PHY1510H

Classical Electromagnetism

Fall 2010
Problem Set 2
22 October 2010

HANDED OUT: Friday, 22 October 2010

DUE: 6:10 PM, Monday, 4 November, 2010
Hand in at

QUESTIONS:

1. A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$.

   (a) Show that the potential inside the spherical surface can be expressed as

   $$\Phi(\vec{x}) = \frac{Q}{8\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{2l+1} \left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right] \frac{r^l}{R^{l+1}} P_l(\cos\theta), \tag{1}$$

   where, for $l = 0$, $P_{l-1}(\cos\alpha) = -1$.

   (b) What is the potential outside the sphere? Plot it as a function of r for $\theta = 0$ and $\theta = \pi/2$.

   (c) Determine the electric field (magnitude and direction!) at the origin.

   (d) Calculate the dipole moment and quadrupole moment tensor of the charge distribution.

   (e) Discuss the limiting forms of the potential within the sphere and the electric field at the origin as

      i. the spherical cap becomes very small ($\alpha \ll 1$);

      ii. the spherical cap becomes so large that the area with the charge on it is now a very small cap at the south pole.

2. Three charges $(q, -2q, q)$ are located in a straight line with separation a and with the middle charge $(-2q)$ at the origin, enclosed by a grounded conducting sphere of radius b also centred on the origin (see figure in Jackson problem 3.7). Use spherical-polar coordinates throughout.

   (a) Calculate the potential of the three charges in the absence of the grounded sphere.

   (b) Find the limiting form of the potential in the absence of the grounded sphere as $a \to 0$, but the product $qa^2 = Q$ remains constant .

   (c) The presence of the grounded sphere of radius b alters the potential for $r < b$. The added potential can be viewed as caused by the surface charge density induced on the inner surface of the sphere, or be image charges located at $r > b$. Find the potential (whatever way you want!) everywhere inside the sphere for $r < a$ and for $r > a$.

(d) Show that in the limit $a \to 0$,

$$\Phi(r,\theta,\phi) \to \frac{Q}{2\pi\epsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right) P_2(\cos\theta). \tag{2}$$

(e) What is the potential outside of the sphere?

3. Consider the "spherical cow" model[1] of a battery connected to an external circuit. A sphere of radius a and conductivity σ is embedded in a uniform medium of conductivity σ'. Inside the the sphere there is a uniform (chemical) force in the $\hat{z}$ direction acting on the charge carriers; its strength as an effective electric field used to apply Ohm's Law is F. In the steady state, electric fields exist inside and outside the sphere and surface charge resides on its surface.

   (a) Find the electric field (in addition to F) and current density everywhere in space.

   (b) Determine the surface charge density of the sphere and show that its electric dipole moment is

   $$p = \frac{4\pi\epsilon_0 \sigma a^3 F}{\sigma + 2\sigma'}. \tag{3}$$

   (c) Show that the total current flowing out through the upper hemisphere of the sphere is

   $$I = \pi a^2 F \frac{2\sigma\sigma'}{\sigma + 2\sigma'}. \tag{4}$$

   (d) Calculate the total power dissipation outside of the sphere. Using the lumped circuit relations $P = I^2 R_e = I V_e$, find the external resistance R_e and voltage V_e.

4. A localized charge density $\rho(x,y,z)$ is place in an external electrostatic field described by the potential $\Phi^{(0)}(x,y,z)$. The external potential varies slowly in space over the region where ρ is non-zero.

   (a) From first principles, calculate the total force acting on the charge distribution as an expansion of multipole moments times derivatives of the field, up to and including quadrupole moments. Show that the force is

   $$\vec{F}(\vec{x}) = q\vec{E}^{(0)}(0) + \left\{\vec{\nabla}\left[\vec{p}\cdot\vec{E}^{(0)}\right]\right\}_0 + \left\{\vec{\nabla}\left[\frac{1}{6}\sum_{j,k} Q_{jk}\frac{\partial E_j^{(0)}}{\partial x_k}(\vec{x})\right]\right\}_0 + \cdots \tag{5}$$

   (b) Jackson calculates the energy of this configuration in Equation (4.24). How are the force calculated here and Jackson's energy expression related?

[1] Do any of you know this joke? See `http://en.wikipedia.org/wiki/Spherical_cow` .