Introduction to Quantum Mechanics: Problem Set 4

1. We start from the definitions given in the notes. Given two linear, hermitian operators A and B we define

$$\begin{aligned}\bar{A} &= A - \langle A \rangle \\ \bar{B} &= B - \langle B \rangle \\ C &= \bar{A} + i\lambda\bar{B}\end{aligned}$$

By definition $(\Delta A)^2 = \left\langle \bar{A}^2 \right\rangle$ and $(\Delta B)^2 = \left\langle \bar{B}^2 \right\rangle$. We construct $\left\langle CC^\dagger \right\rangle$.

$$\begin{aligned}\left\langle CC^\dagger \right\rangle &= \langle\psi| CC^\dagger |\psi\rangle \\ &= \left(C^\dagger |\psi\rangle\right) C^\dagger |\psi\rangle\end{aligned}$$

If we take the hermitian transpose of this and apply rule (5.36) for transposition, we get

$$\left[\left(C^\dagger |\psi\rangle\right) C^\dagger |\psi\rangle\right]^\dagger = \left(C^\dagger |\psi\rangle\right) C^\dagger |\psi\rangle$$

So $\left\langle CC^\dagger \right\rangle$ is equal to its hermitian transpose which means that it is real. Further, since it is the inner product of the vector $C^\dagger |\psi\rangle$ with *itself*, it follows that $\left\langle CC^\dagger \right\rangle$ is non-negative. Therefore $\left\langle CC^\dagger \right\rangle \geq 0$. Expanding $\left\langle CC^\dagger \right\rangle$ in terms of $\bar{A}$ and $\bar{B}$ gives:

$$\begin{aligned}\left\langle CC^\dagger \right\rangle &= \left\langle \left(\bar{A} + i\lambda\bar{B}\right)\left(\bar{A} - i\lambda\bar{B}\right) \right\rangle \\ &= \left\langle \left(\bar{A}\bar{A} + \lambda^2\bar{B}\bar{B} - i\lambda\bar{A}\bar{B} + i\lambda\bar{B}\bar{A}\right) \right\rangle \\ &= \left\langle \bar{A}\bar{A} \right\rangle + \lambda^2 \left\langle \bar{B}\bar{B} \right\rangle - i\lambda \left\langle \left[\bar{A}, \bar{B}\right] \right\rangle \\ &= (\Delta A)^2 + \lambda^2 (\Delta B)^2 - i\lambda \left\langle \left[\bar{A}, \bar{B}\right] \right\rangle\end{aligned}$$

Since we know that the whole quantity is real and the first two terms are real (they are expectation values of hermitian operators), it follows that the last term must also be real, meaning that $\left\langle \left[\bar{A}, \bar{B}\right] \right\rangle$ is purely imaginary.

Now $\left\langle CC^\dagger \right\rangle$ is a real quadratic function of λ. A quadratic function $a\lambda^2 + b\lambda + c$ has a single extremum at $\lambda = -b/2a$. Since we have already shown that $\left\langle CC^\dagger \right\rangle$ is everywhere positive, this extremum must be a minimum (since if a quadratic function has a maximum it extends to $-\infty$). Thus $\left\langle CC^\dagger \right\rangle$ has a minimum at

$$\lambda = \frac{i}{2} \frac{\langle [A, B] \rangle}{(\Delta B^2)}$$

Furthermore, since $\left\langle CC^\dagger \right\rangle$ is positive everywhere, even at the minimum value $\left\langle CC^\dagger \right\rangle$ must be non-zero. Plugging in the minimizing value of lambda and applying this inequality we have that

$$\langle CC\dagger\rangle_{\min} = (\Delta A)^2 + \left(\frac{i}{2}\frac{\langle [A,B]\rangle}{(\Delta B^2)}\right)^2 (\Delta B)^2 - i\frac{i}{2}\frac{\langle [A,B]\rangle}{(\Delta B^2)}\left\langle [\bar{A},\bar{B}]\right\rangle \geq 0$$

