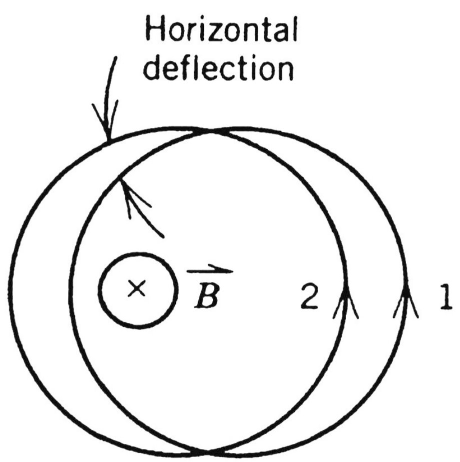
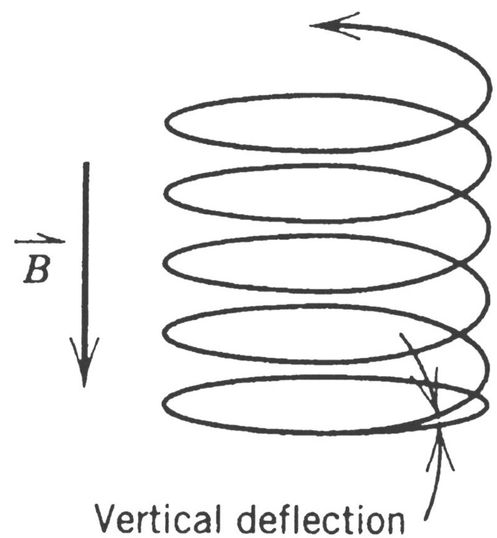
FIRST LOOK AT ORBIT STABILITY

- FIRST CYCLOTRONS AND BETATRONS DID NOT IMMEDIATELY WORK.
- SETTING LORENTZ = CENTRIPETAL ONLY WORKS FOR ONE SPECIAL ORBIT

  EQUILIBRIUM / SYNCHRONOUS ORBIT
- FOR MACHINE TO WORK, ONE WOULD HAVE TO INJECT EXACTLY ONTO THAT ORBIT.





SLIGHT DISPLACEMENT FROM EQUILIB ORBIT

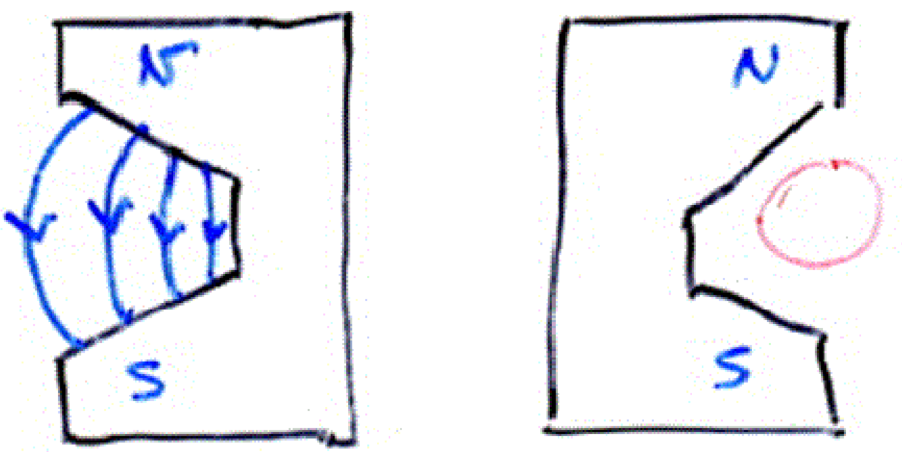
⇓

PARTICLE LOST

SLIGHT DISPLACEMENT FROM EQUILIBRIUM ORBIT

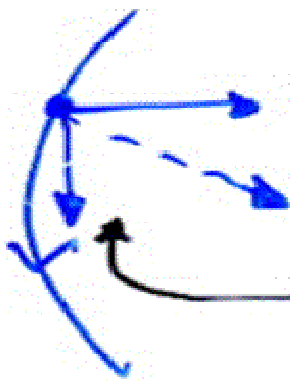
→ LOSS OF PARTICLES

→ NEED HORIZONTAL + VERTICAL FOCUSING



SHAPE POLE-FACES OF GUIDE MAGNET

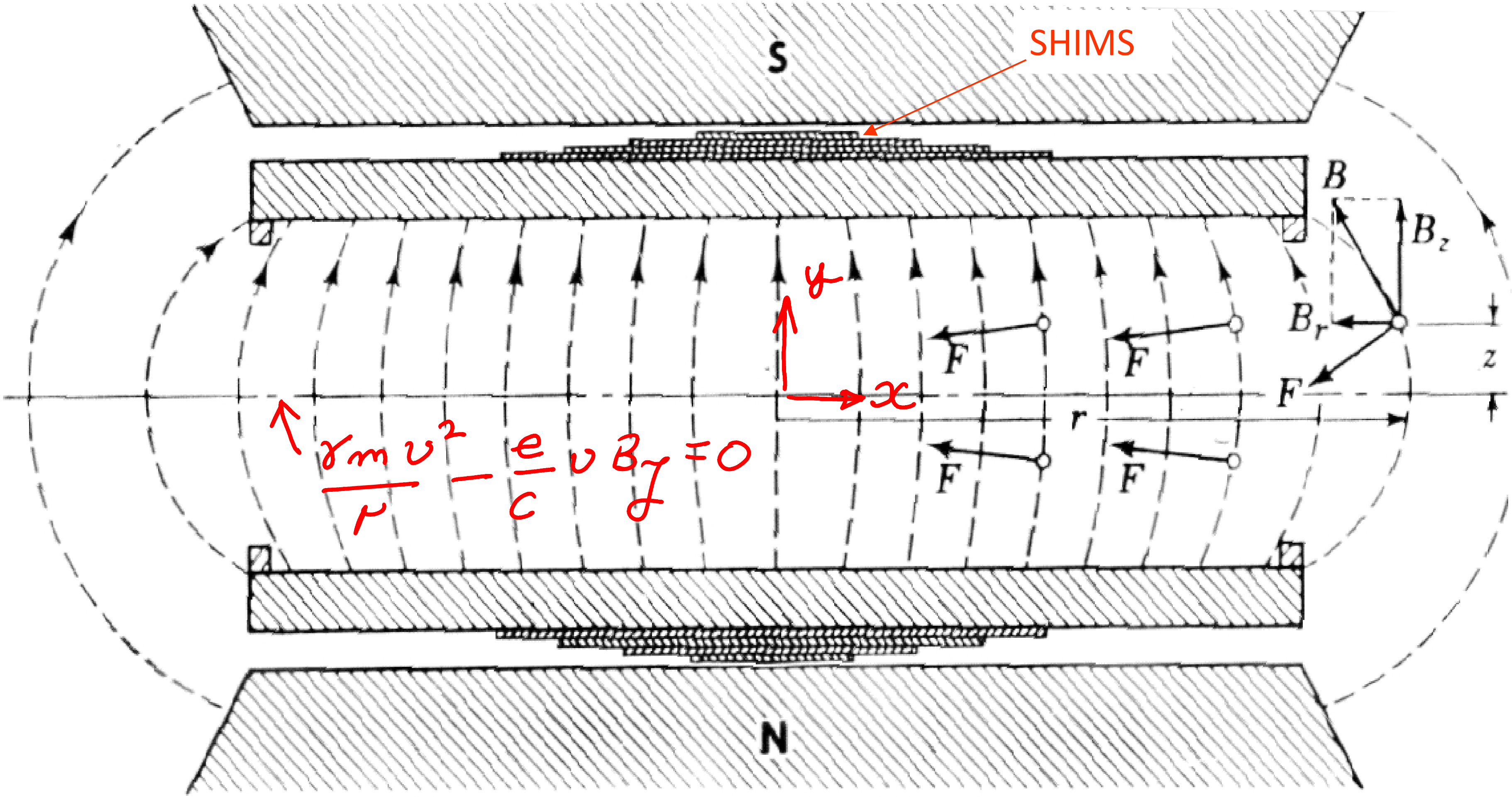
Cf BETATRON



VERTICAL FOCUSING MAGNETIC FIELD

GUIDE FIELD

Orbital Stability in a Cyclotron



$$\overline{F} = \frac{q}{c}\,\overline{v} \times \overline{B}$$

WEAK FOCUSING

CAN WRITE A RADIALLY DECREASING FIELD

$$B_y = B_{y_0}\left(\frac{R}{r}\right)^n$$

n ← FIELD INDEX

R ← RADIUS OF EQUILIBRIUM ORBIT

B_{y_0} ← VERTICAL FIELD AT R

$$\frac{\partial B_y}{\partial r} = B_{y_0}\frac{R^n}{r^n}\cdot\frac{-n}{r} \rightarrow n = \frac{-r}{B_{y_0}}\cdot\frac{r^n}{R^n}\cdot\frac{\partial B_y}{\partial r} = \frac{-r}{B_y}\frac{\partial B_y}{\partial r}$$

