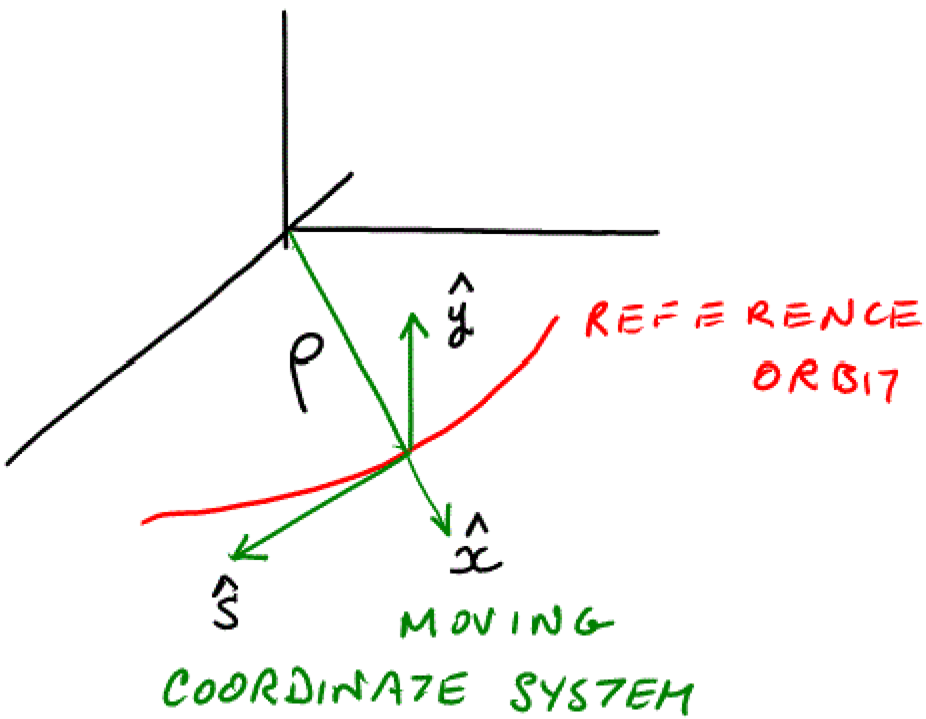
ORBITAL STABILITY

- IN DISCUSSION OF BETATRON OSCILLATIONS IN
- A WEAKLY FOCUSED MACHINE DERIVED CONDITIONS FOR STABILITY AROUND EQUILIBRIUM ORBIT
- FOR STRONG FOCUSING THE FIELD INDEX CAN BE $>>1$. IN FACT, WE CAN IMAGINE THAT IT WILL VARY AROUND THE RING

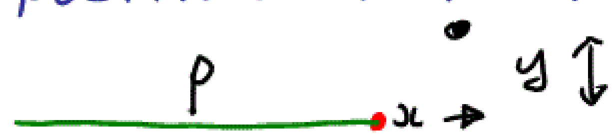
- WANT TO STUDY
  - SPATIAL STABILITY OF BEAM ENVELOPE → BETATRON OSCILLATIONS
  - PHASE STABILITY W.R.T THE PHASE OF ACCELERATING VOLTAGE
    → SYNCHROTRON OSCILLATIONS

WORK IN MOVING COORDINATE SYSTEM

- EQUILIBRIUM ORBIT IN A PARTICULAR MACHINE IS DEFINED BY CHOSEN MAGNET CONFIGURATION
- OUR MAIN INTEREST IS IN UNDERSTANDING HOW PARTICLES MOVE RELATIVE TO EQUILIBRIUM ORBIT
- MOTION EASIER TO TREAT IN COORDINATE SYSTEM THAT MOVES ALONG EQUILIBRIUM ORBIT.



POSITION OF PARTICLE



$$\bar{R} = r\hat{x} + y\hat{y}\,; \quad r = \rho + x$$

EQUATION OF MOTION

$$\frac{d\bar{p}}{dt} = e\,\bar{v} \times \bar{B}$$

MAGNETIC FIELD → RADIAL + VERTICAL COMPONENTS
→ NO COMPONENT IN $\hat{s}$-DIRECTION

$$\bar{v} \times \bar{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{s} \\ v_x & v_y & v_s \\ B_x & B_y & 0 \end{vmatrix} = -v_s B_y \hat{x} + v_s B_x \hat{y} + (v_x B_y - v_y B_x)\hat{s}$$

ASSUME NO ENERGY CHANGE → NO ACCELRATION IN $\hat{S}$
→ NO SYNCHROTRON RADIATION

$$e\,\bar{v} \times \bar{B} = \frac{d\bar{p}}{dt} = \frac{d}{dt}\left(\gamma m \dot{\bar{R}}\right) = \gamma m \ddot{\bar{R}}$$

$$\ddot{\bar{R}} = \frac{e\,\bar{v} \times \bar{B}}{\gamma m}$$

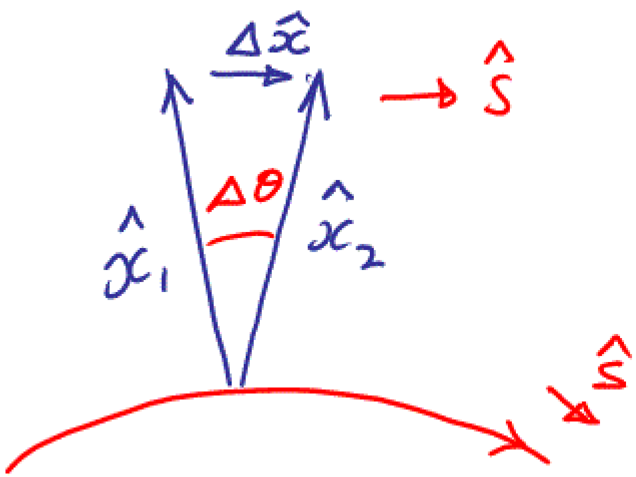
EVALUATE THIS IN LOCAL MOVING COORD SYSTEM

$$\bar{R} = r\hat{x} + y\hat{y} \rightarrow \dot{\bar{R}} = \dot{r}\hat{x} + r\dot{\hat{x}} + \dot{y}\hat{y}$$

VARIES IN TIME

CONSTANT

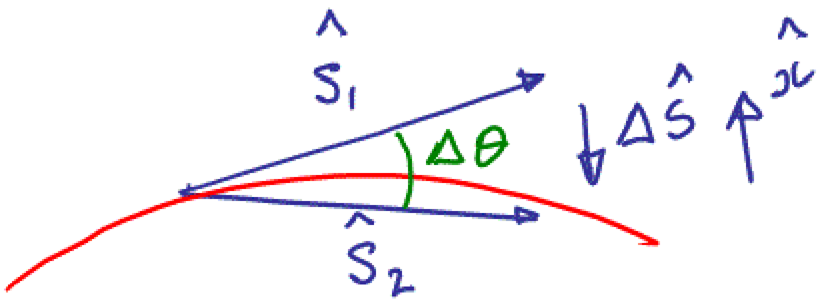
VARIATION IN TIME OF UNIT VECTORS



$$\Delta\hat{x} = \underbrace{\overset{=1}{|\hat{x}|}\,\Delta\theta}_{\text{MAG}} \cdot \underset{\text{DIRECTION}}{\hat{s}}$$

ASSUME $\hat{s}$ CONSTANT

$$\dot{\hat{x}} = \dot{\theta} \cdot \hat{s}$$



$$\Delta\hat{s} = |\hat{s}|\,\Delta\theta\,(-\hat{x})$$

$$\dot{\hat{s}} = -\dot{\theta}\,\hat{x}$$

$$\dot{\theta} = \frac{v_s}{\rho}$$

HAD

$$\dot{\vec{R}} = \dot{r}\,\hat{x} + r\,\dot{\hat{x}} + \dot{y}\,\hat{y}$$

$$\dot{R} = \dot{r}\,\hat{x} + r\underbrace{\dot{\theta}\,\hat{s}}_{\text{VARIES IN TIME}} + \dot{y}\,\hat{y}$$

$$\ddot{\vec{R}} = \ddot{r}\,\hat{x} + \dot{r}\,\dot{\hat{x}} + r\frac{d}{dt}\left(\dot{\theta}\,\hat{s}\right) + \dot{\theta}\,\hat{s}\,\dot{r} + \ddot{y}\,\hat{y}$$

$$= \ddot{r}\,\hat{x} + \dot{r}\,\underset{\downarrow\ \dot{\theta}\,\hat{s}}{\dot{\hat{x}}} + r\dot{\theta}\,\dot{\hat{s}} + r\ddot{\theta}\,\hat{s} + \dot{\theta}\,\hat{s}\,\dot{r} + \ddot{y}\,\hat{y}$$

$$= \ddot{r}\,\hat{x} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{s} + r\dot{\theta}\,\underset{\downarrow\ -\dot{\theta}\,\hat{x}}{\dot{\hat{s}}} + \ddot{y}\,\hat{y}$$

$$\ddot{\vec{R}} = \ddot{r}\,\hat{x} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{s} - r\dot{\theta}^2\hat{x} + \ddot{y}\,\hat{y}$$

$$\ddot{\vec{R}} = \underbrace{\left(\ddot{r} - r\dot{\theta}^2\right)}_{\text{RADIAL MOTION}}\hat{x} + \underbrace{\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)}_{\text{ALONG ORBIT}}\hat{s} + \underbrace{\ddot{y}\,\hat{y}}_{\text{VERTICAL}}$$

X OR RADIAL MOTION FROM

$$\ddot{\bar{R}} = (\ddot{r} - r\dot{\theta}^2)\hat{x} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s} + \ddot{y}\hat{y}$$

$$F = \gamma m \ddot{x}$$

$$e(\bar{v} \times \bar{B})_x = -e v_s B_y \hat{x} = \frac{d\bar{p}}{dt} = \gamma m (\ddot{r} - r\dot{\theta}^2)\hat{x}$$

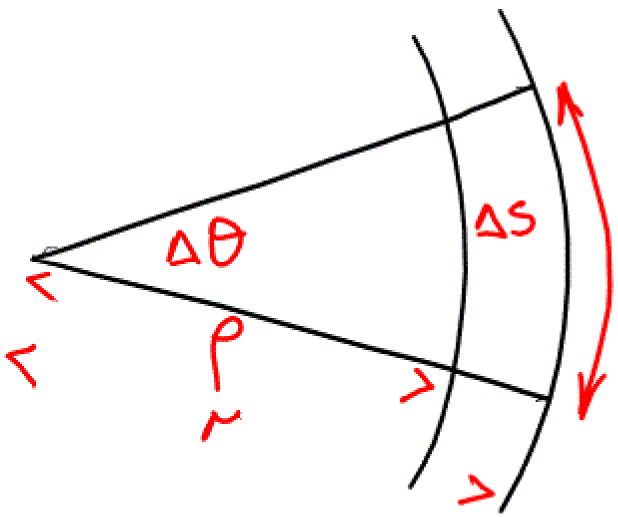
EQUATION OF MOTION $(\ddot{r} - r\dot{\theta}^2) = -\dfrac{e v_s B_y}{\gamma m}$

FOR $v_x \ll v_s$; $v_y \ll v_s$ $p \rightarrow \gamma m v_s$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{e v_s^2 B_y}{p}$$

IN AN ACCELERATOR MORE INTERESTED IN HOW MOTION VARIES ALONG PERIODIC ORBIT.