From which it follows that

$$(\Delta A)^2 - \frac{1}{4}\frac{\langle [A,B]\rangle^2}{(\Delta B^2)} + \frac{1}{2}\frac{\langle [A,B]\rangle^2}{(\Delta B^2)} \geq 0$$
$$(\Delta A)^2 (\Delta B)^2 + \frac{1}{4}\langle [A,B]\rangle^2 \geq 0$$

but since $\langle [A,B]\rangle$ is actually imaginary, $\langle [A,B]\rangle^2$ is negative. It's less confusing to write it as $-\left|\langle [A,B]\rangle\right|^2$ which then gives:

$$(\Delta A)^2 (\Delta B)^2 - \frac{1}{4}\left|\langle [A,B]\rangle\right|^2 \geq 0$$
$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4}\left|\langle [A,B]\rangle\right|^2$$
$$(\Delta A)(\Delta B) \geq \frac{1}{2}\left|\langle [A,B]\rangle\right|$$

For the specific case where $A = x$ and $B = p$, we know that $[x,p] = i\hbar$ so that $|\langle [A,B]\rangle| = \hbar$ and $\Delta x \Delta p = \hbar/2$. We can construct the operator C as

$$C = (x - \langle x\rangle) + i\lambda (p - \langle p\rangle)$$

It is obvious from the previous discussion that the minimum value of $\langle\psi| CC^\dagger |\psi\rangle$ is zero and it will occur when

$$\lambda = \frac{i}{2}\frac{\langle [p,x]\rangle}{(\Delta p)^2} = \frac{i}{2}\frac{-i\hbar}{(\Delta p)^2} = \frac{\hbar}{2(\Delta p)^2}$$

so that $\left\langle CC^\dagger \right\rangle$ is minimized when

$$C = (x - \langle x\rangle) + i\frac{\hbar}{2(\Delta p)^2}(p - \langle p\rangle)$$

and when this occurs $C^\dagger |\psi\rangle = 0$. In the x-basis this looks like

$$(x - \langle x\rangle)\psi(x) + i\frac{\hbar}{2(\Delta p)^2}\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x} - \langle p\rangle\right)\psi(x) = 0$$

$$i\hbar\left(i\hbar\frac{\mathrm{d}}{\mathrm{d}x}+\langle p\rangle\right)\psi(x)=2\left(\Delta p\right)^2\left(x-\langle x\rangle\right)\psi(x)$$
$$\left(-\hbar^2\frac{\mathrm{d}}{\mathrm{d}x}+i\hbar\langle p\rangle\right)\psi(x)=2\left(\Delta p\right)^2\left(x-\langle x\rangle\right)\psi(x)$$

Or, slightly rewritten,

$$\hbar^2\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)+2\left(\Delta p\right)^2x\psi(x)+\left(-i\hbar\langle p\rangle-2\left(\Delta p\right)^2\langle x\rangle\right)\psi(x)=0$$

This is a first-order differential equation for $\psi(x)$ of the form $\psi'(x)+(ax-b)\psi(x)=0$ where a and b are two constants. Such equations can be solved by integration. Let $a=2\Delta p^2/\hbar^2$ and let $b=i\langle p\rangle/\hbar+2\Delta p^2\langle x\rangle/\hbar^2$ then

$$\psi'(x)+(ax-b)\psi(x)=0$$
$$\frac{\psi'(x)}{\psi(x)}=-ax+b$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi(x)=-ax+b$$

Taking the antiderivative of both sides gives

$$\ln\psi(x)=-\frac{1}{2}ax^2+bx$$
$$\psi(x)=\exp\left[-\frac{1}{2}ax^2+bx\right]$$