DEFN of B_{y_0}

FOR SMALL EXCURSIONS FROM EQUILIBRIUM ORBIT

$$r \approx R \rightarrow n = \frac{-R}{B_{y_0}}\cdot\frac{\partial B_y}{\partial r}$$

AT EQUILIBRIUM ORBIT $\quad \dfrac{\gamma m v^2}{R} - \dfrac{e}{c}\cdot v\cdot B_{y_0} = 0$

AWAY FROM EQUILIBRIUM ORBIT

RESTORING FORCE $\quad F_x = \dfrac{\gamma m v^2}{r} - \dfrac{e}{c} v B_y \qquad (1)$

WRITE RADIAL POSITION $r = R + x = R\left(1 + \frac{x}{R}\right)$

ORBIT RADIUS

EQUILIBRIUM ORBIT

SMALL RADIAL EXCURSION

$$B_y = B_{oy} + \frac{\partial B_y}{\partial x} \cdot x = B_{oy}\left(1 + \frac{R}{B_{oy}} \cdot \frac{\partial B_y}{\partial x} \cdot \frac{1}{R} \cdot x\right)$$

SUBST INTO $F_x = \frac{\gamma m v^2}{r} - \frac{e}{c} v B_y$

$$F_x = \frac{\gamma m v^2}{R}\left(1 + \frac{x}{R}\right)^{-1} - \frac{e}{c} v B_{oy}\left(1 + \frac{R}{B_{oy}} \cdot \frac{\partial B_y}{\partial x} \cdot \frac{x}{R}\right)$$

FOR $x << R \rightarrow \left(1 + \frac{x}{R}\right)^{-1} \approx \left(1 - \frac{x}{R}\right)$

AND $n = -\frac{R}{B_{oy}} \frac{\partial B_y}{\partial x}$

$$F_x = \frac{\gamma m v^2}{R}\left(1 - \frac{x}{R}\right) - \frac{e}{c} \cdot v \cdot B_{oy}\left(1 - \frac{n x}{R}\right) \quad (2)$$

$$F_x = \frac{\gamma m v^2}{R}\left(1-\frac{x}{R}\right) - \frac{e}{c}\cdot v\cdot B_{oy}\left(1-\frac{n x}{R}\right) \quad (2)$$

ON EQUILIBRIUM $\quad \frac{\gamma m v^2}{R} = \frac{e}{c}\cdot v B_{oy}$

$$F_x = -\frac{\gamma m v^2}{R}\cdot\frac{x}{R}(1-n)$$

RESTORING FORCE FOR $n<1$

$F_x \propto x$

SIMPLE HARMONIC MOTION $\quad F_x = -kx \,; \quad F_x = \gamma m \ddot{x}$

$\ddot{x} + \omega_x^2 \cdot x = 0 \,; \quad \omega_x^2 = \frac{k}{\gamma m}$

IN THIS CASE HAVE SHM OSCILLATIONS FOR

$$k = \frac{v^2}{R^2}(1-n)\cdot\gamma m \Rightarrow \omega_x = \frac{v}{R}\sqrt{1-n} = \omega_0\sqrt{1-n}$$

$n<1$ FOR STABLE OSCILLATIONS AROUND EQUILIBRIUM ORBIT

BETATRON OSCILLATIONS $\rightarrow$ ANY CIRCULAR ACCELERATOR

THIS TREATMENT IS IN HORIZONTAL PLANE

MAIN RESTORING EFFECT FOR CORRECT FIELD GRADIENT IS

INTERPLAY OF LORENTZ & CENTRIFUGAL

IF A PARTICLE MOVES $R \rightarrow R + a$

B_y DECREASES $\rightarrow$ F_L DECREASES

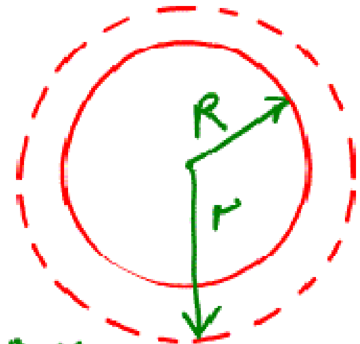
R INCREASES $\rightarrow$ F_c DECREASES

$$F_c = \frac{\gamma m v^2}{r} \quad ; \quad F_L = e B_y \cdot v$$

STABILITY DEPENDS ON WHICH OF THESE FORCES IS GREATER, AFTER A DISPLACEMENT FROM

THE EQUILIBRIUM ORBIT

LIMITING CASE IS WHEN ΔF_c, ΔF_L JUST GIVE EQUILIBRIUM AT LARGER RADIUS



$$\frac{\gamma m v^2}{R} = \frac{e}{c} v B_{0y} \rightarrow \gamma m v = \frac{e}{c} R B_{0y}$$

IF THERE IS NO CHANGE IN MOMENTUM $R \rightarrow r$ AT LARGER RADIUS

$$\gamma m v = \frac{e}{c} \cdot r B_y = \frac{e}{c} R B_{0y} \rightarrow B_y = B_{0y} \frac{R}{r}$$

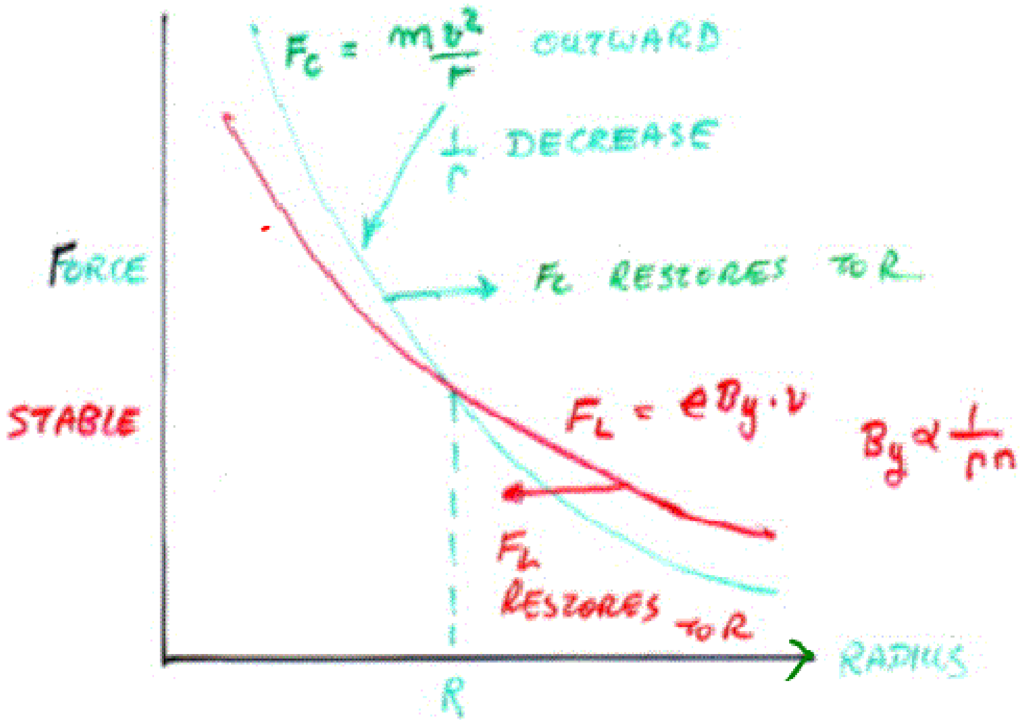
COMPARE THIS TO $B_y = B_{0y} \left(\frac{R}{r} \right)^n$

