Introduction to Quantum Mechanics: Problem Set 5

1. (Textbook 6.13) Consider a free particle of mass μ constrained to move on a ring of radius a.

   (a) Show that the Hamiltonian of this system is

   $$H = L_z^2/2I$$

   where the z-axis is through the center O of the ring and is perpendicular to its plane, and I is the moment of inertia of the particle with respect to the center O.

   (b) Find the energy eigenfunctions for the system and write down a general expression for the solution of the time-dependent Schrodinger equation.

   Solution

   (a) First, we note that the moment of inertia of a particle of mass μ, with respect to a point O, is given by $I = \mu r^2$ where r is the distance from the particle to O. In our case, r is fixed to be the radius of the circle, so $I = \mu a^2$. We will now solve the problem in two different ways.

   1) Since we are dealing with a free particle, the energy is given by the kinetic term only: $E = \mu v^2/2$. The z-component of the angular momentum is $L_z = (\mathbf{r} \times \mathbf{p}) \cdot \hat{\mathbf{z}} = \mu(\mathbf{r} \times \mathbf{v}) \cdot \hat{\mathbf{z}}$. But, for motion in a circle in the x-y plane, $\mathbf{v}$ and $\mathbf{r}$ are perpendicular and their cross product points in the $\hat{\mathbf{z}}$ direction. Also, the magnitude of $\mathbf{r}$ is fixed to be a. Thus, we find that $L_z = \mu|\mathbf{r} \times \mathbf{v}| = \mu v a$, and we can express the energy as

   $$\begin{aligned} E &= \mu v^2/2 \\ &= \mu L_z^2/2\mu^2 a^2 \\ &= L_z^2/2\mu a^2 \\ &= L_z^2/2I. \end{aligned}$$

   So our Hamiltonian operator in the Schrodinger equation will be $\hat{H} = \hat{L}_z^2/2I$.

2) We note that since the particle is constrained to a circle, this is just a **one dimensional** free particle problem. Let x be the distance traveled on the ring (arc length) from some point, which we will denote by $x = 0$. x can take any value between 0 and $2\pi a$, but since 0 and $2\pi a$ are the same point, we demand that $\psi(0) = \psi(2\pi a)$ for any function $\psi(x)$. The hamiltonian for a 1-D free particle of mass μ is given by

$$\hat{H} = \frac{\hat{p}^2}{2\mu} = -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}$$

in the x basis. Now, change coordinates from the length x to the angle ϕ using $x = a\phi$. Then $\frac{d^2}{dx^2} = \frac{1}{a^2}\frac{d^2}{d\phi^2}$, so

$$\hat{H} = -\frac{\hbar^2}{2\mu a^2}\frac{d^2}{d\phi^2} = \frac{\hat{L}_z^2}{2I},$$

where we've made use of $\hat{L}_z = -i\hbar\frac{d}{d\phi}$.

(b) Since $\hat{H}$ is just a function of $\hat{L}_z$, the eigenfunctions $\hat{H}$ and $\hat{L}_z$ will be the same. The eigenfunctions $\Phi_m(\phi)$ of $\hat{L}_z$ are those that satisfy the equation

$$\hat{L}_z\Phi_m(\phi) = -i\hbar\frac{d\Phi_m(\phi)}{d\phi} = m\hbar\Phi_m(\phi).$$

where we have put in the factor of $\hbar$ explicitly in the eigenvalue, anticipating our result. The solution to this simple differential equation is

$$\Phi_m(\phi) = e^{im\phi},$$

for any value of m. However, we have not yet used the boundary conditions (i.e. conditions at the 'endpoints' 0 and 2π). Since this is a ring, we need $\Phi_m(0) = \Phi_m(2\pi)$, as $\phi = 0$ and $\phi = 2\pi$ refer to the same point. So we have

$$\Phi_m(0) = 1 = e^{2im\pi} = \Phi_m(2\pi),$$

which can only be satisfied if m is an integer. The energy eigenvalues are

$$\hat{H}\Phi_m(\phi) = \frac{\hat{L}_z^2}{2I}\Phi_m(\phi) = \frac{(m\hbar)^2}{2I}\Phi_m(\phi) = E_m\Phi_m(\phi).$$

Any general wavefunction $\psi(\phi)$ will be expressed as a linear combination of the eigenfunctions

$$\psi(\phi) = \sum_{m=-\infty}^{m=\infty} a_m \Phi_m(\phi),$$

and the time evolution will add a phase of $\exp(-iE_m t/\hbar)$ to each $\Phi_m(\phi)$

$$\psi(\phi, t) = \sum_{m=-\infty}^{m=\infty} a_m e^{-iE_m t/\hbar} \Phi_m(\phi) = \sum_{m=-\infty}^{m=\infty} a_m e^{im(\phi - m\hbar t/2I)}$$

Note that the a_m are just the coefficients of the fourier series of $\psi(\phi, 0)$.