CHANGE VARIABLE $t \rightarrow s$



$$\Delta S = \rho \, \Delta\theta$$

$$v_s \Delta t = r \, \Delta\theta$$

$$\Delta t = \frac{r}{v_s} \cdot \Delta\theta$$

$$\Delta t = \frac{r}{v_s} \cdot \frac{\Delta s}{\rho}$$

$$\frac{1}{\Delta t^2} = v_s^2 \left(\frac{\rho}{r}\right)^2 \frac{1}{\Delta S^2}$$

$$\frac{d}{dt} = \frac{ds}{dt} \cdot \frac{d}{ds}$$

HAD $\quad \dfrac{d^2 r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 = -\dfrac{e v_s^2}{p} B_y$

$\rightarrow r = \rho + x \quad \rightarrow \rho$ IS CONSTANT

$$\frac{d\theta}{dt} = \frac{v_s}{r}$$

$$\frac{d^2 x}{dt^2} - r\left(\frac{v_s}{r}\right)^2 = -e\frac{v_s^2}{p} \cdot B_y$$

$\dfrac{1}{\Delta t^2} = v_s^2 \left(\dfrac{\rho}{r}\right)^2 \dfrac{1}{\Delta s^2} \longrightarrow \cancel{v_s^2}\dfrac{\rho^2}{r^2} \cdot \dfrac{d^2 x}{ds^2} - r\dfrac{\cancel{v_s^2}}{r^2} = -\dfrac{e\cancel{v_s^2}}{p} B_y$

$\dfrac{e}{p} = \dfrac{1}{B\rho} \rightarrow \quad \dfrac{d^2 x}{ds^2} - \dfrac{r}{\rho^2} = -\dfrac{r^2}{\rho^2} \cdot \dfrac{B_y}{B\rho}$

$r = \rho + x \rightarrow \quad \dfrac{d^2 x}{ds^2} - \left(\dfrac{\rho + x}{\rho^2}\right) = -\left(1 + \dfrac{x}{\rho}\right)^2 \dfrac{B_y}{\rho B}$

EQUATION OF MOTION IN x - DIRECTION

y – DIRECTION, HAD!

$$\ddot{\vec{R}} = (\ddot{r} - r\dot{\theta}^2)\hat{x} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s} + \ddot{y}\,\hat{y}$$

$$F = \gamma m \ddot{y}$$

$$e(\bar{v} \times \bar{B})_y = +e\, v_s B_x \hat{y} = \gamma m\, \ddot{y}\, \hat{y}$$

$$v_x, v_y \ll v_s \qquad p \to \gamma m v_s$$

$$\frac{d^2 y}{dt^2} = +\frac{e\, v_s^2}{p} \cdot B_x$$

CHANGE $dt^2 \to ds^2$

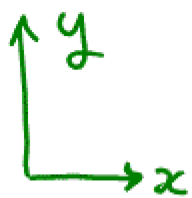
$$\frac{d^2 y}{ds^2} = +\left(1 + \frac{x}{\rho^2}\right)\frac{B_x}{B\rho}$$

EQUATION OF MOTION IN y – DIRECTION

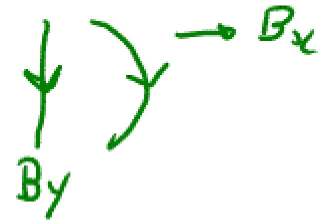
IN ENGINEERING TERMS, THE MAGNETIC FIELDS CAN BE QUITE COMPLEX – NONLINEAR BUT ASSUME LINEAR FOR SIMPLICITY

$$B_x = B_x(0,0) + \frac{\partial B_x}{\partial y}\cdot y + \frac{\partial B_x}{\partial x}\cdot x$$

$$B_y = B_y(0,0) + \frac{\partial B_y}{\partial x}\cdot x + \frac{\partial B_y}{\partial y}\cdot x$$



$B_x(0,0) = 0$



; $B_y(0,0) \rightarrow B$

DON'T WANT x & y MOTION COUPLED

$\therefore$ ASSUME $\dfrac{\partial B_x}{\partial x} = \dfrac{\partial B_y}{\partial y} = 0$

$$B_x = \frac{\partial B_x}{\partial y} \cdot y$$

$$B_y = B + \frac{\partial B_y}{\partial x} \cdot x$$

$$\vec{\nabla} \times \vec{B} = 0 \rightarrow \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) = 0$$

$$B_x = \frac{\partial B_y}{\partial x} \cdot y$$

$$B_y = B + \frac{\partial B_y}{\partial x} \cdot x$$

$$B_y = B + \frac{\partial B_y}{\partial x} \cdot x$$

MOTION IN x - DIRECTION

$$\frac{d^2x}{ds^2} - \frac{\rho + x}{\rho^2} + \left(1 + \frac{x}{\rho}\right)^2 \cdot \frac{1}{B\rho} \qquad = 0$$

FOR SMALL x $\quad \left(1 + \frac{x}{\rho}\right)^2 \sim \left(1 + \frac{2x}{\rho}\right)$

$$\left(B + x\frac{\partial B_y}{\partial x}\right)\left(1 + \frac{2x}{\rho}\right)\frac{1}{B\rho} = \left(B + \frac{2xB}{\rho} + \cancel{2x^2}\frac{\partial B_y}{\partial x} + x\frac{\partial B_y}{\partial x}\right)\frac{1}{B\rho}$$

$$\frac{d^2x}{ds^2} - \frac{1}{\rho} - \frac{x}{\rho^2} + \frac{1}{\rho} + \frac{2x}{\rho^2} + \frac{x}{B\rho}\frac{\partial B_y}{\partial x}$$

FOR SMALL OSCILLATIONS, x & y EQUATIONS OF MOTION

x $\leftrightarrow$
$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2} + \frac{1}{B\rho}\frac{\partial B_y(s)}{\partial x}\right]\cdot x = 0 \quad (1)$$

$\updownarrow$ y
$$\frac{d^2y}{ds^2} - \frac{1}{B\rho}\cdot\frac{\partial B_y}{\partial x}\cdot y = 0 \quad (2)$$

IN (1) GO BACK TO FORM

$$\frac{d^2x}{ds^2} - \frac{\rho + x}{\rho^2} = \frac{-B_y}{B\rho}\left(1 + \frac{x}{\rho}\right)^2$$

ON EQUILIBRIUM ORBIT $\frac{d^2x}{ds^2} = 0$; $x = 0$

$$\frac{1}{\rho} = \frac{B_y}{B\rho} \longrightarrow \frac{1}{\rho} = \frac{e}{p}\cdot B_y \longrightarrow$$ CIRCULAR MOTION IN DIPOLE FIELD

EQUATIONS OF MOTION

x $\longleftrightarrow$

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2} + \frac{1}{B\rho}\frac{\partial B_y(s)}{\partial x}\right]\cdot x = 0 \qquad (1)$$

OSCILLATIONS $\longrightarrow$ $\frac{1}{\rho^2} \ll \frac{1}{B\rho}$

$\updownarrow$ y

$$\frac{d^2y}{ds^2} - \frac{1}{B\rho}\cdot\frac{\partial B_y}{\partial x}\cdot y = 0 \qquad (2)$$

BOTH ARE OF FORM

$$x'' + K(s)\,x = 0$$

HILL'S EQUATION

SIMPLE HARMONIC MOTION WITH VARIABLE SPRING CONSTANT

TRANSVERSE OSCILLATIONS VARY IN AMPLITUD & FREQUENCY ALONG ORBIT

OFF MOMENTUM TRAJECTORIES

DERIVED EQUATION FOR RADIAL MOTION

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho} + \frac{1}{(B\rho)}\frac{\partial B_y}{\partial x}\right] \cdot x(s) = 0$$

THIS ASSUMES THAT WHILE A PARTICLE MAY NOT BE ON THE EQUILIBRIUM ORBIT – IT DOES HAVE THE EQUILIBRIUM MOMENTUM

ASSUMING LINEAR BEHAVIOUR, WE GOT

$$\frac{d^2x}{ds^2} - \frac{\rho+x}{\rho^2} = -\frac{B_y}{B\rho}\left(1+\frac{x(s)}{\rho}\right)^2 = 0$$

REWRITE

$$\frac{d^2x}{ds^2} - \frac{\rho+x}{\rho^2} = -B_y \cdot \frac{e}{p_0}\left(1+\frac{x}{\rho}\right)^2 \quad \text{— (1)}$$

IN DISCUSS OF WEAK FOCUSING, FOR A SMALL EXCURSION FROM THE EQUILIBRIUM ORBIT

RESTORING FORCE

$$F_x = \frac{\gamma m v^2}{\rho} - e \cdot v \cdot B_y \qquad \text{MKS}$$

$$F_x = \frac{p_o v}{\rho} - e \cdot v \cdot B_y$$

$$\frac{F_x}{p_o v} = \frac{1}{\rho} - B_y \cdot \frac{e}{p_o}$$

ASSUME $F_x \propto x$ $\quad \frac{k \cdot x}{p_o v} = \frac{1}{\rho} - \frac{e}{p_o} B_y$

ASSUME $p_o v$ CONSTANT → SLOW ACCELERATION

→ ABSORB → $k \cdot x = \frac{1}{\rho} - \frac{e}{p_o} \cdot B_y$

$$\frac{e}{p_o} \cdot B_y = \frac{1}{\rho} - k \cdot x \quad \text{— (2)}$$

$$\frac{e}{p_0} \cdot B_y = \frac{1}{\rho} - k \cdot x$$

CONSIDER A PARTICLE AT x WHICH DOES NOT HAVE EQUILIBRIUM MOMENTUM p_0

$$p = p_0 + \Delta p \,, \quad p = p_0\left(1 + \frac{\Delta p}{p_0}\right) \,, \quad \frac{1}{p} = \frac{1}{p_0}\left(1 - \frac{\Delta p}{p}\right) \quad \Delta p \ll p_0$$

