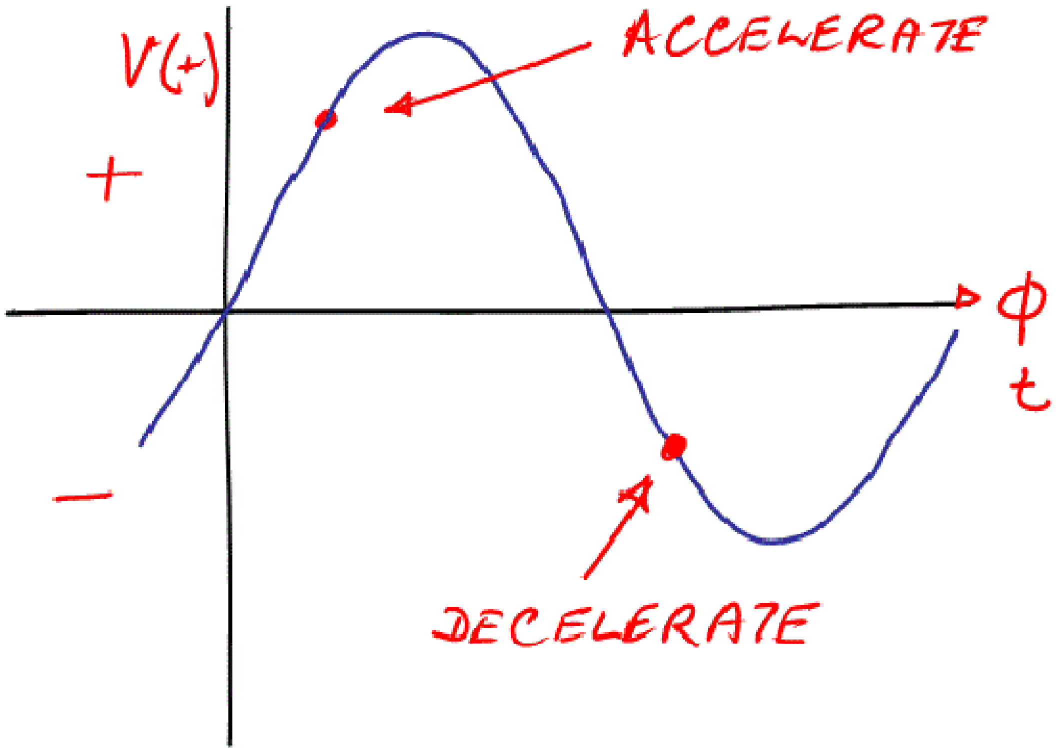
PHASE STABILITY

HAVE DISCUSSED STABILITY OF LATERAL MOTION AROUND EQUILIBRIUM ORBIT.

DOES AN NON-IDEAL ORBIT PARTICLE GET CONTINUALLY ACCELERATED → LOOK BACK AT WIDEROE LINAC.