2. A particle in a spherically symmetric potential is in the following state:

$$\psi(x, y, z) = C(xy + yz + zx)e^{-\alpha r^2}$$

(a) What is the probability that a measurement of L^2 will give

i) 0 ?
ii) $6\hbar^2$?

(b) If we find $l = 2$, what are the probabilities of finding $m = 2, 1, 0, -1, -2$?

Solution

(a) This part of the question can be solved without doing any math. We know that $\hat{L}^2$ acts only on the angular part of the wavefunction, so we only look at terms in θ and ϕ. What we are going to do is match up these terms with the spherical harmonics $Y_{l,m}$. We know that in spherical coordinates, we have

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta. \end{aligned}$$

This means that the xy term will be of the form $\sin^2\theta$(ϕ terms), and the yz and xz terms will be of the form $\sin\theta\cos\theta$(ϕ terms). But, comparing with the expressions for $Y_{l,m}$ (see page 285 in the textbook or this chart), we see that the only $Y_{l,m}$ of these forms are $Y_{2,\pm2}$ (for $\sin^2\theta$) and $Y_{2,\pm1}$ (for $\sin\theta\cos\theta$). So l is *always* 2, and $\hat{L}^2\psi(x, y, z)$ will give back $6\hbar^2\psi(x, y, z)$. Thus we have a 0 percent chance of measuring $L^2 = 0$ and a 100 percent chance of measuring $L^2 = 6\hbar^2$.

(b) For this part of the problem, we will have to write out everything in terms of the $Y_{l,m}$. Using

$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2} \; ; \; \sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i},$$

we have, looking at just the θ and ϕ terms

$$\frac{\psi(r,\theta,\phi)}{Cr^2e^{-\alpha r^2}} = \sin^2\theta\frac{e^{2i\phi}-e^{-2i\phi}}{4i} + \sin\theta\cos\theta\left(\frac{e^{i\phi}+e^{-i\phi}}{2}+\frac{e^{i\phi}-e^{-i\phi}}{2i}\right).$$

Note that we used $2\cos\phi\sin\phi = \sin 2\phi$ in the expression above. These are almost in the form of spherical harmonics. What we are missing are the normalizations. Putting them in, and using $1/i = -i$, we find that

$$\psi(r,\theta,\phi) = C\sqrt{\frac{2\pi}{15}}r^2e^{-\alpha r^2}\left(iY_{2,-2} - iY_{2,2} + (i-1)Y_{2,1} + (i+1)Y_{2,-1}\right).$$

First, we note that there is no $m = 0$ component, so the probability of measuring $m = 0$ is 0. For the rest, we can compare the square of the coefficients of the $Y_{l,m}$'s to see their relative probabilities. $|i|^2 = |-i|^2 = 1$ and $|i-1|^2 = |i+1|^2 = 2$, so we see that the probability of measuring $m = 2$ is the same as that of measuring $m = -2$, and similarly for $m = 1$ and $m = -1$. We also see that the probability of measuring $m = \pm 1$ is twice as much as the probability of measuring $m = \pm 2$. Since all the probabilities have to add to one, we see that the probability P_m of measuring m is $1/6$ for $m = \pm 2$, $2/6$ for $m = \pm 1$ and 0 for $m = 0$. Note we did not have to find C, because the radial part and angular part of ψ are normalized separately.

3. A system with orbital momentum $l = 1$ is in the following initial state:

$$|\psi\rangle = \frac{1}{\sqrt{14}}\begin{pmatrix} -\sqrt{3} \\ 2\sqrt{2} \\ \sqrt{3} \end{pmatrix}$$

(a) Calculate $\langle L_y\rangle$ if the system is in $l_x = -\hbar$

(b) Calculate $\langle L_y^2\rangle$ if the system is in $l_x = -\hbar$

(c) Calculate ΔL_y if the system is in $l_x = -\hbar$

(d) If we measure L_y with the system in the initial state above, what values will we obtain and with what probabilities?