TALKING ABOUT OFF MOMENTUM PARTICLE, SO

$$\frac{d^2x}{ds^2} - \frac{1}{\rho} - \frac{x}{\rho^2} = -B_y \frac{e}{p}\left(1 + \frac{x}{\rho}\right)^2$$

$$\frac{d^2x}{ds^2} - \frac{1}{\rho} - \frac{x}{\rho^2} = -B_y \frac{e}{p_0}\left(1 - \frac{\Delta p}{p_0}\right)\left(1 + \frac{x}{\rho}\right)^2$$

$$\frac{d^2x}{ds^2} - \frac{1}{\rho} - \frac{x}{\rho^2} = -\left(\frac{1}{\rho} - kx\right)\left(1 - \frac{\Delta p}{p_0}\right)\left(1 + \frac{x}{\rho}\right)^2$$

$$\frac{d^2x}{ds^2} - \frac{1}{\rho} - \frac{x}{\rho^2} = -\left(\frac{1}{\rho} - kx\right)\left(1 - \frac{\Delta p}{p_0}\right)\left(1 + \frac{x}{\rho}\right)^2$$

KEEP LINEAR TERMS, $p_0 \to p$

$$\frac{d^2x}{ds^2} + \left(\frac{1}{\rho^2} - k\right)x = \frac{1}{\rho}\frac{\Delta p}{p} \qquad (4)$$

COMPARE TO

$$\frac{d^2x}{ds^2} + \left(\frac{1}{\rho^2} + \frac{1}{(B\rho)}\frac{\partial B_y}{dx}\right)x(s) = 0$$

WHICH IS

$$\frac{d^2x}{ds^2} + \left(\frac{1}{\rho^2} + k\right)x = 0$$

HAVE EQUATION OF MOTION FOR OFF MOMENTUM PARTICLE

TRANSFER MATRICES FOR MAGNETS

$$x''(s) + \left[\frac{1}{\rho(s)} - k(s)\right] x(s) = \frac{1}{\rho(s)} \frac{\Delta p}{p}$$

SIGN OF k is ARBITRARY

ASSUME INSIDE A MAGNET $\frac{1}{\rho}$, k CONSTANT

$\Delta p/p = 0$

QUADRUPOLE: $\frac{1}{\rho} = 0$ k CONSTANT

$$x''(s) - k\, x(s) = 0$$

$k < 0$ = FOCUSSING

FOR $k < 0$ $x''(s) + k\, x(s) = 0$ SHM

SOLUTION IS

$$x(s) = A \cos\left(\sqrt{k} \cdot s\right) + B \sin\left(\sqrt{k} \cdot s\right)$$

OSCILLATING SOLUTION = FOCUSING

FOR $k > 0$ "EXPONENTIAL" SOLUTION — DEFOCUS

$$x(s) = A \cosh\left(\sqrt{k} \cdot s\right) + B \sinh\left(\sqrt{k} \cdot s\right)$$

$$x'(x) = \sqrt{k} \cdot A \cdot \cosh\left(\sqrt{k} \cdot s\right) + \sqrt{k} \cdot B \cosh\left(\sqrt{k} \cdot s\right)$$

ASSUME PARTICLE STARTS AT $s=0$ AND HAS DISPLACEMENT AND ANGLE x_0 x_0'

$$\bar{X}_0 = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}$$

APPLY INITIAL CONDITIONS

$$x(s) = x_0 \cosh \sqrt{k} \cdot s + \frac{x_0'}{\sqrt{k}} \sinh \sqrt{k} \cdot s$$

$$x'(x) = x_0 \sqrt{k} \sinh \sqrt{k} \cdot s + \underline{x_0'} \cosh \sqrt{k} \cdot s$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos \Omega & \frac{1}{\sqrt{k}} \sinh \Omega \\ \sqrt{k} \sinh \Omega & \cosh \Omega \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \qquad \Omega = \sqrt{k} \cdot s$$

$$M = \begin{cases} \begin{pmatrix} \cos\Omega & \frac{1}{\sqrt{k}}\sin\Omega \\ -\sqrt{|k|}\sin\Omega & \cos\Omega \end{pmatrix} & k < 0 \quad \text{FOCUS} \\[2ex] \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} & k = 0 \quad \text{DRIFT} \\[2ex] \begin{pmatrix} \cosh\Omega & \frac{1}{\sqrt{k}}\sinh\Omega \\ \sqrt{k}\sinh\Omega & \cosh\Omega \end{pmatrix} & k > 0 \quad \text{DEFOCUS} \end{cases}$$

note $\det M = 1$

DIPOLE MAGNET: $k = 0$ & $\frac{\Delta p}{p} = 0$

FROM ④ $x''(s) + \frac{1}{\rho(s)} \cdot x = 0$

THIS AGAIN IS JUST SHM $\omega^2 = \frac{1}{\rho^2}$

SO SOLUTION SHOULD LOOK LIKE

$$x(s) = A\cos\frac{s}{\rho} + B\sin\frac{s}{\rho}$$

APPLY INITIAL CONDITIONS

$$M_{DIPOLE} = \begin{pmatrix} \cos\frac{s}{\rho} & \rho\sin\frac{s}{\rho} \\ -\frac{1}{\rho}\sin\frac{s}{\rho} & \cos\frac{s}{\rho} \end{pmatrix}$$

THIS IS ACTUALLY FOCUSING

— INTERESTING

THIS IS A DIFFERENT WAY OF SEEING THE CONDITIONS FOR WEAK FOCUSING.

EQUATIONS OF MOTION ARE

$$\ddot{x} + \left(\frac{1}{\rho^2} - k\right) x(s) = 0$$

$$\ddot{y}(s) + k\, y(s) = 0$$

DEFINE $n = \rho^2 k$

$$x'' + \left(\frac{k}{n} - k\right) x = x'' + \left(\frac{1-n}{n}\right) k\, x = 0$$

$$y'' + \frac{n}{\rho^2} y = 0$$

$$\left.\begin{array}{l} \text{IF } \left(\frac{1-n}{n}\right) > 0 \text{ FOCUSING IN } x \\ n > 0 \text{ FOCUSING IN } y \end{array}\right\} \text{WEAK FOCUSING CONDITION}$$

DISPERSION & MOMENTUM COMPACTION

SO FAR HAVE ONLY LOOKED AT PARTICLES ON THE EQUILIBRIUM MOMENTUM.

WHAT ABOUT $\Delta p/p \neq 0$

IN A PERFECT DIPOLE $k=0$, THEN (4) $\rightarrow$

$$x''^{2} + \frac{1}{\rho^2} \cdot x(s) = \frac{1}{\rho} \Delta p/p$$

DEFINE A SPECIAL TRAJECTORY $\Delta p/p = 1$ AND CALL THIS $D(s) \rightarrow$ DISPERSION FUNCTION

$$D''(s) + \frac{1}{\rho^2} D(s) = \frac{1}{\rho}$$

UM – AN INHOMOGENEOUS EQUATION

$$D''(s) + \frac{1}{\rho_2} D(s) = \frac{1}{\rho} \quad — ①$$

HAVE ALREADY SOLVED HOMOGENEOUS :-

$$D(s) = A\cos\frac{s}{\rho} + B\sin\frac{s}{\rho}$$

$$D'(s) = A\sin\frac{s}{\rho} + B\cos\frac{s}{\rho}$$

TO SOLVE THE INHOMOGENEOUS EQUATION, WE JUST NEED TO FIND ONE PARTICULAR SOLUTION & ADD IT TO THE INHOMOGENEOUS SOLUTION.

ONE SOLUTION IS CLEARLY $D_p = C$ ← CONSTANT

ρ IS CONSTANT

FROM ①

$$0 + \frac{1}{\rho^2} D_p = \frac{1}{\rho} \rightarrow 0 + \frac{1}{\rho^2} \cdot C = \frac{1}{\rho}$$

$$C = \rho$$

SO GENERAL SOLUTION OF INHOMOGENEOUS

$$D(s) = A\cos\frac{s}{\rho} + B\sin\frac{s}{\rho} + \rho$$

$$D'(s) = -\frac{A}{\rho}\sin\frac{s}{\rho} + B\cos\frac{s}{\rho}$$

AT $s = 0$ $D(0) = D_0$; $D'(0) = D_0'$

$$A = D_0 - \rho \quad ; \quad B = \rho D_0'$$

$$D(s) = D_0\cos\frac{s}{\rho} + D_0'\rho\sin\frac{s}{\rho} + \rho\left(1 - \cos\frac{s}{\rho}\right)$$

$$D'(s) = -\frac{D_0}{\rho}\sin\frac{s}{\rho} + D_0'\cos\frac{s}{\rho} + \sin\frac{s}{\rho}$$

THIS CAN BE WRITTEN

IN MATRIX FORM

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\frac{s}{\rho} & \rho\sin\frac{s}{\rho} & \rho\left(1-\cos\frac{s}{\rho}\right) \\ -\frac{1}{\rho}\sin\frac{s}{\rho} & \cos s/\rho & \sin s/\rho \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

THE SAME MATRIX ALSO GIVES EVOLUTION OF THE VECTOR

$$\begin{pmatrix} x(s) \\ x'(s) \\ \Delta p/p \end{pmatrix}$$

→ CONVINCE YOURSELF THAT THIS IS TRUE

A PARTICLE WITH THE NOMINAL BEAM MOMENTUM OBEYS

$$x(s)_N = x_0 \cos\frac{s}{\rho} + \rho\, x_0' \sin\frac{s}{\rho}$$

AND AN OFF MOMENTUM PARTICLE OBEYS

$$x(s)_D = x_0 \cos\frac{s}{\rho} + \rho\, x_0' \sin\frac{s}{\rho} + \rho\,\frac{\Delta p}{p}\left(1-\cos\frac{s}{\rho}\right)$$