THIS SITUATION JUST CORRESPONDS TO THE LIMITING CASE OF $n = 1$

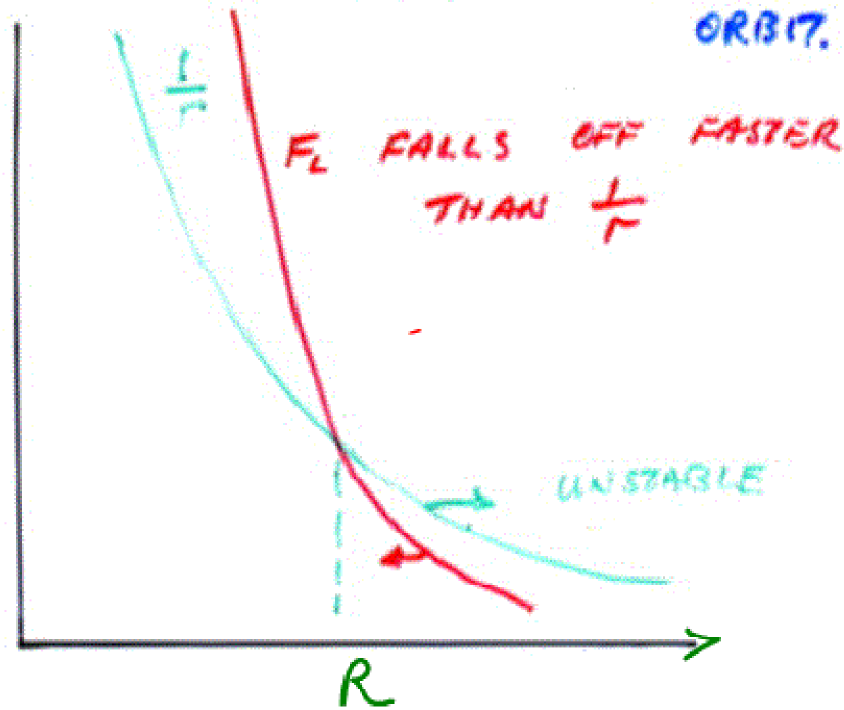
OBVIOUS FROM $\omega_x = \frac{v}{R} \sqrt{1-n}$

$$B_y = B_{oy}\left(\frac{R}{r}\right)^n$$

$n < 1$ STABLE



$n > 1 \rightarrow$ UNSTABLE ORBIT.



VERTICAL BETATRON OSCILLATIONS

$$\gamma m \ddot{y} = \frac{e}{c} v B_x$$

NO CENTRIFUGAL FORCE

$$\bar{\nabla} \times \bar{B} = 0 \rightarrow \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} = 0 \rightarrow B_x = \int \frac{\partial B_x}{\partial y} dy$$

USE $n = -\frac{R_0}{B_{y_0}} \cdot \frac{\partial B_y}{\partial x}$ $\rightarrow$ $-n \int \frac{B_{0y}}{R} \cdot dy = -n \frac{B_{0y}}{R} \cdot y$

$$\gamma m \ddot{y} = -n \frac{B_{0y}}{R_0} \cdot \frac{e}{c} \cdot v \cdot y$$

USE $\frac{\gamma m v^2}{R} = \frac{e}{c} \cdot v B_{0y}$

$$\gamma m \ddot{y} + n \gamma m \frac{v^2}{R^2} y = 0 \rightarrow \ddot{y} + n \frac{v^2}{R^2} \cdot y = 0$$

SHM FOR $n > 0$

VERTICAL BETATRON OSCILLATIONS

$$\omega^2 = n \frac{v^2}{R^2} = n \omega_0^2$$

$$\omega = \omega_0 \sqrt{n}$$

Betatron Oscillations

Horizontal

$$B_z = B_{z_0}\left(\frac{R}{r}\right)^n$$

Field Index

Equilibrium Orbit

$$\frac{\gamma m v^2}{R} - \frac{e}{c} v B_{z_0} = 0$$

Centrifugal = Lorentz
on equilibrium orbit

$$F_x = \frac{\gamma m v^2}{r} - \frac{e}{c} v B_z$$

Restoring Force

$$F_x = -\frac{\gamma m v^2}{R}\frac{x}{R}(1-n)$$

Simple Harmonic

$$\omega_x = \frac{v}{R}\sqrt{1-n}$$

Stable Oscillations around
Equilibrium orbit

Vertical

$$\gamma m \frac{d^2 y}{dt^2} = \frac{e}{c} v B_x$$

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = 0$$

$$\gamma m \frac{d^2 y}{dt^2} = -n \frac{B_z}{R}\frac{e}{c} v = F_z$$

$$\gamma m \frac{d^2 y}{dt^2} + n \gamma m \frac{v^2}{R^2} = 0$$

$$\omega_z = n \frac{v^2}{R^2}$$

Weak Focusing

$$1 > n > 0$$

WEAK FOCUSING

- CANNOT MAKE FIELD INDEX n ARBITRARILY LARGE IN BOTH HORIZONTAL & VERTICAL
- CANNOT MAKE RESTORING FORCES ARBITRARILY LARGE IN BOTH DIRECTIONS
- CANNOT MAKE AMPLITUDE OF HORIZONTAL AND VERTICAL BETATRON OSCILLATIONS ARBITRARILY SMALL

LARGE CROSS SECTION OF VACUUM PIPE $\rightarrow$ LARGE MAGNETS $\rightarrow$ \$\$\$ \$\$\$