Subsituting in for a and b we get

$$\begin{aligned}\psi(x)&=C\exp\left[i\frac{\langle p\rangle}{\hbar}x+\frac{2\Delta p^2\langle x\rangle x}{\hbar^2}-\frac{\Delta p^2}{\hbar^2}x^2\right]\\&=C\exp\left[i\frac{\langle p\rangle}{\hbar}x\right]\exp\left[-\frac{\Delta p^2}{\hbar^2}\left(x-\langle x\rangle\right)^2\right]\exp\left[\frac{\Delta p^2}{\hbar^2}\langle x\rangle^2\right]\\&=C'\exp\left[i\frac{\langle p\rangle}{\hbar}x\right]\exp\left[-\frac{\Delta p^2}{\hbar^2}\left(x-\langle x\rangle\right)^2\right]\end{aligned}$$

In the last step the constant exponential obtained by completing the square was rolled into the normalization constant. This is the desired result, so we're done.

2. (Bransden and Joachain 5.11)

The Hamiltonian operator H for a certain physcial system is represented by the matrix

$$\mathbf{H}=\hbar\omega\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}$$

while two other observables A and B are represented by the matrices

$$\mathbf{A} = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2\mu & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \mu & 0 \end{pmatrix}$$

where λ and μ are real (non-zero) numbers.

(a) Find the eigenvalues and eigenvectors of $\mathbf{A}$ and $\mathbf{B}$.

(b) If the system is in a state described by the state vector

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$

where c_1, c_2 and c_3 are complex constants and

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(i) find the relationship between c_1, c_2 and c_3 such that $\mathbf{u}$ is normalised to unity; and

(ii) find the expectation values of H, A and B

(iii) What are the possible values of the energy that can be obtained in a measurement when the system is described by the state vector $\mathbf{u}$? For each possible result find the wave function in the matrix representation immediately after the measurement.

Solution:

(a) The n^{th} eigenvector $\mathbf{v}_n$ of $\mathbf{A}$ will have the property that

$$\mathbf{A}\mathbf{v}_n = a_n\mathbf{v}_n \tag{1}$$

where a_n is the corresponding eigenvalue. By inspection we can see that $\mathbf{H}$, $\mathbf{A}$ and $\mathbf{B}$ are all symmetric matrices so we expect both their eigenvalues and their eigenvectors to be real. To find the eigenvectors we must find solutions to equation (1), or, equivalently to:

$$\mathbf{A} - a_n\mathbb{I}\mathbf{v}_n = 0$$

Now this has an obvious trivial solution, namely $\mathbf{v}_n = \mathbf{0}$, but that's not very interesting. To have non-trivial eigenvectors the equation system must be *degenerate* in which case there will be an infinite number of solution instead of just one. The equation system can only be degenerate if the determinant of the matrix $(\mathbf{A} - a_n\mathbb{I})$ vanishes.

$$
\begin{aligned}
|\mathbf{A} - a_n\mathbb{I}| &= 0 \\
\begin{vmatrix} -a_n & \lambda & 0 \\ \lambda & -a_n & 0 \\ 0 & 0 & 2\lambda - a_n \end{vmatrix} &= 0 \\
(2\lambda - a_n)\left(a_n^2 - \lambda^2\right)^2 &= 0 \\
(2\lambda - a_n)(a_n - \lambda)(a_n + \lambda) &= 0
\end{aligned}
$$

Thus the determinant will vanish if and only if a_n is either λ, $-\lambda$ or 2λ, so these are the eigenvalues. We can find the corresponding eigenvectors by solving equation (1).

$$
\begin{aligned}
\mathbf{A}\mathbf{v}_n &= a_n\mathbf{v}_n \\
\begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} a_n v_1 \\ a_n v_2 \\ a_n v_3 \end{pmatrix}
\end{aligned}
$$

Which is really just three coupled linear equations in the components of $\mathbf{v}$, namely

$$
\begin{aligned}
\lambda v_2 &= a_n v_1 \\
\lambda v_1 &= a_n v_2 \\
2\lambda v_3 &= a_n v_3
\end{aligned}
$$