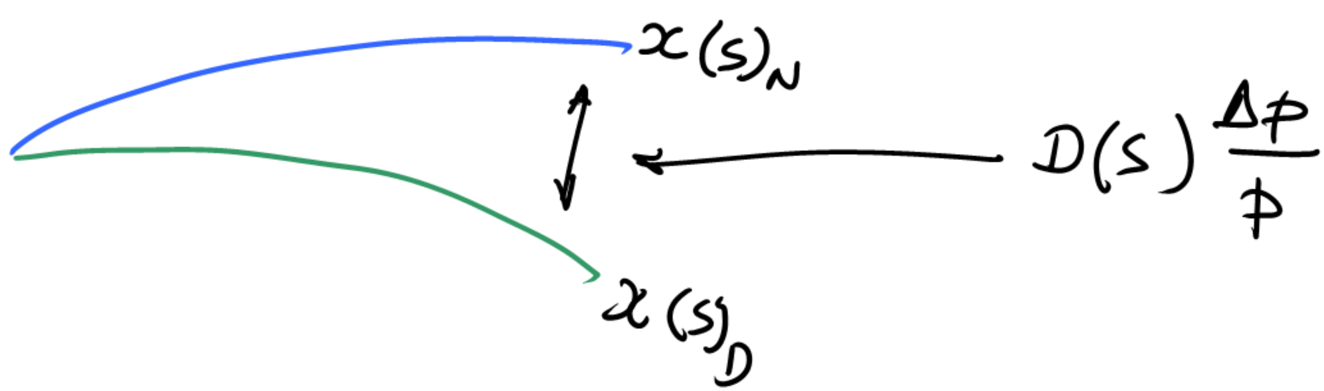
IF WE ASSUME THAT BOTH PARTICLES ARE AT THE SAME POINT AT $s=0$, THEN $D(0)=D'(0)=0$

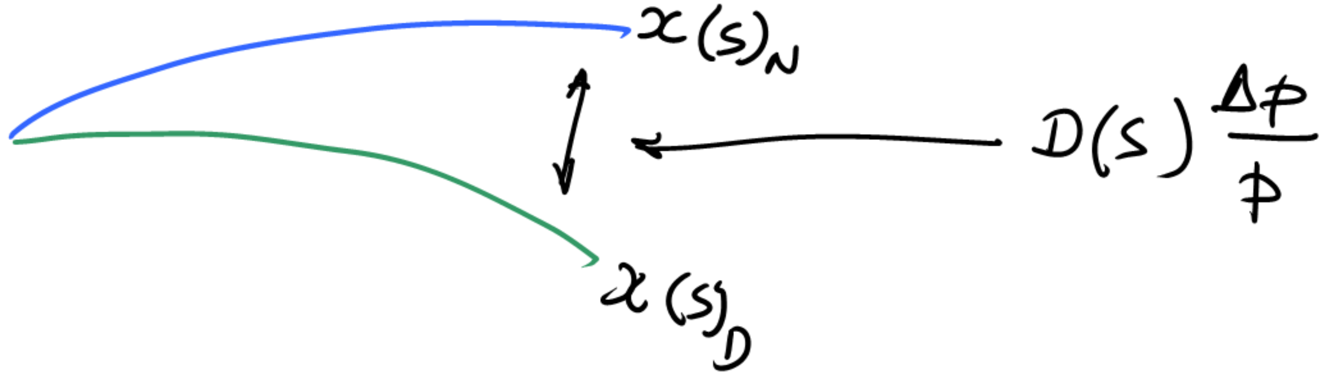
AND

$$D(s) = \rho\left(1-\cos\frac{s}{\rho}\right)$$

SO $$x(s)_D = x(s)_N + D(s)\,\frac{\Delta p}{p}$$

DISPERSION FUNCTION





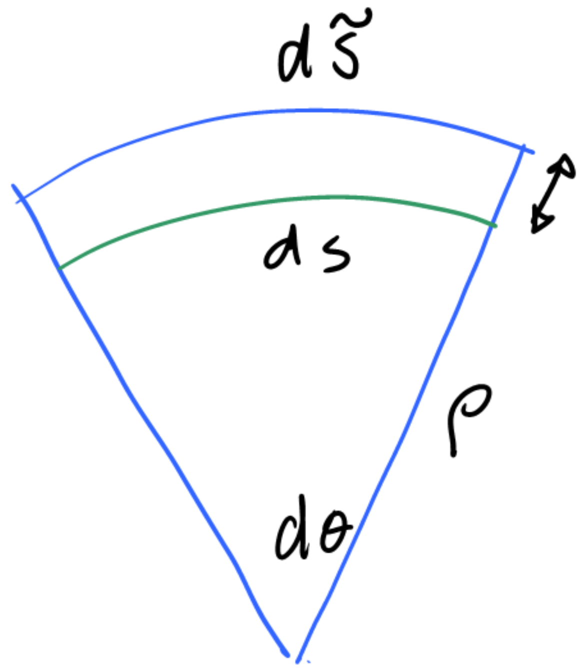
IF WE REFER EVERYTHING TO $x(s)_N$, I.E. THE EQUILIBRIUM ORBIT, THEN THE TRAJECTORY OF THE OFF MOMENTUM PARTICLE IS

$$x(s)_D = D(s)\,\Delta p/p$$

THIS WILL NOT, IN GENERAL HAVE SAME ORBIT LENGTH AS REFERENCE PARTICLE

THE RATIO OF CHANGE IN ORBIT LENGTH TO CHANGE IN MOMETUM IS MOMENTUM COMPACTION

$$\alpha = \frac{\Delta L/L}{\Delta p/p}$$



$x = \frac{D \Delta p}{p}$

BENDING MAGNETS PRODUCE A DIFFERENCE IN PATH LENGTH

$$\frac{d\tilde{s}}{\rho + x} = \frac{ds}{\rho} \; ; \quad d\tilde{s} = \frac{\rho + x}{\rho} ds$$

PATH LENGTH OF OFF MOMENTUM PARTICLE

$$L = L_0 + \Delta L = \oint \frac{\rho + x_D}{\rho} ds = \oint ds + \frac{\Delta p}{p} \oint \frac{D(s)}{\rho(s)} ds$$

$$\Delta L = \frac{\Delta p}{p} \oint \frac{D(s)}{\rho(s)} ds$$

MOMENTUM COMPACTION

$$\alpha = \frac{\Delta L / L}{\Delta p / p} = \frac{1}{L_0} \oint \frac{D(s)}{\rho(s)} ds$$

ANALYTICAL SOLUTION OF HILL'S EQUATION

WANT SOME FORMALISM WHICH DOES NOT JUST DESCRIBE ONE PARTICLE

- WANT TO DESCRIBE BEAM ENVELOPE → PHASE AMPLITUDE

$$x'' + K(s)\,x = 0$$

FOR A PERIODIC STRUCTURE $K(s) = K(s+c)$

SOLUTION $x = A\,\omega(s)\cos\left(\phi(s) + \delta\right)$

A, δ → INITIAL CONDITIONS

K CONSTANT → SHM $x = A\cos\left(\phi(s) + \delta\right)$

- A CONSTANT
- $\phi(s) = s\sqrt{K}$ INCREASES LINEARLY

HERE A VARIES WITH s, $\phi(s)$ DOES NOT INCREASE LINEARLY WITH s

TRY SOLUTION $x(s) = A\omega(s)\cos(\phi(s)+\delta)$

IN $\ddot{x} + K(s)\cdot x = 0$

$$\ddot{x} + Kx = -A(2\dot{\omega}\phi' + \omega\phi'')\sin(\phi+\delta) +$$

$$A(\omega'' - \omega\phi'^2 + K\omega)\cos(\phi+\delta) = 0$$

IF δ ARBITRARY, COEFFICIENTS OF SIN, COS $= 0$

$$\omega(2\dot{\omega}\phi' + \omega\phi'') = 0 \rightarrow (\omega^2\phi')' = 0$$

$$\phi' = k/\omega^2$$

k CONSTANT OF INTEGRATION

ABSORB INTO A, $k \rightarrow 1$

$$\Phi' = \frac{1}{\omega^2}$$

COEFFICIET OF COSINE TERM

$$\omega'' - \omega\phi'^2 + K(s)\omega \rightarrow \omega'' - \omega \cdot \frac{1}{\omega^4} + K(s)\omega = 0$$

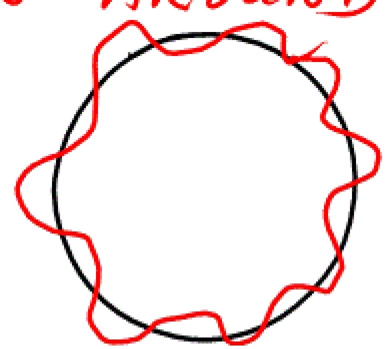
$$\omega^3\left(\omega'' + K(s)\omega\right) = 1$$

NOW HAVE AN EQUATION FOR $\omega(s)$ AS A FUNCTION OF POSITION AROUND ACCELERATOR RING

$$x(s) = A\,\omega(s)\cos\left(\phi(s) + \delta\right)$$

$$\omega'' + K(s)\omega = \frac{1}{\omega^3} \quad ; \quad \omega^2\phi' = 1$$

AMPLITUDE OF TRANSVERSE OSCILLATIONS AS A FUNCTION OF POSITION AROUND ACCELERATOR RING



$$\omega'' + K(s)\,\omega = \frac{1}{\omega^3} \quad ; \quad \omega^2 \phi' = 1$$

DEFINE COURANT - SNYDER PARAMETERS

$$\beta(s) = \omega^2(s)$$ AMPLITUDE FUNCTION

$$\alpha(s) = -\frac{1}{2}\beta'(s)$$

$$\gamma(s) = \left(1 + \alpha^2(s)\right) / \beta(s)$$

SO $\phi'(s) = \dfrac{1}{\beta(s)} \rightarrow \Delta\phi(s) = \displaystyle\int_0^s \frac{ds}{\beta(s)}$

$$x = A\,\omega(s)\cos\left(\phi(s) + \delta\right)$$

PHASE ADVANCE ALONG PATH S.