MUST BE A BETTER WAY

$\rightarrow$ SHORT DIGRESSION INTO PARTICLE DYNAMICS

DYNAMICS OF PARTICLES IN MAGNETIC FIELDS

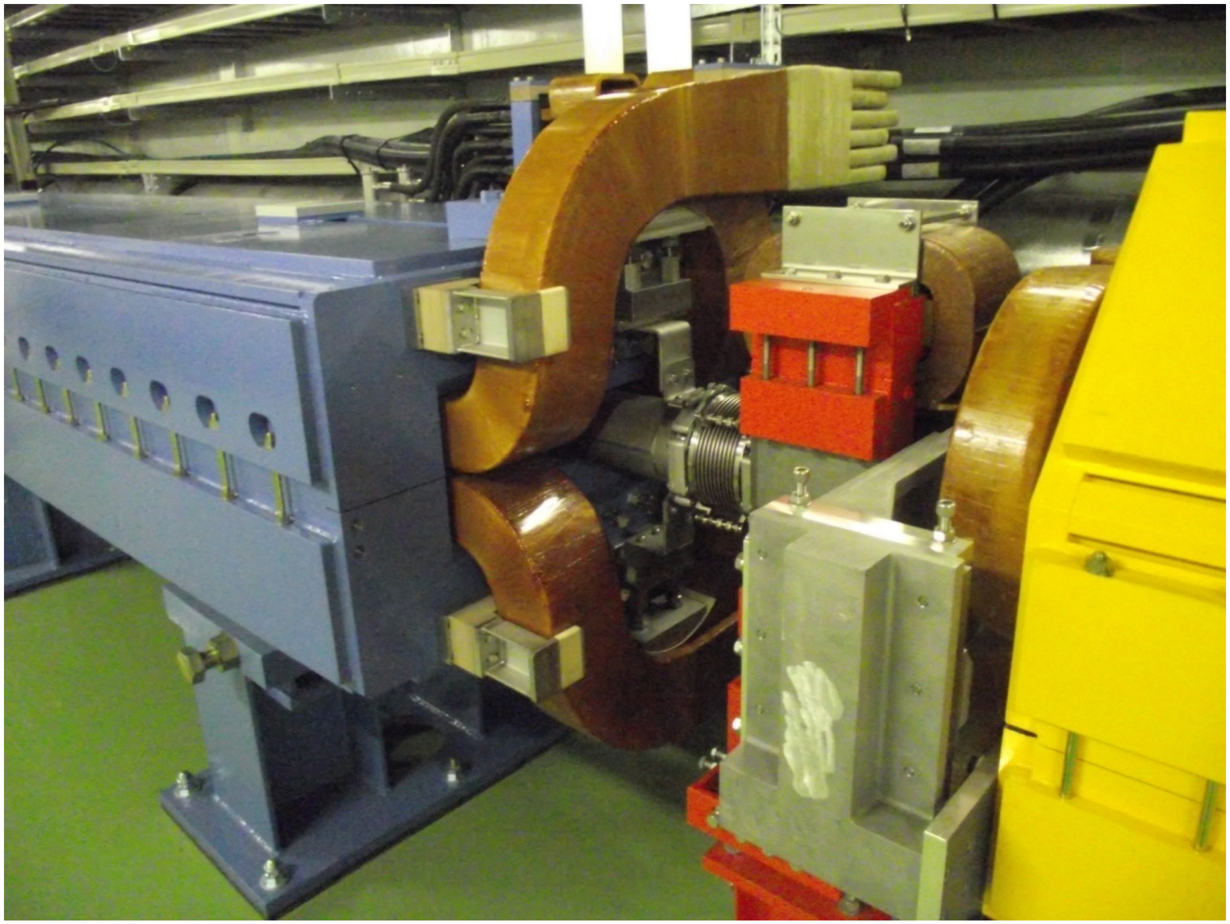
MAGNETS OF MANY MULTIPOLES USE IN ACCELERATOR

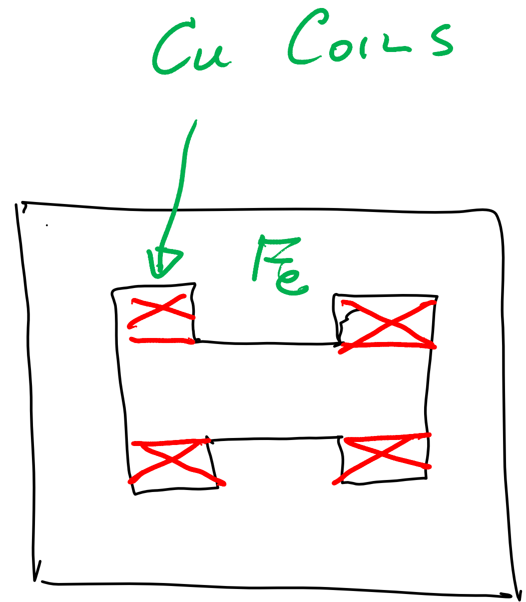
DIPOLE — BENDING MAGNET

QUADRUPOLE — FOCUSING

6-POLE
12-POLE } REMOVE CHROMATIC ABERATIONS

DIPOLE — BEND

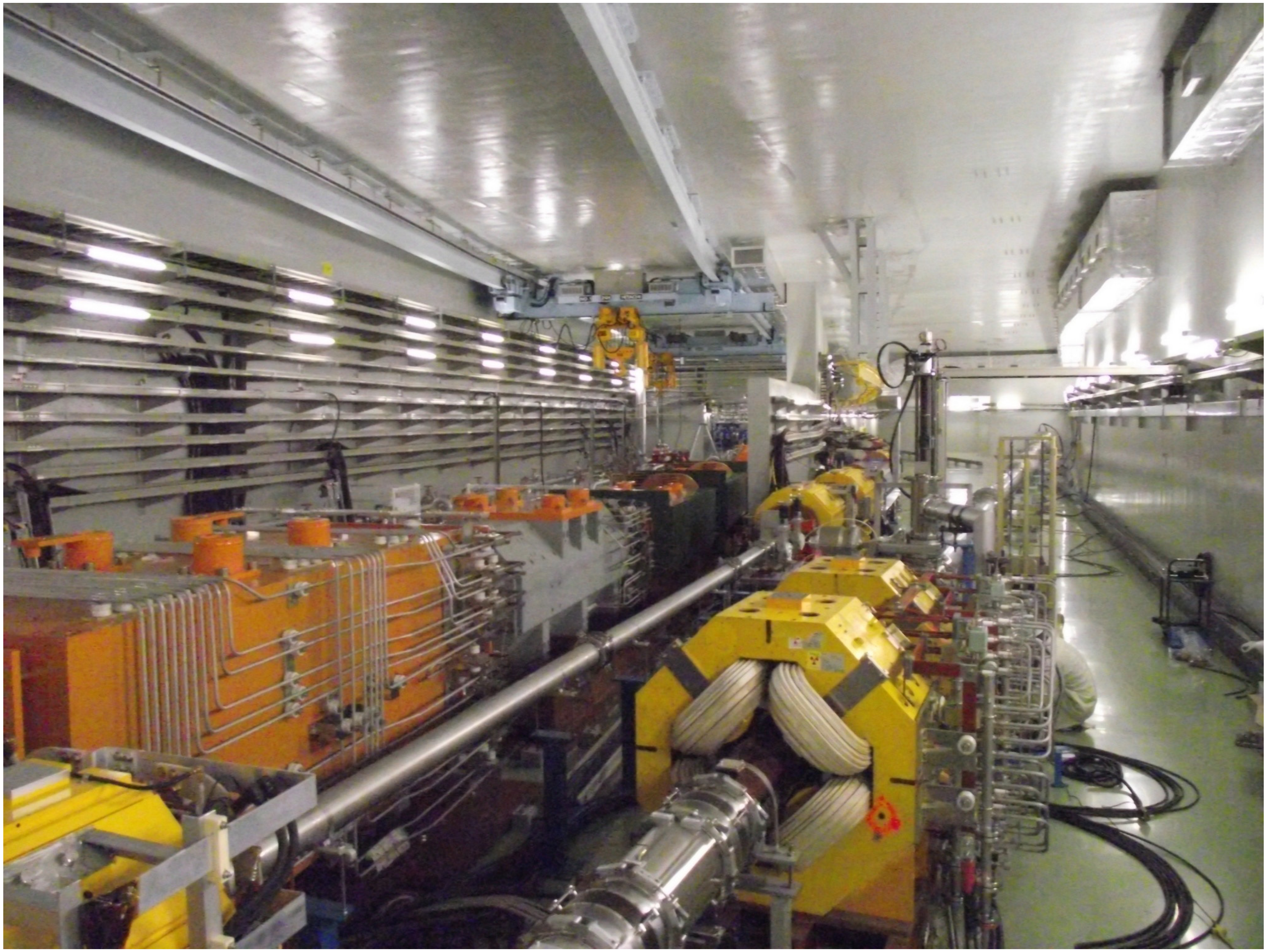




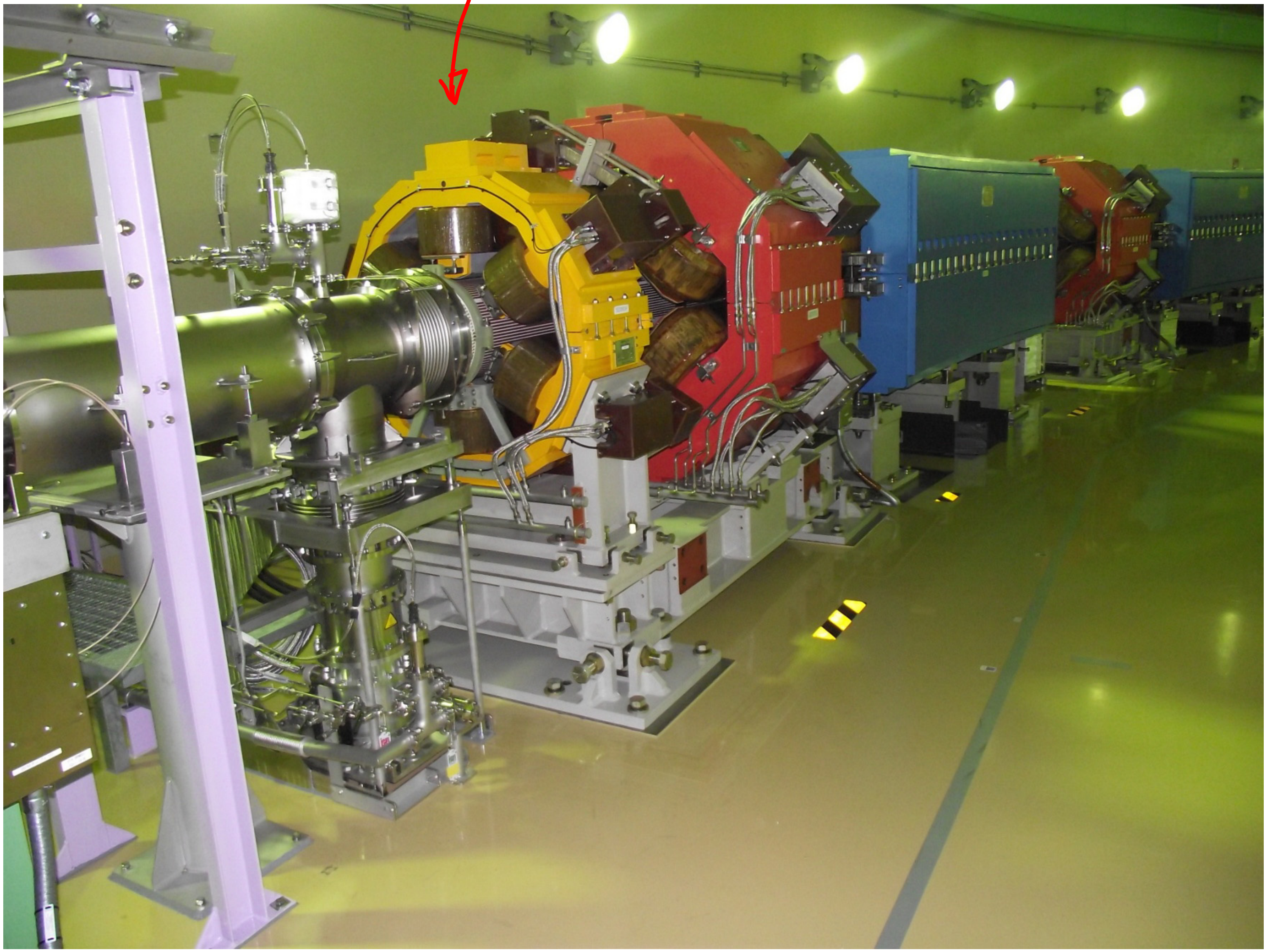
$$B = \frac{2\mu_0 N I}{h}$$

GAP BETWEEN POLES

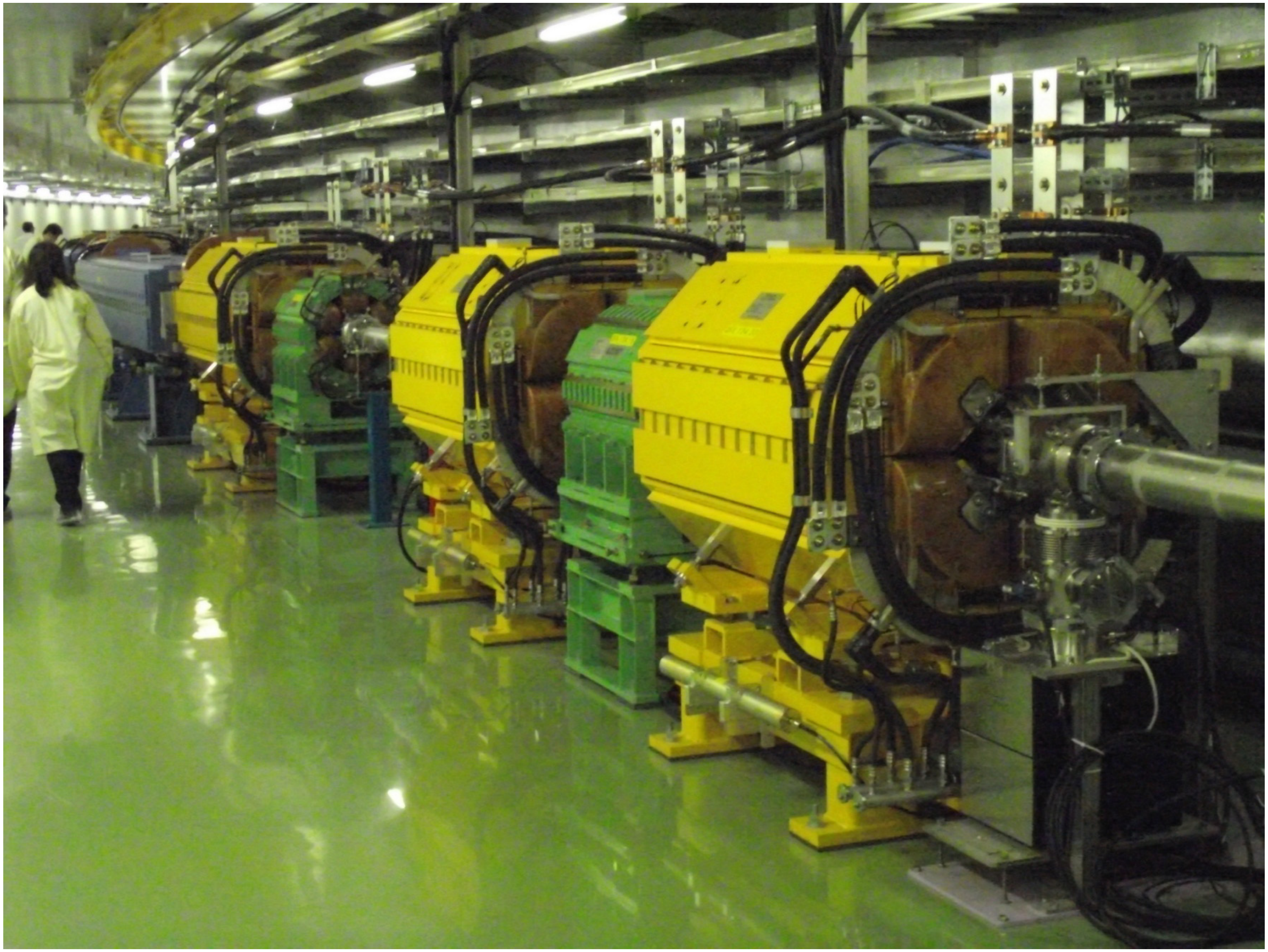
QUADRUPOLES — FOCUSING

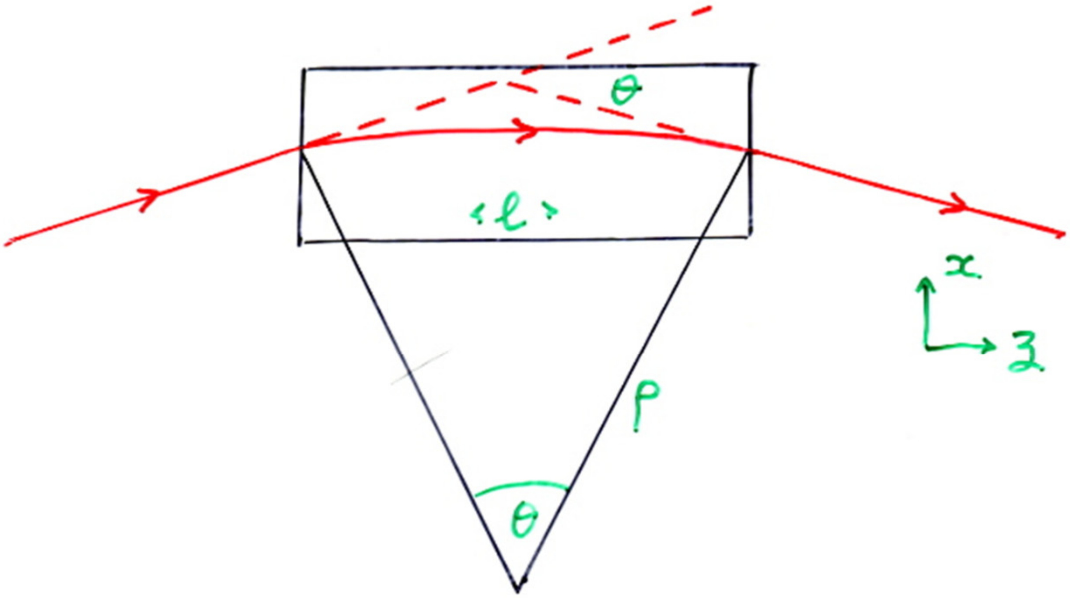


6-POLE — CORRECTIONS