When $a_n = \lambda$ this is

$$
\begin{aligned}
\lambda v_2 &= \lambda v_1 \\
\lambda v_1 &= \lambda v_2 \\
2\lambda v_3 &= \lambda v_3
\end{aligned}
$$

or

$$
\begin{aligned}
v_2 &= v_1 \\
v_1 &= v_2 \\
2v_3 &= v_3
\end{aligned}
$$

We can see immediately that $v_3 = 0$, but the equations in v_1 and v_2 are degenerate (they're supposed to be — that's what a zero determinant means). Set $v_1 = 1$ then $v_2 = 1$ and $v_3 = 0$.

Now since this is a quantum state we should be normalizing the vector to unity. $1^2 + 1^2 + 0^2 = 2$, so the normalized vector is $\mathbf{v}_1 = 1/\sqrt{2}\,(1,1,0)$. Similarly if we use the $-\lambda$ eigenvalue we get the eigenvector $\mathbf{v}_2 = 1/\sqrt{2}\,(1,-1,0)$ and with the $a_3 = 2\lambda$ eigenvalue we get $\mathbf{v}_3 = (0,0,1)$.

The eigenvalues and eigenvectors for $\mathbf{B}$ can be found in the same way. We solve the characteristic equation

$$
\begin{aligned}
|\mathbf{B} - b_n \mathbb{I}| &= 0 \\
\begin{vmatrix} 2\mu - b_n & 0 & 0 \\ 0 & -b_n & \mu \\ 0 & \mu & -b_n \end{vmatrix} &= 0 \\
(2\mu - b_n)\left(b_n^2 - \mu^2\right) &= 0 \\
(2\mu - b_n)(b_n - \mu)(b_n + \mu) &= 0
\end{aligned}
$$

Which give eigenvalues $b_1 = 2\mu$,$b_2 = \mu$ and $b_3 = -\mu$. The eigenvectors can be found by solving

$$
\begin{aligned}
2\mu v_1 &= b_n v_1 \\
\mu v_3 &= b_n v_2 \\
\mu v_2 &= b_n v_3
\end{aligned}
$$

Which give the normalized eigenvectors $\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = 1/\sqrt{2}\,(0,1,1)$ and $\mathbf{v}_3 = 1/\sqrt{2}\,(0,1,-1)$. In summary:

Matrix $\mathbf{A}$	
Eigenvalue	Eigenvector
λ	$1/\sqrt{2}\,(1,1,0)$
$-\lambda$	$1/\sqrt{2}\,(1,-1,0)$
2λ	$(0,0,1)$
Matrix $\mathbf{B}$	
Eigenvalue	Eigenvector
2μ	$(1,0,0)$
μ	$1/\sqrt{2}\,(0,1,1)$
$-\mu$	$1/\sqrt{2}\,(0,1,-1)$

(2) (i) By definition $\mathbf{u}$ is normalized if $\mathbf{u}^\dagger \cdot \mathbf{u} = 1$.

$$
\begin{aligned}
\mathbf{u}^\dagger \cdot \mathbf{u} &= 1 \\
\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} &= 1 \\
|c_1|^2 + |c_2|^2 + |c_3|^2 &= 1
\end{aligned}
$$

So $\mathbf{u}$ is normalized if $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$.