$\omega^2 \rightarrow$ OSC FREQUENCY

$2\pi\beta \rightarrow$ OSCILLATION WAVE LENGTH

$\lambda = 2\pi\beta(s)$

HAD $\quad \omega'' + K(s)\,\omega - \frac{1}{\omega^3} = 0$

$$\omega = \sqrt{\beta} \longrightarrow \omega' = \frac{1}{2\sqrt{\beta}}\,\beta'$$

$$\omega'' = -\frac{1}{4}\,\beta^{-3/2}(\beta')^2 + \frac{1}{2}\,\beta^{-\frac{1}{2}}\beta''$$

$$\rightarrow \quad \frac{1}{2}\,\beta\,\beta'' - \frac{1}{4}(\beta')^2 + K\beta^2 - 1 = 0$$

$$\left(2\beta\beta'' - \beta'^2 + 4K\beta^2\right) = 4$$

$$K\beta = \gamma + \alpha'$$

FOR A PERIODIC ACCELERATOR PARTICLE ORBITS ARE SOLUTIONS OF

$$\left.\begin{aligned} x'' + K(s)\,x &= 0 \\ y'' + K(s)\,y &= 0 \end{aligned}\right\} \quad 1$$

AND $$2\beta\beta'' - \beta'^2 + 4\beta^2 K(s) = 4 \qquad 2$$

1 DESCRIBES INDIVIDUAL PARTICLE

2 DESCRIBES BEAM ENVELOPE

CALL x or y y

$y'' + K(s)\,y = 0$ SOLN $y(s) = A\sqrt{\beta(s)}\cos(\phi(s)+\delta)$ ①

DIFFERENTIATE SOLUTION:

$$y'(s) = A\frac{d}{ds}\sqrt{\beta}\cos(\phi+\delta) + A\sqrt{\beta}\frac{d}{ds}\{\cos(\phi+\delta)\}$$

$\alpha(s) = -\frac{1}{2}\frac{d\beta}{ds}$, $\frac{d\sqrt{\beta}}{ds} = -\frac{1}{\sqrt{\beta}}\cdot\alpha$

DROP δ FOR NOW

$$y' = -\frac{A\alpha}{\sqrt{\beta}}\cos\phi - A\sqrt{\beta}\frac{d\phi}{ds}\sin\phi$$

$\frac{d\phi}{ds} \leftarrow \frac{1}{\beta}$

$$y' = -\frac{A\alpha}{\sqrt{\beta}}\cos\phi - \frac{A}{\sqrt{\beta}}\sin\phi$$

$$\beta y' + \alpha y = -A\sqrt{\beta}\,\alpha\cos\phi - A\sqrt{\beta}\sin\phi + A\sqrt{\beta}\,\alpha\cos\phi$$

$$= -A\sqrt{\beta}\sin\phi \quad (2)$$

TAKE $(1)^2 + (2)^2$

$$(\beta y' + \alpha y)^2 + y^2 = A^2\beta\sin^2\phi + A^2\beta\cos^2\phi$$

$$\beta^2 y'^2 + \alpha^2 y^2 + y^2 + 2\alpha\beta y' y = A^2\beta$$

$$\underbrace{\frac{1+\alpha^2}{\beta}}_{\gamma} y^2 + \beta y'^2 + 2\alpha y' y = A^2$$

$$\beta y'^2 + 2\alpha y y' + \gamma y^2 = A^2$$

COURANT – SNYDER INVARIANT — BEAM EMITTANCE

$$(1)\quad \underbrace{\beta y'^2 + 2\alpha y y' + \gamma y^2 = A^2}_{\text{THIS IS AN ELLIPSE IN } y\, y' \text{ SPACE}} \leftarrow$$ INITIAL CONDITIONS INVARIANT OF MOTION

THIS IS AN ELLIPSE IN $Y\ Y'$ SPACE

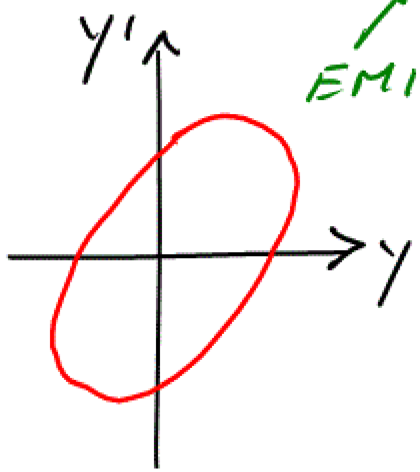
ELLIPSE $\quad a x^2 + 2bxy + c y^2 = d$

$$\text{AREA} = \pi d \Big/ \sqrt{ac - b^2}$$

IN (1) $\quad \text{AREA} = \pi A^2 \Big/ \underbrace{\sqrt{\beta\gamma - \alpha^2}}_{=1} = \pi A^2$

DEFINE $\varepsilon = \pi A^2$

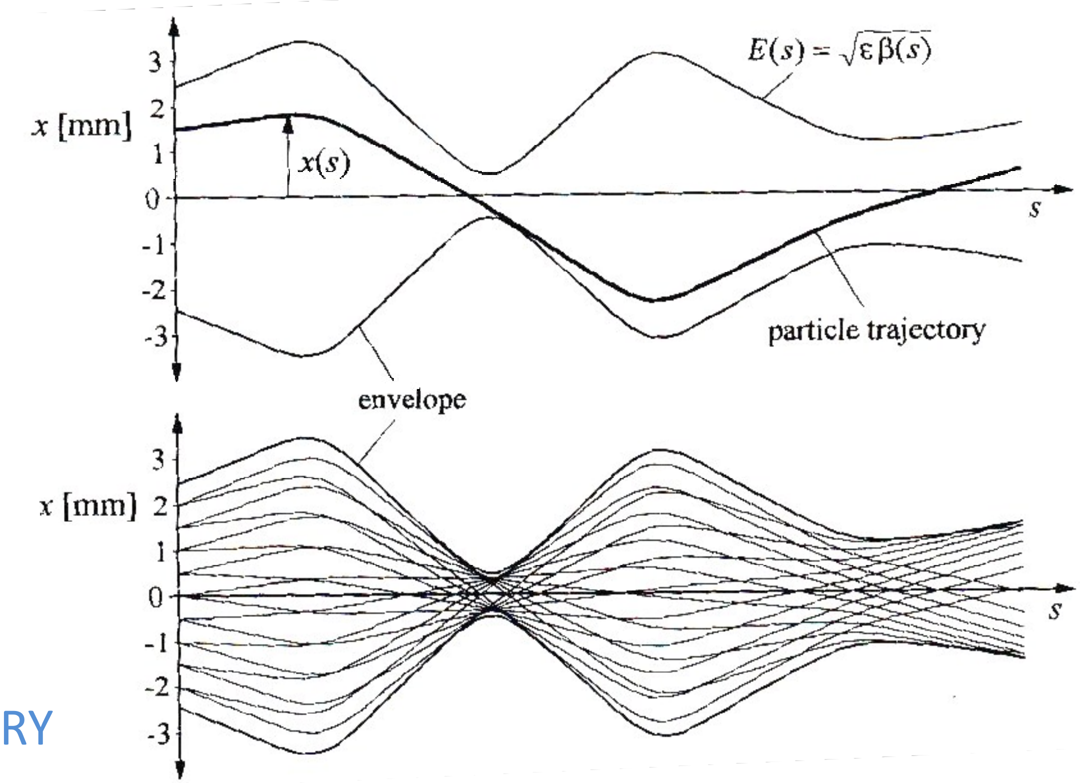
EMITTANCE



$$\beta y'^2 + 2\alpha y y' + \gamma y^2 = \frac{\varepsilon}{\pi}$$

$$y(s) = A\,w(s)\cos(\phi(s) + \delta)$$

$$y(s) = \sqrt{\frac{\varepsilon \beta(s)}{\pi}}\cos(\phi(s) + \delta)$$



TRAJECTORY

$$\frac{d^2Y}{ds^2} + K(s)Y = 0$$

$$Y(s) = A\omega(s)\cos(\phi(s) + \delta)$$

$$Y(s) = \sqrt{\frac{\varepsilon}{\pi}\beta(s)}\cos(\phi(s) + \delta)$$

BEAM ENVELOPE

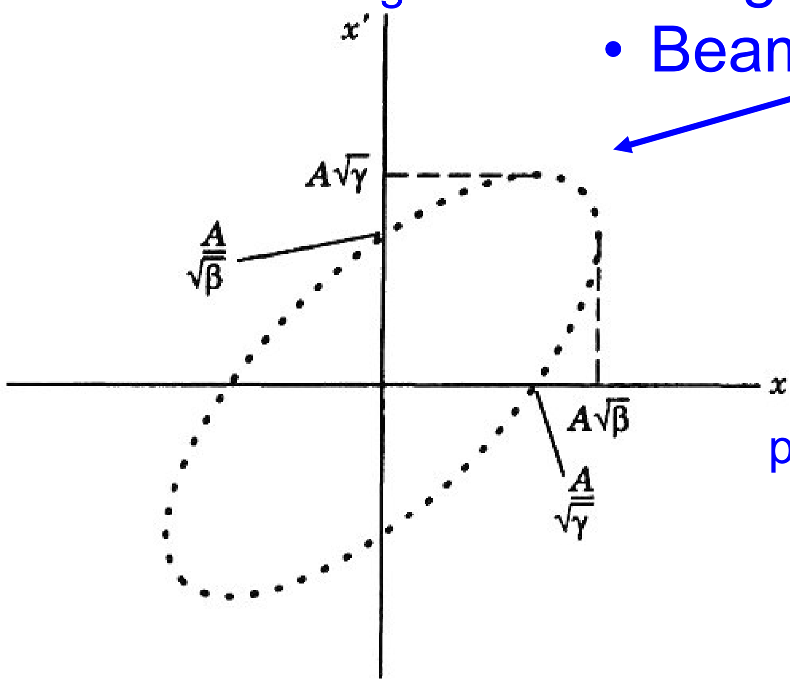
$$\frac{d^2\omega}{ds^2} + K(s)\omega = \frac{1}{\omega^3}$$

$$\beta(s) = \omega^2(s)$$

Amplitude of betatron oscillations

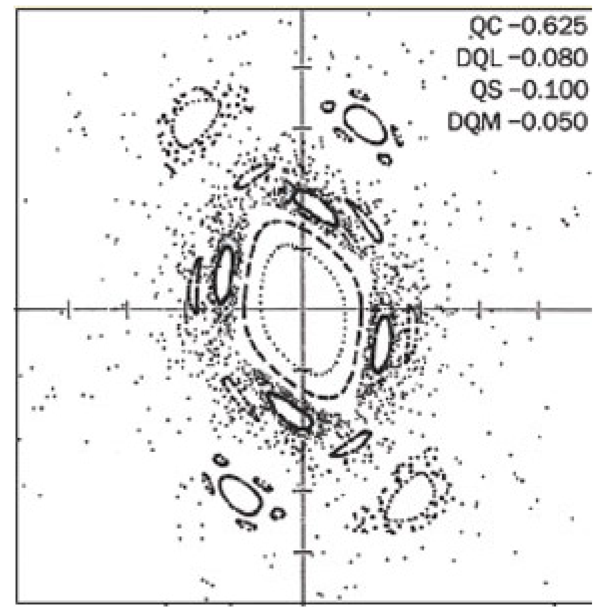
- Single Particle Phase Space
- Beam Envelope