DIPOLE → QUAD → 6 → QUAD → 6 → QUAD





BENDING MAGNET

$$\frac{1}{\rho} = \frac{eB}{p}$$

PASSING THRU DIPOLE, CHANGE IN ANGLE $x' = \dfrac{dx}{dz}$

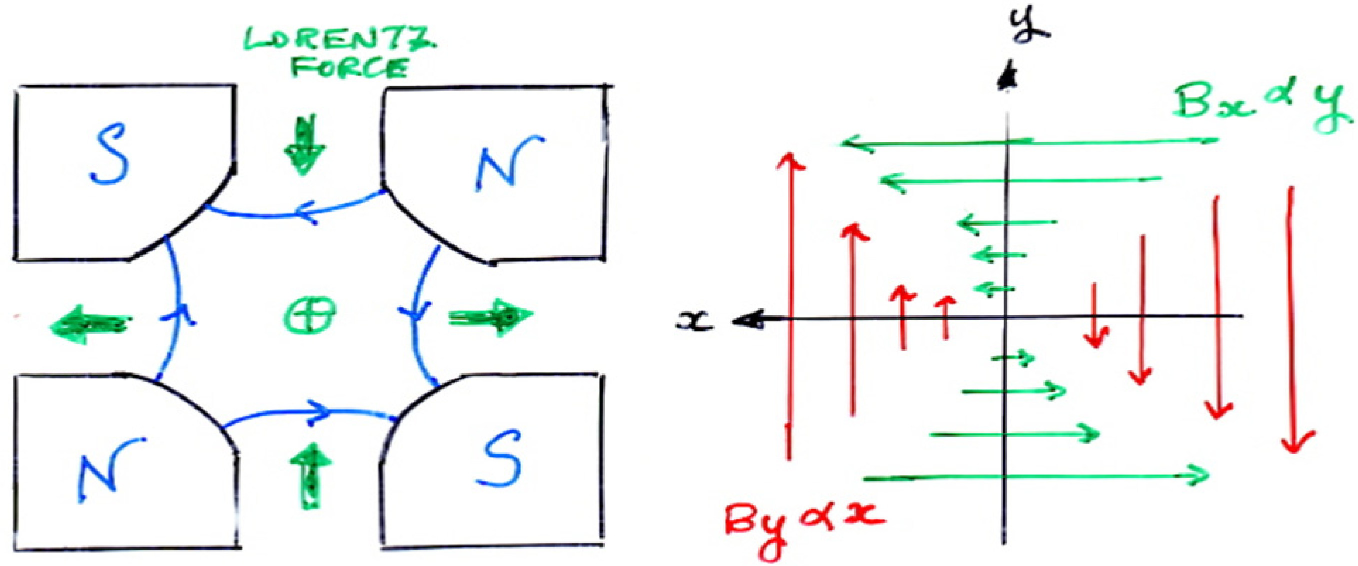
$$\Delta x' = \frac{\ell}{\rho} = \frac{\ell\,(e B_y)}{p}$$

FIELD GRADIENT

BUT $B_y = \dfrac{\partial B_y}{\partial x} \cdot x = -k\,x$

$$\Delta x' = -\frac{\ell e}{p} \cdot k \cdot x = \theta$$

QUADRUPOLE



POLE FACES — EQUIPOTENTIAL SURFACE

FOR HYPERBOLIC POLE FACES

MAGNETIC SCALAR POTENTIAL $\quad \phi = k \cdot x \cdot y$

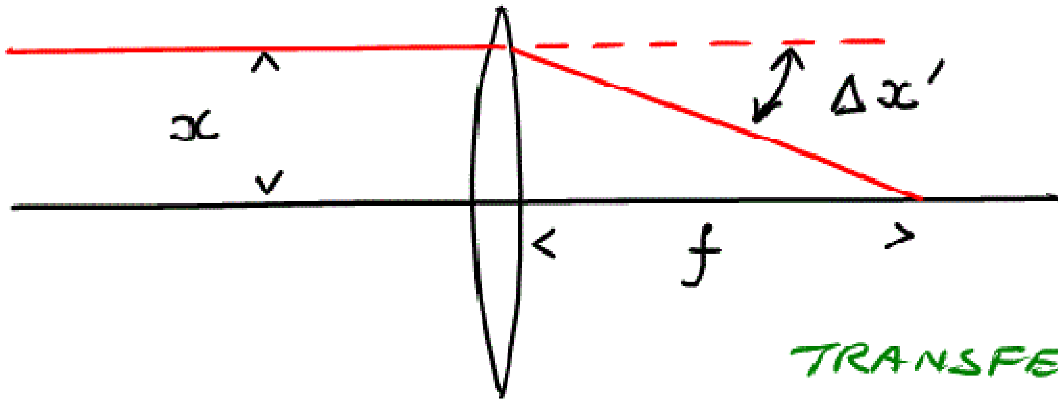
$$\vec{B} = -\vec{\nabla}\phi \rightarrow \begin{array}{l} B_x = -ky \\ B_y = -kx \end{array}$$

FOCUS

DEFOCUS

WARM IRON $\quad k \sim 12\ \mathrm{T\,m^{-1}}$ FOR 10 GeV e^-

SUPERCONDUCTING $\quad k \sim 75\ \mathrm{T\,m^{-1}}$ FOR 1 TeV p



TRANSFER MATRIX

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{OUT} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{IN}$$

THIN LENS

DEFINITION OF FOCAL LENGTH

$$\Delta x' = -\frac{x}{f}$$

$$\frac{1}{f} = -\frac{\Delta x'}{x} = \frac{e}{p} \cdot k \cdot l$$

FIELD GRADIENT OF QUADRUPOLE

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{o} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{i}$$