NOTE: Some of you may have intepreted questions a) through c) as finding either i) $\langle L_z\rangle$, etc. if the system is in $l_y = -\hbar$ or ii) $\langle L_y\rangle$, etc. if the system is in $l_y = -\hbar$. In case i), the answers are the same as those given below. In case ii), we are in an eigenstate of L_y, so the expectation value is just the eigenvalue $-\hbar$, the expectation value of L_y^2 is its square, $\hbar^2$, and $\Delta L_y = 0$. If you chose any of these interpretations, you'll be graded based on the pertinent answers.

Solution

It is useful to start out by writing out the matrix representation for $\hat{L}_x$ and $\hat{L}_y$ in the $\hat{L}_z$ basis:

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \hat{L}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

(a) To find the eigenstate $|\phi\rangle$ with $l_x = -\hbar$ in our basis, we use $\hat{L}_x|\phi\rangle = -\hbar|\phi\rangle$, or

$$\hat{L}_x + \hbar\hat{I}|\phi\rangle = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} |\phi\rangle = 0.$$

Just by looking at the top and bottom row, we can see that the first and third components of $|\phi\rangle$ must be equal. Then, from the top row, the second component must be $-\sqrt{2}$ times the first. Normalizing, we get

$$|\phi\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

Now, all that is left is matrix multiplication:

$$\langle\phi|\hat{L}_y|\phi\rangle = \frac{1}{2}(1, \ -\sqrt{2}, \ 1)\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = 0.$$

Note this is the same result we got for $\langle\hat{L}_z\rangle$. This is actually to be expected, since what we call the y axis could easily be relabeled as the z axis (we can do this by rotating about the x axis), without changing the problem (under rotations about the x-axis, eigenstates of L_x pick up a phase). In fact, you'll see we get the same answers for (b) and (c) as well.

(b) First we calculate $\hat{L}_y^2$ in the matrix representation.

$$\hat{L}_y^2 = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Then, we have, as promised,

$$\langle\phi|\hat{L}_y^2|\phi\rangle = \frac{1}{2}(1, \ -\sqrt{2}, \ 1)\frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \frac{\hbar^2}{2}.$$

(c) $\Delta L_y = \sqrt{\langle L_y^2 \rangle - \langle L_y \rangle^2} = \hbar/\sqrt{2}$

(d) To do this problem, we must expand $|\psi\rangle$ in terms of the eigenvectors of $\hat{L}_y$. We know that the eigenvalues of $\hat{L}_y$ will be $\hbar, 0$ and $-\hbar$ (eigenvalues are measurable quantities, so regardless of how we choose our axes, the components of $\mathbf{L}$ must all have the same eigenvalues). Let the corresponding eigenvectors be, respectively, $|1\rangle, |0\rangle$, and $|-1\rangle$. Then

$$\begin{aligned}
\hat{L}_y + \hbar\hat{I}|-1\rangle &= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & -i & 0 \\ i & \sqrt{2} & -i \\ 0 & i & \sqrt{2} \end{pmatrix} |-1\rangle &= 0 \\
\hat{L}_y|0\rangle &= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} |0\rangle &= 0 \\
\hat{L}_y - \hbar\hat{I}|1\rangle &= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} -\sqrt{2} & -i & 0 \\ i & -\sqrt{2} & -i \\ 0 & i & -\sqrt{2} \end{pmatrix} |1\rangle &= 0
\end{aligned}$$

By inspection we can see that the normalized eigenvectors are

$$|-1\rangle = \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{2}i \\ 1 \end{pmatrix}; |0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; |1\rangle = \frac{1}{2} \begin{pmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{pmatrix}.$$

From this, we can get our probabilities:

$$|\langle\psi|-1\rangle|^2 = 1/2; |\langle\psi|0\rangle|^2 = 0; |\langle\psi|1\rangle|^2 = 1/2.$$

Thus we will only obtain the values $\hbar$ and $-\hbar$, each with 50 percent probability.

For completion, here is $|\psi\rangle$ in terms of the eigenstates

$$|\psi\rangle = \frac{\sqrt{3} - 2i}{\sqrt{14}}|-1\rangle + \frac{\sqrt{3} + 2i}{\sqrt{14}}|1\rangle.$$