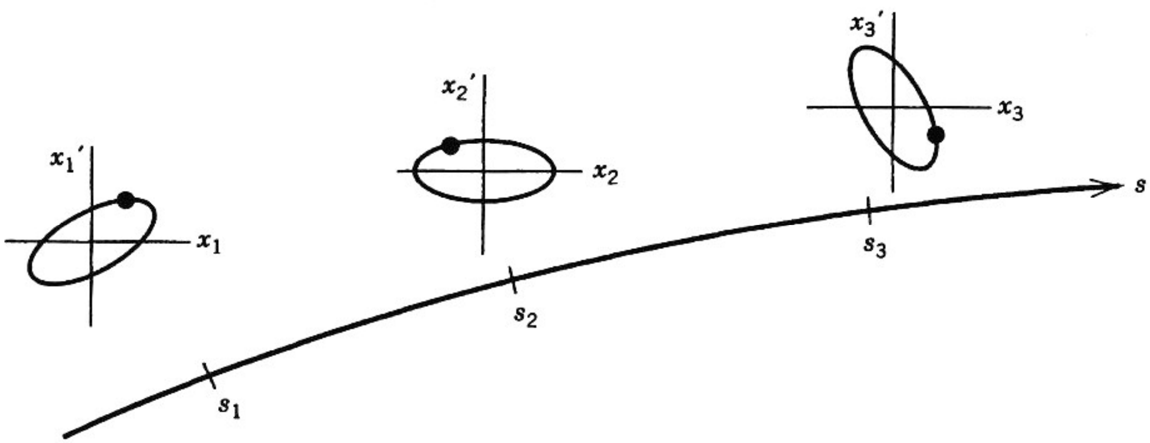
angle



position

- Real Accelerator
- Non-linear





Shape of phase space changes along accelerator lattice
Area constant -> Liouville

FROM ELLIPSE PLOT, AT ANY POINT IN ACCELERATOR MAXIMUM $x = A\sqrt{\beta}$

AT SOME POINT IN THE LATTICE WILL HAVE APERTURE $2a$ WHICH DEFINES MAXIMUM ELLIPSE WHICH WILL FIT THRU MACHINE ADMITTANCE

$$2a(s) = 2A\sqrt{\beta} \rightarrow A^2 = a^2(s)/\beta$$

$$\beta y'^2 + 2\alpha y y' + \gamma y^2 = \frac{\varepsilon}{\pi} = A^2$$

$$\text{ADMITTANCE} = \frac{\pi a^3}{\beta_{MAX}}$$

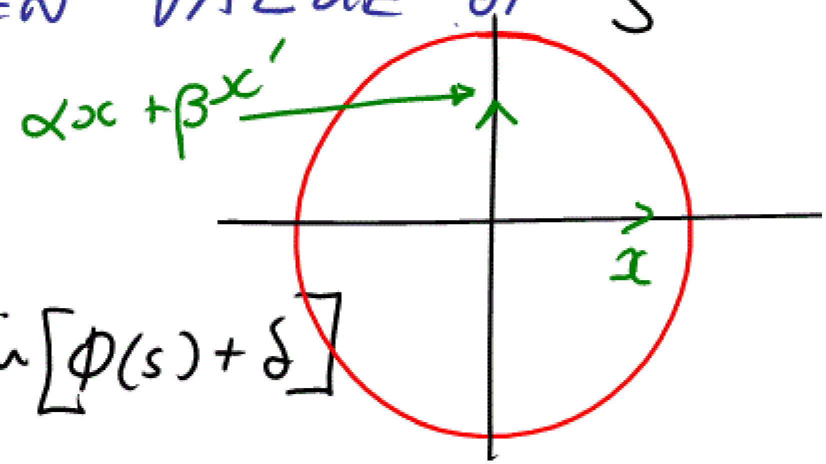
IN A REAL ACCELERATOR THERE ARE MANY PARTICLES

USUALLY ADMITTANCE / EMITTANCE DEFINED IN TERM OF PHASE SPACE BOUNDARY WHICH CONTAINS FRACTION F OF BEAM

NUMBER DISTRIBUTION ASSUMED GAUSSIAN

$$n(x)\,dx = \frac{1}{\sqrt{2\pi}\cdot\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx$$

ASSUME THIS DISTRIBUTION IS CONSTANT IN TIME AT A GIVEN VALUE OF S



$$x(s) = A\sqrt{\beta}\cos[\phi(s)+\delta]$$

$$\alpha(s)x(s)+\beta(s)x'(s) = -A\sqrt{\beta}\sin[\phi(s)+\delta]$$

IN THIS TRANSFORMED PHASE SPACE WHERE BOUNDARY IS A CIRCLE

$$n(x, \alpha x+\beta x')\,dx\,d(\alpha x+\beta x')$$

$$= \frac{1}{2\pi\sigma^2} \exp\left\{-\left[x^2+(\alpha x+\beta x')^2\right]/2\sigma^2\right\} \, dx \, d(\alpha x+\beta x')$$

CHANGE TO POLAR COORDINATES

$$r^2 = x^2 + (\alpha x + \beta x')^2$$

$$n(r,\theta)\, r\, dr\, d\theta = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \, r\, dr\, d\theta$$

IF FRACTION F OF BEAM IS WITHIN RADIUS a

$$F = \int_0^{2\pi}\int_0^a n\, r\, dr\, d\theta = \int_0^a e^{-r^2/2\sigma^2} \frac{r\,dr}{\sigma^2}$$

$$\int x e^{-x^2} dx = \frac{1}{2} e^{-x^2}$$

$$F = \int_0^a e^{-r^2/2\sigma^2} \frac{r\,dr}{\sigma^2} \rightarrow a^2 = -2\sigma^2 \ln(1-F)$$

AND $$\frac{\beta\varepsilon}{\pi} = \beta\left(\frac{1+\alpha^2}{\beta}\right)x^2 + 2\alpha\beta x x' + \beta^2 x'^2$$

$$= x^2 + (\alpha x + \beta x')^2 = a^2$$

F CONTAINED IN $a^2 = \frac{\beta\varepsilon}{\pi} = -2\sigma^2 \ln(1-F)$

$$\varepsilon = -\frac{2\pi\sigma^2}{\beta} \ln(1-F)$$

ε	E %
σ^2/β	15
$\pi\sigma^2/\beta$	39
$4\pi\sigma^2/\beta$	87

This is just some details

Emittance & Admittance

Admittance is phase space area associated with the largest ellipse the accelerator will accept.

maximum value of x is $A\sqrt{\beta}$

half aperture available to the beam is $a(s)$

somewhere there will be a minimum in $\frac{a(s)}{\sqrt{\beta(s)}}$

$\rightarrow$ ie a maximum in $\sqrt{\beta(s)}$

then the admitance is $\left(\pi \dfrac{a^2}{\beta}\right)_{m}$

$\pi A^2 \rightarrow$ area of ellipse

when there is a minimum in $\dfrac{a}{\sqrt{\beta}}$ $\rightarrow$ ie when

β becomes a maximum with respect to a, the area of the ellipse is

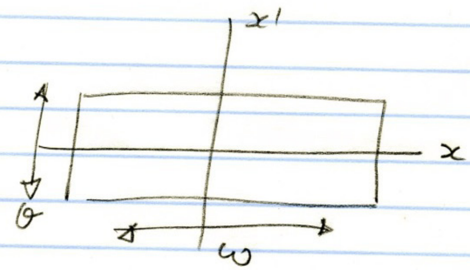
$$\pi A^2 \rightarrow \left(\pi \frac{a^2}{\beta}\right)_{mi}$$

$A = \dfrac{x}{\sqrt{\beta}}$ $\rightarrow$ aperture of mag.

governed by smallest aperture

<u>Emittance</u>

Source has width ω within angle θ



phase space area is $\omega\theta$

phase space is generally not a uniform rectangle.

If a beam in a synchrotron has emittance ε then phase space is bounded by curve

$$\frac{\varepsilon}{\pi} = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

convenient to speak of emittance for a particular particle distribution in terms of RMS beam size

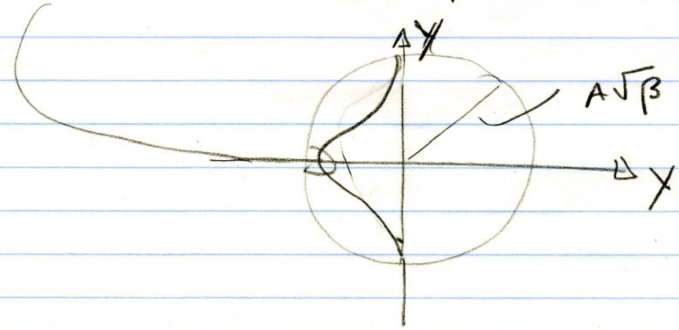
Suppose distribution in transverse coord x is given by density function

$$n(x)\,dx = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx$$

Trajectories in $x - (\alpha x + \beta x')$ phase space are circles

$X = \quad x(s) = A\sqrt{\beta} \cos[\psi(s) + \delta]$

$Y = \quad \alpha(s) x(s) + \beta(s) x'(x) = -A\sqrt{\beta} \sin[\psi(s) + \delta]$



$X^2 + Y^2 = A^2\beta$

distribution in the coordinate $\alpha x + \beta x'$ would be a Gaussian. standard deviation of σ

population in a given circle just rotates thru an angle corresponding to the PHASE ADVANCE

equilibrium distribution is independent of position ~~~~ along circle depends only on radius

2-dimensional phase space distribution

$$n(x, \alpha x + \beta x')\,dx\,d(\alpha x + \beta x')$$

$$= \frac{1}{2\pi\sigma^2} \exp - [x^2 + (\alpha x + \beta x')^2]/2\sigma^2$$

$$dx\,d(\alpha x + \beta x')$$

switch to polar coordinates

$$r = x^2 + (\alpha x + \beta x')^2$$

then distribution is:

$$n(r, \theta)\,r\,dr\,d\theta = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2}\, r\,dr\,d\theta$$

Define a radius a within which a fraction F of the particles are contained

$$F = \int_0^{2\pi}\int_0^a n\,r\,dr\,d\theta = \int_0^a e^{-\frac{r^2}{2\sigma^2}} \frac{r\,dr}{\sigma^2}$$

$$\left[-\frac{1}{2} e^{-\frac{r^2}{2\sigma^2}}\right]_0^a$$

$$-\frac{1}{2} e^{-a^2/2\sigma^2} + 1 = F$$

$$-\frac{1}{2} e^{-a^2/2\sigma^2} = 1 - F$$

$-\frac{1}{2} \ln$

$$a^2 = -2\sigma^2 \ln(1-F)$$

$$\frac{\beta\varepsilon}{\pi} = x^2 + (\alpha x + \beta x')^2$$

If this emittance is the $x - x'$ phase space which contains F, then $\frac{\beta\varepsilon}{\pi} = a$ by def.

$$\pi a^2 = \beta\varepsilon^2 = -2\pi\sigma^2 \ln(1-F)$$

$$\varepsilon = -\frac{2\pi\sigma^2}{\beta} \ln(1-F)$$

$$x_{max} = \sqrt{\frac{\varepsilon \beta_{max}}{\pi}}$$

$$x'_{max} = \sqrt{\frac{\varepsilon \gamma_{max}}{\pi}}$$

IN DISCUSING HILL'S EQUATION USE THE SOLUTION

$$x(s) = A\,\omega(s)\cos\left(\phi(s)+\delta\right);\quad \phi' = \frac{1}{\omega^2(s)}$$