(ii) The expectation value of an operator $\mathbf{A}$ for a system in state $\mathbf{u}$ is defined to be

$$\langle \mathbf{A} \rangle = \mathbf{u}^\dagger \mathbf{A} \mathbf{u}$$

The expectation values can be found directly then as

$$\begin{aligned}
\langle \mathbf{A} \rangle &= \mathbf{u}^\dagger \mathbf{A} \mathbf{u} \\
&= \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
&= \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} \lambda c_2 \\ \lambda c_1 \\ 2\lambda c_3 \end{pmatrix} \\
&= \lambda c_1^* c_2 + \lambda c_2^* c_1 + 2\lambda c_3^* c_3 \\
&= 2\lambda \left(\Re \left\{ c_1^* c_2 \right\} + |c_3|^2 \right)
\end{aligned}$$

Notice that the expectation value is real. This has to be the case for Hermitian operators.

$$\begin{aligned}
\langle \mathbf{B} \rangle &= \mathbf{u}^\dagger \mathbf{B} \mathbf{u} \\
&= \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} 2\mu & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \mu & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
&= \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} 2\mu c_1 \\ \mu c_3 \\ \mu c_2 \end{pmatrix} \\
&= 2\mu c_1^* c_1 + \mu c_2^* c_3 + \mu c_3^* c_2 \\
&= 2\mu \left(\Re \left\{ c_2^* c_3 \right\} + |c_1|^2 \right)
\end{aligned}$$

Again, this is manifestly real. Finally we'll do the Hamiltonian:

$$\begin{aligned}
\langle \mathbf{H} \rangle &= \mathbf{u}^\dagger \mathbf{H} \mathbf{u} \\
&= \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
&= \hbar\omega \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} c_1 \\ 2c_2 \\ 2c_3 \end{pmatrix} \\
&= \hbar\omega c_1^* c_1 + 2\hbar\omega c_2^* c_2 + 2\hbar\omega c_3^* c_3 \\
&= \left(\hbar\omega |c_1|^2 + 2\hbar\omega |c_2|^2 + 2\hbar\omega |c_3|^2 \right)
\end{aligned}$$

(iii) According to postulate 4 "The only result of a precise measurement of the dynamical variable A is one of the eigenvalues a_n of the linear operator A associated with A." It follows that for any state the only possible values of the energy that can be measured are the eigenvalues of $\mathbf{H}$. Further, for the specific state $\mathbf{u}$ a given energy eigenvalue can only be measured if there is a non-zero probability of finding the state to be in the corresponding eigenvector. As it happens, the vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are the eigenvectors of $\mathbf{H}$.

$$|\mathbf{A} - a_n \mathbb{I}| = 0$$

$$\begin{vmatrix} \hbar\omega - E_n & 0 & 0 \\ 0 & 2\hbar\omega - E_n & 0 \\ 0 & 0 & 2\hbar\omega - E_n \end{vmatrix} = 0$$

$$(\hbar\omega - E_n)(2\hbar\omega - E_n)(2\hbar\omega - E_n) = 0$$

Which give eigenvalues $E_1 = \hbar\omega$,$E_2 = 2\hbar\omega$ and $E_3 = 2\hbar\omega$. The eigenvectors can be found by solving

$$\begin{aligned} \hbar\omega v_1 &= E_n v_1 \\ 2\hbar\omega v_2 &= E_n v_2 \\ 2\hbar\omega v_3 &= E_n v_3 \end{aligned}$$

This can be solved to give $\mathbf{v}_1 = (1, 0, 0)$. The second eigenvector is degenerate, so any vector orthogonal to $\mathbf{v}_1$ will be an eigenvector with eigenvalue $2\hbar\omega$ it's convenient to pick $\mathbf{v}_2 = (0, 1, 0)$ and $\mathbf{v}_3 = (0, 0, 1)$.
So the possible outcomes of an energy measurement are $\hbar\omega$, so long as c_1 is non-zero and $2\hbar\omega$ as long as either c_2 or c_3 are non-zero.
Immediately after a measurement that finds the energy of the system to be E_n the system will be in the corresponding eigenstate. If the energy is measured to be $\hbar\omega$ then the state of the system will be $\mathbf{u}_1 = (1, 0, 0)$. If the energy is measured to be $2\hbar\omega$ then we exclude the possibility of the state being in $\mathbf{u}_1 = (1, 0, 0)$, but we have obtained no information about whether the state is $\mathbf{v}_2 = (0, 1, 0)$ or $\mathbf{u}_2 = (0, 0, 1)$. If a measurement yields no information then there is no collapse, so the final state of the system is $(c_2\mathbf{u}_2 + c_3\mathbf{u}_3)/\sqrt{|c_2|^2 + |c_3|^2}$ where we have *renormalized* the state by dividing by the new norm of the vector.

