0+P)c\sin^2\frac{\theta}{2}}{E - Pc}
\end{aligned}$$

Which (finally) is the desired result.

(b) To find the maximum value of the electron momentum we need to go back to the equation derived from conservation of momentum:

$$P_f^2 = p_0^2 + P^2 + \frac{h^2}{\lambda_f^2} + 2Pp_0 - \frac{2(p_0+P)h}{\lambda_f}\cos\theta$$

By inspection we can see that this is maximal when $\theta = \pi$ since then the last term will make a maximal positive contribution and all the other terms are positive. This makes sense since if the photon is back-scattered then the electron will get a push in the forward direction by conservation of momentum.

When $P = 0$ the maximum momentum transfer also occurs when $\theta = \pi$, which is qualitatively the same.

(c) Nowhere in the solution to part (a) did we require that P be a positive number. Consequently our derivation works equally well whether the electron and the photon move in the same direction or opposite directions. We don't need to go through the whole derivation again with negative P, we merely need to substitute $P \to (-P)$ into the solution to (a)

$$\begin{aligned}\Delta\lambda &= \frac{2\lambda_0\left(p_0 + (-P)\right) c \sin^2\frac{\theta}{2}}{E - (-P)c} \\ &= \frac{2\lambda_0\left(p_0 - P\right) c \sin^2\frac{\theta}{2}}{E + Pc}\end{aligned}$$

As desired.

1.29 The spacing of the Bragg planes in a NaCl crystal is $d = 2.82\mathring{A}$. Calculate the angular position of the first- and second-order diffraction maximum for 100 eV electrons incident on the crystal surface at the Bragg angle θ_B.

Solution: To find when the Bragg condition for constructive interference is met we must first calculate the wavelength of the electron. The electron rest mass-energy is 511 keV, over 5000 times bigger than the given kinetic energy of 100 eV. We are therefore justified in using the non-relativistic approximation that $E_k = p^2/2m$. That being the case,

$$\begin{aligned}p &= \sqrt{2mE_k} \\ p &= \sqrt{2(9.11\times10^{-31}kg)(100eV)(1.602\times10^{-19}J/eV)} \\ p &= 5.24\times10^{-24}kg\cdot m/s\end{aligned}$$

Dividing this momentum by the electron mass gives a velocity of 5.9×10^6 m/s. This is 2% of the speed of light, which is consistent with our choice to use the non-relativistic approximation.

To find the wavelength we use the de Broglie relation

$$\begin{aligned}\lambda &= \frac{h}{p} &(2)\\ &= \frac{6.626\times10^{-34}J\cdot s}{5.24\times10^{-24}kg\cdot m/s} &(3)\\ &= 1.26\times10^{-10}m &(4)\end{aligned}$$

So the electron wavelength is 1.26 Å. This seems reasonable since atoms are on the order of $1\mathring{A}$ in size. Now using this wavelength we can substitute into the Bragg formula. We want to use the version containing the Bragg angle θ_B, namely

$$2d\sin\theta_B = n\lambda$$

For the first order diffraction we use $n = 1$ and obtain

$$\begin{aligned}2(2.82\mathring{A})\sin\theta_B &= 1\left(1.26\mathring{A}\right) \\ \sin\theta_B &= 0.223 \\ \theta_B &= 12.9^o\end{aligned}$$

The second-order diffraction can be obtained by setting $n = 2$

$$\begin{aligned} 2(2.82\mathring{A})\sin\theta_B &= 2\left(1.26\mathring{A}\right) \\ \sin\theta_B &= 0.447 \\ \theta_B &= 26.5^o \end{aligned}$$

So the angular positions of the first and second order peaks will be 12.9^o and 26.5^o respectively.