CAN ALSO WRITE A!

$$x = \omega(s)\left(A_1\cos\phi(s) + A_2\sin\phi(s)\right)$$

AND

$$x' = \left(A_1\omega' + A_2/\omega\right)\cos\phi + \left(A_2\omega' - \frac{A_1}{\omega}\right)\sin\phi$$

INITIAL CONDITIONS $x_0,\ x_0',\ s = s_0,\ \phi = 0$

$$A_1 = \frac{x_0}{\omega};\quad A_2 = x_0'\omega - x_0\omega'$$

GOING FROM $S_0 \to S_0 + C$ PERIOD OF SOLUTION $\omega(s_0+c) = \omega(s_0)$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+c} = \begin{pmatrix} \cos\Delta\phi_c - \omega\omega'\sin\Delta\phi_c & \omega^2\sin\Delta\phi_c \\ \frac{-1+(\omega\omega')^2}{\omega^2}\sin\Delta\phi_c & \cos\Delta\phi_c + \omega\omega'\sin\Delta\phi_c \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

IN TERMS OF COURANT-SNYDER, LAST BECOMES

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{S_0+C} = \begin{pmatrix} \cos\Delta\phi_c + \alpha \sin\Delta\phi_c & \beta \sin\Delta\phi_c \\ -\gamma \sin\Delta\phi_c & \cos\Delta\phi_c - \alpha \sin\Delta\phi_c \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{S_0}$$

$$\phi(S_0 \to S_0 + C) \equiv \Delta\phi_c = \int_{S_0}^{S_0+C} \frac{ds}{w^2(s)} \equiv \int_{S_0}^{S_0+C} \frac{ds}{\beta(s)}$$

THIS TRANSFER MATRIX CAN BE WRITTEN

$$M = I \cos\Delta\phi_c + J \sin\Delta\phi_c$$

$$J \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \; ; \quad J^2 = -I$$

OR

$$M = e^{J\Delta\phi_c}$$

SUPPOSE PRODUCT OF ALL TRANSFER MATRICES IN REPEAT PERIOD $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\Delta\phi_c + \alpha \sin\Delta\phi_c & \beta \sin\Delta\phi_c \\ -\gamma \sin\Delta\phi_c & \cos\Delta\phi_c - \alpha \sin\Delta\phi_c \end{pmatrix} \quad (1)$$

$$\cos\Delta\phi_c = \frac{1}{2}(a+d) = \frac{1}{2}\,\mathrm{Tr}\, M$$

IN DISCUSSION OF STABILITY CRITERION IN FODO

$$-1 \leq \frac{1}{2}\mathrm{Tr}\, M \leq 1 \rightarrow -1 \leq \cos\mu \leq 1$$

COMPARING μ IN $e^{i\mu}$ IS $\Delta\phi_c$

$\rightarrow$ PHASE ADVANCE THROUGH REPEAT PERIOD.

FROM 1 ON LAST PAGE

$$\beta = \frac{b}{\sin \Delta\phi_c} \quad ; \quad \alpha = \frac{a-d}{2\sin \Delta\phi_c}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ IS TRANSFER MATRIX AT THIS POINT IN LATTICE

IF ONE KNOWS M CAN GET α, β WHICH ALLOW ONE TO CALCULATE PARTICLE MOTION

EQ M COULD BE

$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \quad \text{OR} \quad \begin{pmatrix} \cos\sqrt{K}L & \frac{1}{\sqrt{K}}\sin(KL) \\ -\sqrt{K}\sin(KL) & \cos\sqrt{K}L \end{pmatrix} \quad \text{OR etc} \ldots$$

DRIFT FOCUSING QUAD

IF ONE DETERMINES β AT EVERY POINT ON LATTICE, MOTION FROM $1 \to 2$.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M(s_1 \to s_2) \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

GET EXPLICIT FORM OF M

START WITH

$$x = w(s)\left(A_1 \cos\psi + A_2 \sin\psi\right)$$

$$x' = \left(A_1 w'(s) + \frac{A_2}{w}\right)\cos\psi + \left(A_2 w'(s) - \frac{A_1}{w(s)}\right)\sin\psi$$

INITIAL CONDITIONS $x_1, x_1'; \ s = s_1$

$$\rightarrow A_1 = \frac{x_1}{w_1}; \quad A_2 = x_1' w_1 - x_1 w_1'$$

$$w_1 = \sqrt{\beta_1} \quad \text{AND} \quad \alpha_1 = -\frac{\beta_1'}{2}$$

$$x_2 = \sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\Delta\psi + \alpha_1 \sin\Delta\psi\right)\cdot x_1 + \sqrt{\beta_1\beta_2}\,\sin\psi \cdot x_1'$$

$$x_2' = -\left\{\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\psi + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\psi\right\}\cdot x_1$$

$$+ \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\Delta\psi - \alpha_2\sin\Delta\psi\right)\cdot x_1'$$

$$M = \begin{pmatrix} \left(\frac{\beta_2}{\beta_1}\right)^{1/2}\left(\cos\Delta\psi + \alpha_1\sin\Delta\psi\right) & \left(\beta_1\beta_2\right)^{\frac{1}{2}}\sin\Delta\psi \\ -\frac{1+\alpha_1\alpha_2}{(\beta_1\beta_2)^{\frac{1}{2}}}\sin\Delta\psi + \frac{\alpha_1-\alpha_2}{(\beta_1\beta_2)^{\frac{1}{2}}}\cos\Delta\psi & \left(\frac{\beta_1}{\beta_2}\right)^{1/2}\left(\cos\Delta\psi - \alpha_2\sin\Delta\psi\right) \end{pmatrix}$$

$$\Delta\psi(s_1 \to s_2) = \int_{s_1}^{s_2}\frac{ds}{\beta(s)}; \quad \nu = \frac{1}{2\pi}\oint\frac{ds}{\beta(s)} =$$ NUMBER OF OSCILLATIONS AROUND ACCELERATOR

ν ↑ TUNE

This is just details

$$\omega^2(\omega'' + K\omega) = k$$

$$x = \omega(s)\,(A_1\cos\psi + A_2\sin\psi)$$

$$x' = \left(A_1\omega' + \frac{A_2 k}{\omega}\right)\cos\psi$$

$$+ \left(A_2\omega' - A_1\frac{k}{\omega}\right)\sin\psi$$

$$\frac{dx}{ds} = \frac{d\omega}{ds}\cdot A_1\cos\psi - \omega(s)A_1\sin\psi$$

$$+ \frac{d\omega}{ds}A_2\sin\psi + \omega(s)A_2\cos\psi$$

$$= \left(A_1\frac{d\omega}{ds} + A_2\omega\psi'\right)\cos\psi$$

$$+ \left(A_2\frac{d\omega}{ds} - \omega\psi' A_1\right)\sin\psi$$

$$\psi' = \frac{k}{\omega^2}$$

$$-\omega(s)A_1\psi'\sin\psi$$

$$\left(A_1\frac{d\omega}{ds} + A_2\frac{k}{\omega}\right)\cos\psi$$

$$+ \left(A_2\frac{d\omega}{ds} - \frac{k}{\omega}A_1\right)\sin\psi$$

initial conditions ②

$$x_0,\ x_0' \qquad s = s_0$$

$$x_0 = \omega(s_0) A_1 \cos 0 \longrightarrow A_1 = \frac{x_0}{\omega}$$

$$x_0' = \left(A_1 \omega' + \frac{A_2 k}{\omega}\right) \cos \phi$$

$$x_0' = x_0 \frac{\omega'}{\omega} + \frac{A_2 k}{\omega}$$

$$x_0' - x_0 \frac{\omega'}{\omega} = A_2 \frac{k}{\omega} \qquad \omega(s_0) = \omega(s)$$

$$A_2 = \frac{\omega}{k} x_0' - \frac{x_0 \omega'}{k} \qquad \begin{pmatrix} \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\boxed{A_2 = \frac{x_0' \omega - x_0 \omega'}{k}}$$

$$x = \omega\left(A_1 \frac{x_0}{\omega} \cos \psi_c + \frac{x_0' \omega - x_0 \omega'}{k} \cdot \sin \psi_c\right)$$

$$x' = \left(\frac{x_0}{\cancel{\omega}} \cdot \omega' + \frac{x_0' \omega - x_0 \omega'}{\cancel{k}} \cdot \frac{k}{\cancel{\omega}}\right) \cos \psi_c$$

$$+ \left[\left(\frac{x_0' \omega - x_0 \omega'}{k}\right) \omega' - \frac{x_0}{\cancel{\omega}} \cdot \frac{k}{\omega}\right] \sin \psi_c$$

$$x = A_1 x_0 \cos\psi_c + \frac{x_0' \omega^2}{k} \sin\psi_c - \frac{x_0 \omega'}{k} \overset{\omega}{} \cdot \sin\psi_c$$

$$x' = \cancel{A_1} \Big[\cancel{\frac{x_0 \omega'}{\omega} \cdot \cos\psi_c} - \cancel{\frac{x_0 \omega'}{k} \cdot \frac{k}{\omega}} \cos\psi_c - \frac{x_0 \omega' \omega'}{k} \sin\psi_c$$

$$- \frac{x_0 \cdot k}{\omega^2} \sin\psi_c \Big]$$

$$x_0' \left[\cos\psi_c + \frac{\omega\omega'}{k} \sin\psi_c \right] -$$

$$-\frac{\omega'^2}{k} - \frac{k}{\omega^2} = -\left(\frac{\omega'^2}{k} + \frac{k}{\omega^2} \right)$$

$$= -\left(\frac{\omega'^2 \omega^2 + k^2 \cdot}{k\omega^2} \right)$$

$$\frac{\frac{\omega'^2 \omega^2}{k^2} + 1}{\omega^2 / k}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\Delta\psi_c - \frac{ww'}{k}\sin\Delta\psi_c & \frac{w^2}{k}\sin\Delta\varphi \\ -\frac{1+(ww'/k)^2}{w^2/k}\sin\Delta\psi_c & \cos\Delta\psi_c + \frac{ww'}{k}\sin\Delta\psi \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\Delta\psi_c = \int_{s_0}^{s_0+C} \frac{k\,ds}{w^2(s)}$$

$$\psi' \quad \frac{d\psi}{ds} = \frac{k}{w(s)^2}$$

$$\text{so} \quad \Delta\psi_c = \int \frac{d\psi}{ds}\cdot ds$$

$$= \int \frac{k}{w^2}\,ds$$

Can rewrite this as

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\Delta\psi_c + \alpha\sin\Delta\psi_c & \beta\sin\Delta\psi_c \\ -\gamma\sin\Delta\psi & \cos\Delta\psi_c - \alpha\sin\Delta\psi_c \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\text{and} \quad \Delta\psi_c = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$$

$$x(s) = A\sqrt{\beta(s)}\cos\left[\psi(s) + \delta\right]$$

$$\omega^3(\omega'' + K\omega) = k^2$$

$$\frac{\omega^2}{k^2} \cdot \frac{\omega\omega''}{k} + \frac{\omega^4}{k^2} \cdot K = 1$$