FIELD FREE REGION

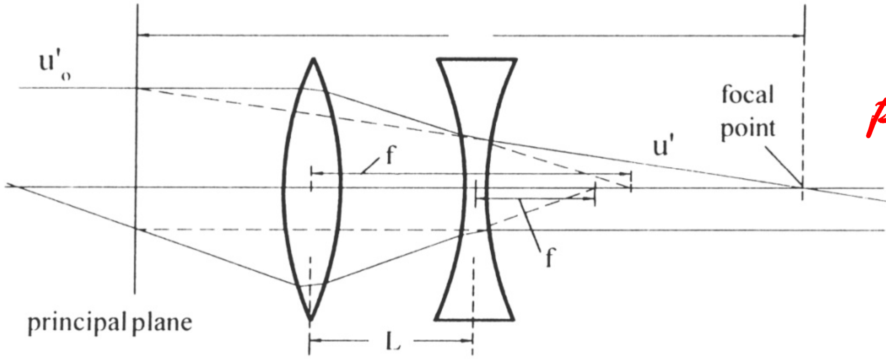


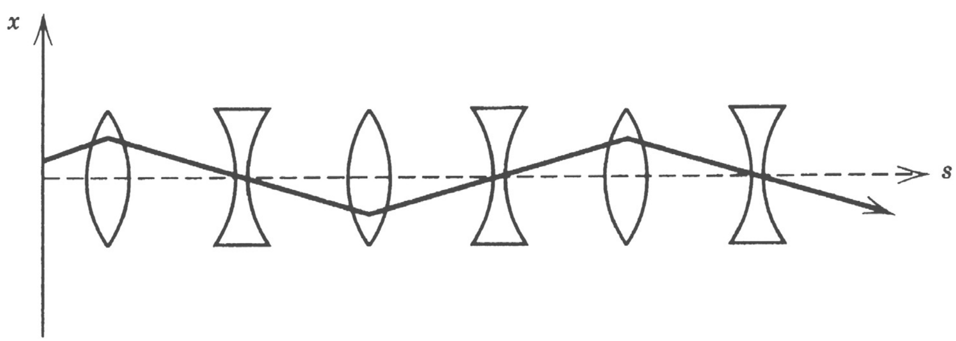
Fig. 4.14. Focusing in a quadrupole doublet

CAN HAVE STRONG FOCUSING IN BOTH PLANES.

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ +\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1+\frac{L}{f} & L \\ -\frac{L}{f^2} & 1-\frac{L}{f} \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 0 \\ -\frac{L}{f^2} & 1 \end{pmatrix} \qquad L \ll f$$

Net Focusing



- FODO Lattice
- Strong Focusing

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{f_1} & L \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1-\frac{L}{f_2} & L \\ \underbrace{\frac{1}{f_1}-\frac{1}{f_2}-\frac{L}{f_1 f_2}} & 1+\frac{L}{f_1} \end{pmatrix}$$

$f_1 = f_2 \longrightarrow -\frac{L}{f^2}$

CONDITION FOR FOCUSING

THIN LENS

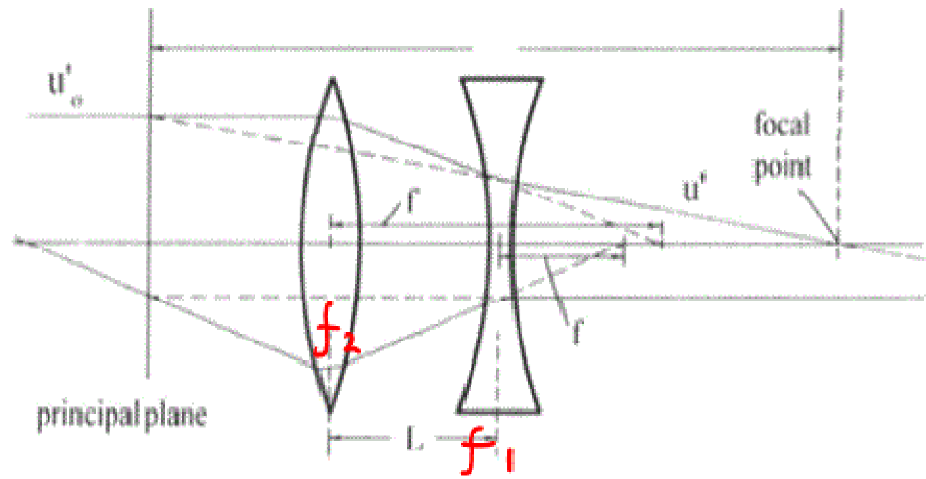


Fig. 4.14. Focusing in a quadrupole doublet

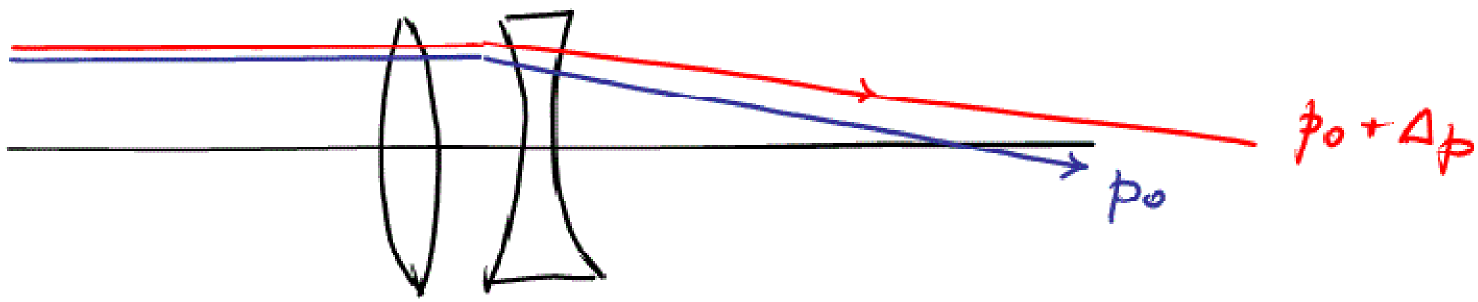
FINITE LENGTH

$$\begin{pmatrix} \cos\phi & \sin\phi/\sqrt{K} \\ -\sqrt{K}\sin\phi & \cos\phi \end{pmatrix}_F$$

$$\begin{pmatrix} \cosh\phi & \sinh\phi/\sqrt{K} \\ \sqrt{K}\sinh\phi & \cosh\phi \end{pmatrix}_D$$

$$\phi = \sqrt{K} \cdot L$$

NON EQUILIBRIUM ORBIT



IN REAL LIFE HAVE FINITE RANGE OF MOMENTA

→ CHROMATIC ABERATION

→ CORRECT WITH SEXTAPOLES

STABILITY OF NONEQUILIBRIUM ORBITS IN FODO LATTICE

- PARTICLE TRAVERSES A SERIES OF ELEMENTS
  TRANSFER MATRICES $M = M_1 M_2 \text{------} M_n$
- PERIODIC STRUCTURE — I.E. TRAVERSE SEQUENCE REPETITIVELY
- AFTER n TRAVERSALS, TRANSFER MATRIX

$$\begin{pmatrix} x \\ x' \end{pmatrix}_n = M^n \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

FOR STABILITY, THIS MUST BE FINITE FOR LARGE n

$M \rightarrow \quad V_1 \quad V_2$ EIGEN VECTOR

$\lambda_1 \quad \lambda_1$ EIGEN VALUES

ANY INITIAL

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{in} = \overbrace{A\,V_1 + B}^{\text{CONST}}V_2 \rightarrow$$

n TIMES THROUGH PERIODIC STRUCTURE

$$M^n \begin{pmatrix} x \\ x' \end{pmatrix}_{IN} = A\lambda_1^n V_1 + B\lambda_2^n V_2$$

• STABILITY REQUIRES λ_1^n λ_2^n NOT GROW WITH n

$$\det(M_i) = 1 \rightarrow \det(M) = 1 \rightarrow \lambda_2 = 1/\lambda_1$$

WRITE $\left.\begin{aligned} \lambda_1 &= e^{i\mu} \\ \lambda_2 &= e^{-i\mu} \end{aligned}\right\}$ μ IS REAL FOR STABLE OSCILLATIONS

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ THEN CHARACTERISTIC EQUATION $\det(M - \lambda I) = 0$

BECOMES $$\underbrace{(ad - bc)}_{\det M = 1} - (a+d)\lambda + \lambda^2 = 0$$

$$\underbrace{\lambda^{-1} + \lambda}_{e^{i\mu} + e^{-i\mu} = 2\cos\mu = \mathrm{Tr}\, M} = a + d = \mathrm{Tr}\, M$$