3. (Bransden and Joachain 5.12)

Prove the commutation relations

(i)

$$[a_-, a_+] = 1$$

and

(ii)

$$[H, a_\pm] = \pm\hbar\omega a_\pm$$

Solution:

(i) $a_\pm$ are defined in equation (5.189) of the book as

$$a_\pm = \frac{1}{\sqrt{2}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} \mp i\frac{\hat{p_x}}{(m\omega\hbar)^{1/2}}\right]$$

Using this definition we can rewrite the commutator as:

$$[a_-, a_+] = \frac{1}{\sqrt{2}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} + i\frac{\hat{p_x}}{(m\omega\hbar)^{1/2}}\right]\frac{1}{\sqrt{2}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} - i\frac{\hat{p_x}}{(m\omega\hbar)^{1/2}}\right] - \frac{1}{\sqrt{2}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} - i\frac{\hat{p_x}}{(m\omega\hbar)^{1/2}}\right]\frac{1}{\sqrt{2}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} + i\frac{\hat{p_x}}{(m\omega\hbar)^{1/2}}\right]$$

$$\begin{aligned}
[a_-, a_+] &= \frac{1}{2}\left[\left(\frac{m\omega}{\hbar}\right)\hat{x}^2 + \frac{\hat{p_x}^2}{(m\omega\hbar)} + i\frac{1}{(m\omega\hbar)^{1/2}}\left(\frac{m\omega}{\hbar}\right)^{1/2}(-\hat{x}\hat{p_x} + \hat{p_x}\hat{x})\right] - \\
&\quad \frac{1}{2}\left[\left(\frac{m\omega}{\hbar}\right)\hat{x}^2 + \frac{\hat{p_x}^2}{(m\omega\hbar)} + i\frac{1}{(m\omega\hbar)^{1/2}}\left(\frac{m\omega}{\hbar}\right)^{1/2}(-\hat{p_x}\hat{x} + \hat{x}\hat{p_x})\right] \\
&= \frac{1}{2}\left(\frac{i}{\hbar}(\hat{p_x}\hat{x} - \hat{x}\hat{p_x} - \hat{x}\hat{p_x} + \hat{p_x}\hat{x})\right) \\
&= \frac{i}{\hbar}(\hat{p_x}\hat{x} - \hat{x}\hat{p_x})
\end{aligned}$$

But the quantity in brackets is just the negative of the commutator of $\hat{x}$ and $\hat{p}$, and we know from (3.77) that $[\hat{x}, \hat{p}] = i\hbar$. So

$$\begin{aligned}
[a_-, a_+] &= \frac{i}{\hbar}(-i\hbar) \\
&= 1
\end{aligned}$$

which is the desired result.