β

$$\beta \cdot \frac{\omega\omega''}{k} + \beta^2 \cdot K = 1$$

$$\beta = \frac{\omega^2}{k}$$

$$\beta' = \frac{2\omega\omega'}{k} \qquad \beta'^2 = \frac{4(\omega\omega')^2}{k^2}$$

$$\beta'' = \frac{2\omega\omega''}{k} + \frac{2\omega'\omega'}{k}$$

$$\beta'' = \frac{2\omega\omega''}{k} + \frac{2\omega'^2}{k}$$

$y \quad x^2$

$\frac{dy}{dx} = 2x \quad \therefore \quad \frac{2\omega\omega''}{k} = \beta'' - 2\frac{\omega'^2}{k}$

$$\frac{\omega\omega''}{k} = \frac{\beta''}{2} - \frac{\omega'^2}{k}$$

$$\beta \cdot \left(\frac{\beta''}{2} - \frac{\omega'^2}{k}\right) + \beta^2 \cdot K = 1$$

$$\frac{\beta\beta''}{2} - \frac{\omega^2\omega'^2}{k^2} + \beta^2 K = 1$$

$$\frac{\beta\beta''}{2} - \frac{\beta'^2}{4} + \beta^2 k = 1$$

$$2\beta\beta'' - \beta'^2 + 4\beta^2 k = 1$$

$$M = I\cos\Delta\psi_s + J\sin\Delta\psi_c$$

$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$J^2 = -1$$

$$M = e^{J\Delta\psi_c}$$

All matrices in repeat period give

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\Delta\psi_c + \alpha\sin\Delta\psi_c & \beta\sin\Delta\psi_c \\ -\gamma\sin\Delta\psi_c & \cos\Delta\psi_c - \alpha\sin\Delta\psi_c \end{pmatrix}$$

$$a + d = 2\cos\Delta\psi_c \rightarrow \cos\Delta\psi_c = \tfrac{1}{2}\mathrm{Tr}\, M$$

$\cos\Delta\psi_c \rightarrow$ gives magnitude of $\sin\Delta\psi_c$ but not the sign.

$\triangleright$ β is +ve $\therefore$ sign same as b

$$\beta = \frac{b}{\sin\Delta\psi_c}$$

$$a - d = 2\alpha\sin\Delta\psi_c \; ; \; \alpha = \frac{a-d}{\sin\Delta\psi}$$

So, if you know transfer Matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then you can get α, β

PERIODIC λ

this procedure works between any two points on the lattice → so can find β at all points on lattice

So going between any 2 arbitrary points

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M(s_1 \rightarrow s_2)\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

Suppose x_1 & x_1' are initial conditions at $s = s_1$

$$x_1 = w(s_1)(A_1 \cos\psi_1 + A_2 \sin\psi_1)$$

$$x_1' = \left(A_1 w_1' + \frac{A_2 k}{w_1}\right)\cos\psi_1 + \left(A_2 w'(s_1) - \frac{A_1 k}{w(s_1)}\right)\sin\psi_1$$

By the same procedure we get

$$A_1 = \frac{x_1}{w_1}$$

$$A_2 = \frac{x_1' w_1 - x_1 w_1'}{k}$$

Do same thing again

this looks like a LOT of algebra!

you get

$$\begin{pmatrix} \left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}}(\cos\Delta\psi + \alpha_1 \sin\Delta\psi & (\beta_1\beta_2)^{\frac{1}{2}} \sin\Delta\psi \\ -\frac{1+\alpha_1\alpha_2}{(\beta_1\beta_2)^{\frac{1}{2}}}\sin\Delta\psi + \frac{\alpha_1-\alpha_2}{(\beta_1\beta_2)^{\frac{1}{2}}}\cos\Delta\psi & \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}(\cos\Delta\psi - \alpha_2 \sin\Delta\psi) \end{pmatrix}$$

$$\Delta\psi(s_1 \to s_2) = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}$$

$$\nu = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

Emittance & Admittance

Admittance is phase space area associated with the largest ellipse the accelerator will accept.

maximum value of x is $A\sqrt{\beta}$

half aperture available to the beam is $a(s)$

somewhere there will be a minimum $\frac{a(s)}{\sqrt{\beta(s)}}$

→ ie a maximum in $\sqrt{\beta(s)}$

CALCULATING HOW β-FN CHANGES THRU LATTICE

TRANSFER MATRIX

$$M_c = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

CAN HAVE SPECIFIC TRANSFER MATRIX eg QUAD

$$M_c = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

So

$$\cos\mu = \frac{1}{2}\left(m_{11} + m_{22}\right), \quad \beta = \frac{m_{12}}{\sin\mu}, \quad \alpha = \frac{m_{11} - m_{22}}{2\sin\mu}, \quad \gamma = -\frac{m_{21}}{\sin\mu}$$

- THE β-FN ALLOWS US TO CALCULATE THE BEAM DIMENSIONS, AND PHASE OF MOTION AT ANY POINT ON THE LATTICE
- WE NEED TO BE ABLE TO CALCULATE THE EVOLUTION OF THE β-FUNCTION AROUND THE LATTICE

GIVEN $\beta_{s1} \longrightarrow$ FIND β_{s2}

THERE ARE TWO APPROACHES

$\longrightarrow$ SEE WILLE P. 83

METHOD #1

SAY WE GO FROM POINT 1 $\longrightarrow$ POINT 2

AT 1 $\quad \bar{X}_1 = \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} \qquad \bar{X}_1^T = (x_1 \;\; x_1')$

TRANSFER MATRIX $M_{12} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad M_{12}^T \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$

DEFINE β-MATRIX $\quad B_1 \equiv \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix} \qquad \det B_1 = 1$

$$\bar{X}_1^T B_1^{-1} \bar{X}_1 = (x_1 \;\; x_1') \begin{pmatrix} \gamma_1 & \alpha_1 \\ \alpha_1 & \beta_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

$$= \gamma_1 x_1^2 + 2\alpha_1 x_1 x_1' + \beta_1 x'^2 = \frac{\varepsilon}{\pi} = \text{INVARIANT}$$

THE TRAJECTORY OF THE PARTICLE CAN BE CALCULATED $\longrightarrow$ USING TRANSFER MATRIX

$$\bar{X}_2 = M_{12}\,\bar{X}_1$$

HAD $\quad \varepsilon/\pi = \bar{X}_1^T\, B_1^{-1}\, \bar{X}_1$

USE $\quad M^{-1}M = 1 \qquad M^T(M^T)^{-1} = 1$ & INSERT

$$\varepsilon/\pi = \bar{X}_1^T\, \underbrace{M_{12}^T\big((M_{12}^T)^{-1}}\, B_1^{-1}\, \underbrace{M_{12}^{-1}\big)\, M_{12}}\, \bar{X}_1$$

USE $\quad A^T B^T = (BA)^T$ & $A^{-1}B^{-1} = (BA)^{-1}$

$$= \bar{X}_1^T\, M_{12}^T\, \left(M_{12}\, B_1\, M_{12}^T\right)^{-1} M_{12}\, \bar{X}_1$$

$$= \left(M_{12}\,\bar{X}_1\right)^T \left(M_{12}\, B_1\, M_{12}^T\right)^{-1} M_{12}\,\bar{X}_1$$

$\downarrow$

$$\frac{\mathcal{E}}{\pi} = (M_{12}\bar{X}_1)^T (M_{12} B_1 M_{12}^T)^{-1} M_{12}\bar{X}_1$$

USE $\bar{X}_2^T = (M_{12}\bar{X}_1)^T$

$$\frac{\mathcal{E}}{\pi} = \bar{X}_2^T (M_{12} B_1 M_{12}^T)^{-1} \bar{X}_2$$

HAD $\frac{\mathcal{E}}{\pi} = \bar{X}_1^T B_1^{-1} \bar{X}_1 \longrightarrow \frac{\mathcal{E}}{\pi} = \bar{X}_2^T B_2^{-1} \bar{X}_2$

$$B_2 = M_{12} B_1 M_{12}^T$$

$$\begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix} = M_{12} \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix} M_{12}^T$$

EQ – DRIFT $\begin{pmatrix} \beta_L & -\alpha_L \\ -\alpha_L & \gamma_L \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ L & 1 \end{pmatrix}$

METHOD #2

$1 \rightarrow 2$ EMITTANCE INVARIANT

$$\frac{\varepsilon}{\pi} = \beta_1 x_1'^2 + 2\alpha_1 x_1' x_1 + \gamma_1 x_1^2 = \beta_2 x_2'^2 + 2\alpha_2 x_2' x_2 + \gamma_2 x_2^2$$

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = M \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} \quad ; \quad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = M^{-1} \begin{pmatrix} x_2 \\ x_2' \end{pmatrix} \Bigg\} \quad \begin{aligned} x_1 &= m_{22} x_2 - m_{12} x_2' \\ x_1' &= -m_{21} x + m_{11} x' \end{aligned}$$

MULTIPLY OUT AND COMPARE COEFFICIENTS

$$\beta_2 = m_{11}^2 \beta_1 - 2 m_{12} m_{11} \alpha_1 + m_{12}^2 \gamma_1$$

$$\alpha_2 = -m_{21} m_{11} \beta_1 + (m_{22} m_{11} + m_{12} m_{21}) \alpha_1 - m_{22} m_{12} \gamma_1$$

$$\gamma_2 = m_{21}^2 \beta_1 - 2 m_{22} m_{21} \alpha_1 + m_{22}^2 \gamma_1$$

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2 m_{11} m_{12} & m_{12}^2 \\ -m_{11} m_{21} & m_{11} m_{22} + m_{12} m_{21} & -m_{22} m_{12} \\ m_{21}^2 & -2 m_{22} m_{21} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

DRIFT

$$\begin{pmatrix} \beta(L) \\ \alpha(L) \\ \gamma(L) \end{pmatrix} = \begin{pmatrix} 1 & -2L^2 & L^2 \\ 0 & 1 & -L \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$