REAL

STABILITY FOR $-1 \leq \frac{1}{2}\mathrm{Tr}\, M \leq 1$

FOR FODO WITH EQUALLY SPACED MAGNETS

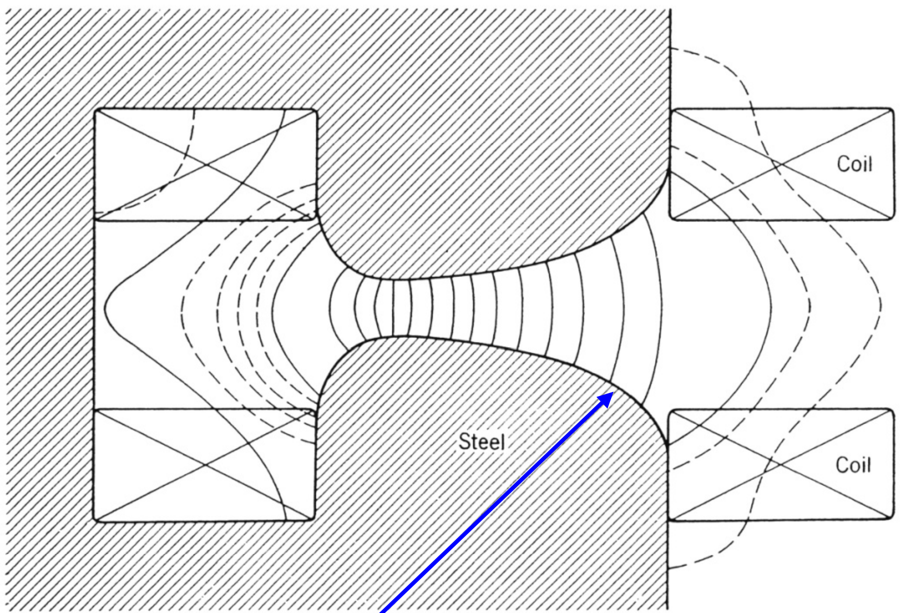
REPEATING SEQUENCE

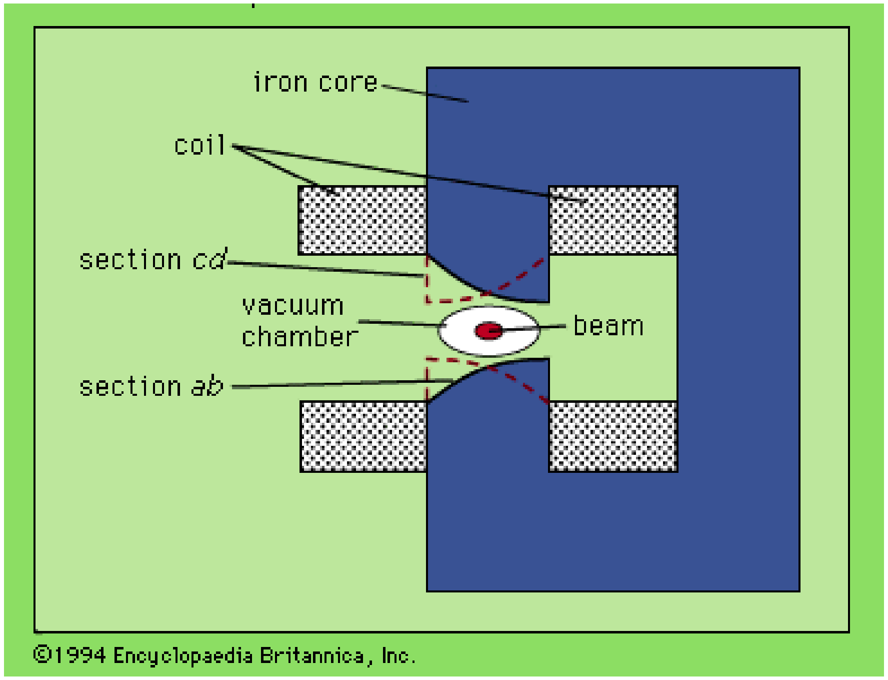
$$M = \begin{pmatrix} 1 + L/f & 2L + L^2/f \\ -L/f^2 & 1 - \frac{L}{f} - \left(\frac{L}{f}\right)^2 \end{pmatrix}$$

STABILITY $-1 \leq \frac{1}{2} \mathrm{Tr}\, M \leq 1 \rightarrow -1 \leq \frac{1}{2}\left(2 - \left(\frac{L}{f}\right)^2\right) \leq 1$

$\rightarrow \left|\frac{L}{2f}\right| \leq 1$ FOCAL LENGTH $>$ $\frac{1}{2}$ ELEMENT SPACING

Strong Focusing

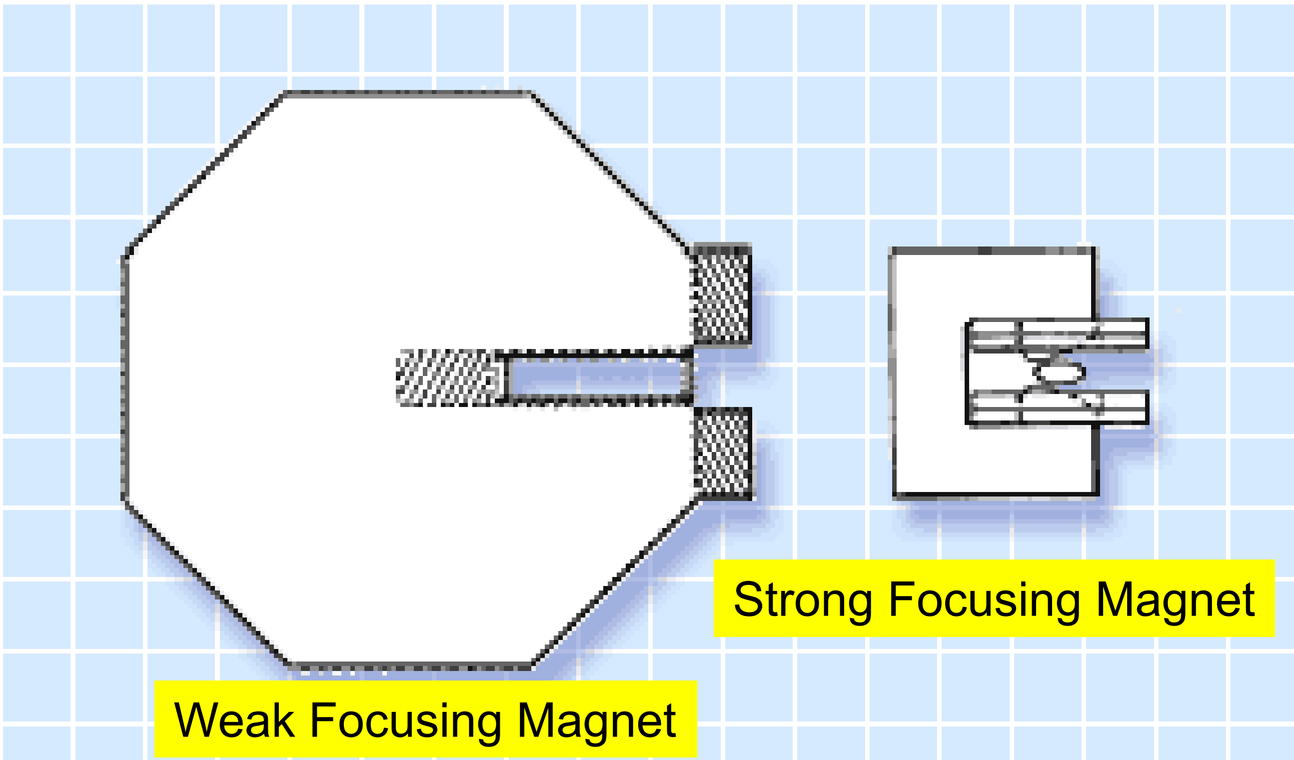




- Field Index set by Pole Face Shape
- Weak, n = 0.5
- Strong, n = 3500

- Strong Focusing = Alternating Gradient
- “Combined Function” Magnet

Enormous Cost Saving



- Strong Focusing = Alternating Gradient
- Reduce amplitude of betatron oscillations
- Reduce diameter of vacuum pipe
- Reduce Aperture of Magnets

- 35 GeV (CERN PS, AGS) costs same as 7 GeV (NIMROD)