(ii) The simple harmonic oscillator Hamiltonian is defined in (5.188) to be

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

so the commutator $[H, a_\pm]$ is

$$\begin{aligned}
[H, a_\pm] &= \left(\frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2\right)\frac{1}{\sqrt{2}}\left(\left(\frac{m\omega}{\hbar}\right)^{1/2} x \mp i\frac{p_x}{(m\hbar\omega)^{1/2}}\right) \\
&- \frac{1}{\sqrt{2}}\left(\left(\frac{m\omega}{\hbar}\right)^{1/2} x \mp i\frac{p_x}{(m\hbar\omega)^{1/2}}\right)\left(\frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{\omega^{1/2}}{2m^{1/2}\hbar^{1/2}}p_x^2 x + \frac{m^{3/2}\omega^{5/2}}{2\hbar^{1/2}}x^3 \mp i\frac{m^{1/2}\omega^{3/2}}{2\hbar^{1/2}}x^2 p_x \mp \frac{i}{2m^{3/2}\hbar^{1/2}\omega^{1/2}}p_x^3\right) \\
&- \frac{1}{\sqrt{2}}\left(\frac{\omega^{1/2}}{2m^{1/2}\hbar^{1/2}}x p_x^2 + \frac{m^{3/2}\omega^{5/2}}{2\hbar^{1/2}}x^3 \mp i\frac{m^{1/2}\omega^{3/2}}{2\hbar^{1/2}}p_x x^2 \mp \frac{i}{2m^{3/2}\hbar^{1/2}\omega^{1/2}}p_x^3\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{\omega^{1/2}}{2m^{1/2}\hbar^{1/2}}\left(p_x^2 x - x p_x^2\right) \mp i\frac{m^{1/2}\omega^{3/2}}{2\hbar^{1/2}}\left(x^2 p_x - p_x x^2\right)\right)
\end{aligned}$$

Now we can simplify the operators using the commutation relation between $x$and p_x, $[x, p_x] = i\hbar$.

$$\begin{aligned}
p_x^2 x - x p_x^2 &= p_x\left(p_x x\right) - \left(x p_x\right) p_x \\
&= p_x\left(x p_x - [x, p_x]\right) - \left([x, p_x] + p_x x\right) p_x \\
&= p_x\left(x p_x - i\hbar\right) - \left(i\hbar + p_x x\right) p_x \\
&= -2i\hbar p_x
\end{aligned}$$

Similarly,

$$\begin{aligned}
x^2 p_x - p_x x^2 &= x\left(x p_x\right) - \left(p_x x\right) x \\
&= x\left([x, p_x] + p_x x\right) - \left(x p_x - [x, p_x]\right) x \\
&= x\left(i\hbar + p_x x\right) - \left(x p_x - i\hbar\right) x \\
&= 2i\hbar x
\end{aligned}$$

so

$$
\begin{aligned}
[H, a_\pm] &= \frac{1}{\sqrt{2}} \left(\frac{\omega^{1/2}}{2m^{1/2}\hbar^{1/2}} \left(-2i\hbar p_x\right) \mp i \frac{m^{1/2}\omega^{3/2}}{2\hbar^{1/2}} \left(2i\hbar x\right) \right) \\
&= \hbar\omega \left(\frac{1}{\sqrt{2}} \left(\frac{-i}{m^{1/2}\hbar^{1/2}\omega^{1/2}} p_x \mp i^2 \frac{m^{1/2}\omega^{1/2}}{\hbar^{1/2}} x \right) \right) \\
&= -\hbar\omega \left(\frac{1}{\sqrt{2}} \left(\frac{i}{m^{1/2}\hbar^{1/2}\omega^{1/2}} p_x \pm (-1) \frac{m^{1/2}\omega^{1/2}}{\hbar^{1/2}} x \right) \right) \\
&= -\hbar\omega \left(\frac{1}{\sqrt{2}} \left(\frac{i}{m^{1/2}\hbar^{1/2}\omega^{1/2}} p_x \mp \frac{m^{1/2}\omega^{1/2}}{\hbar^{1/2}} x \right) \right) \\
&= - \mp \hbar\omega \left(\frac{1}{\sqrt{2}} \left(\mp \frac{i}{m^{1/2}\hbar^{1/2}\omega^{1/2}} p_x + \frac{m^{1/2}\omega^{1/2}}{\hbar^{1/2}} x \right) \right) \\
&= \pm\hbar\omega \left(\frac{1}{\sqrt{2}} \left(\left(\frac{m\omega}{\hbar} \right)^{1/2} x \mp \frac{i}{(m\hbar\omega)^{1/2}} p_x \right) \right) \\
&= \pm\hbar\omega a_\pm
\end{aligned}
$$

As desired.