DO PARTICLES CLOSE TO IDEAL REMAIN CLOSE IN PHASE AND ENERGY

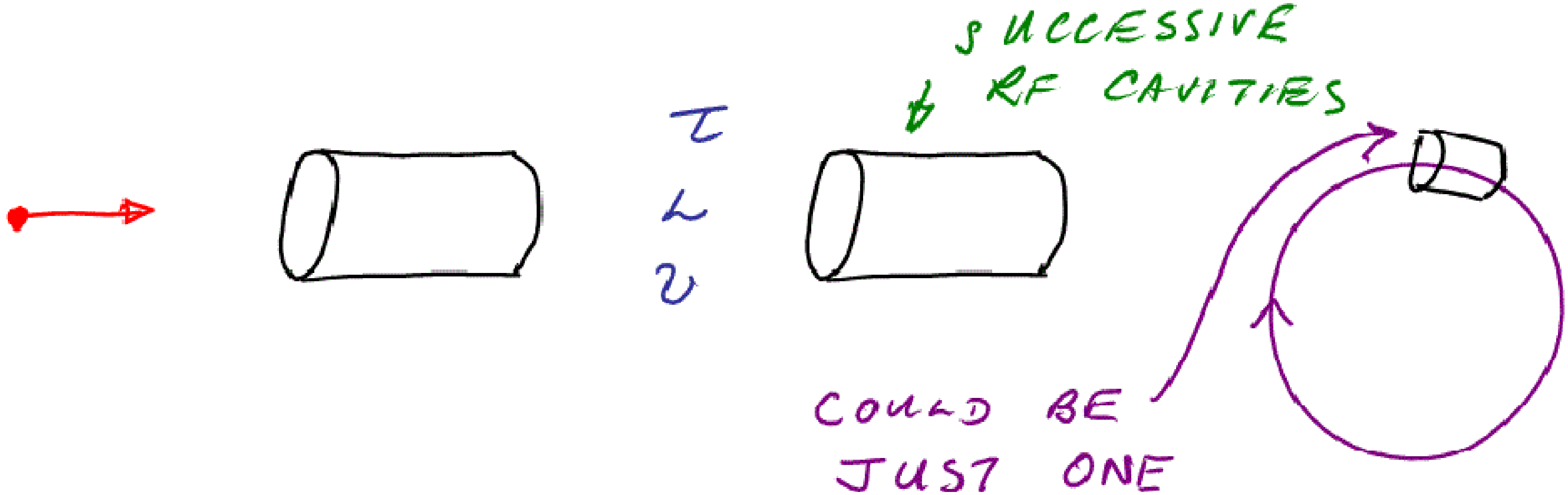
LONGITUDIAL OSCILLATIONS AROUND IDEAL PHASE

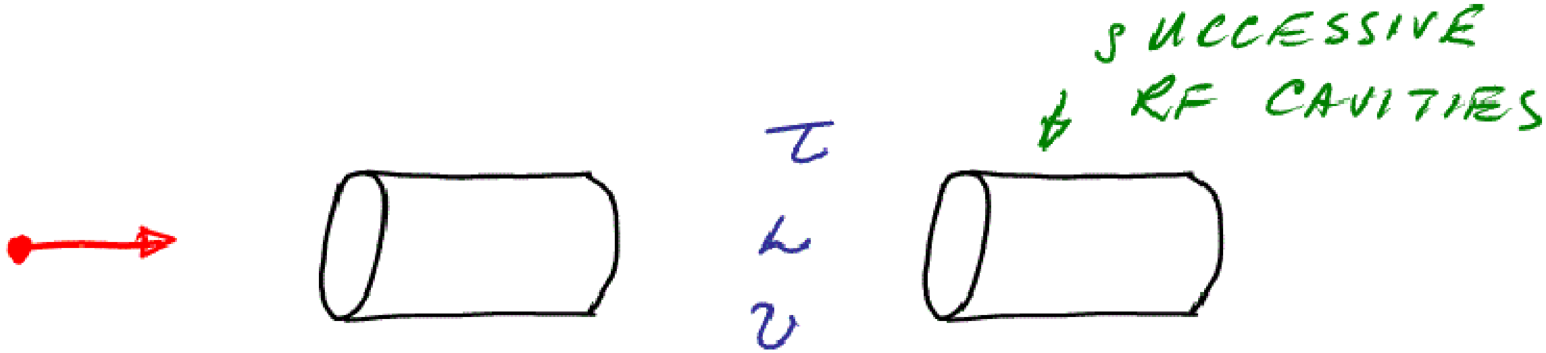
SYNCHROTRON OSCILLATIONS

SOPHISTICATED WAY TO SOLVE THIS IS TO CONSTRUCT A HAMILTONIAN FOR THE MOTIO

→ TACKLE IN HEURISTIC FASHION

- RF CAVITY EXCITED AT FREQUENCY ω
- BY DEFINITION IDEAL PARTICLE ARRIVES AT SAME PHASE ON EACH PASS THROUGH THE CAVITY $\phi, (\phi+2\pi), (\phi+4\pi), \ldots$
- RECEIVES SAME ENERGY INCREMENT ON EACH PASS





τ TIME FOR IDEAL PARTICLE TO PASS SUCCESSIVE CAVITIES

L DISTANCE BETWEEN CAVITIES

$$\tau = \frac{L}{v}$$

"DIFFERENTIATE" AS A QUOTIENT

FRACTIONAL CHANGE IN τ FOR NON-IDEAL PARTICLE

$$\frac{\Delta\tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta v}{v}$$

$$\frac{\Delta \tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta v}{v}$$

$v > v_{IDEAL}$ LESS TIME FOR TRANSIT

$L > L_{IDEAL}$ MORE TIME TO TRANSIT

HAVE $p = \gamma m v \rightarrow \frac{\Delta v}{v} = \frac{1}{\gamma^2}\left(\frac{\Delta p}{p}\right)$

NOW $\Delta L/L$ COULD ALSO DEPEND ON p

IN A CIRCULAR ACCELERATOR $p\uparrow \rightarrow R\uparrow \rightarrow L\uparrow$

SO DEFINE, BY ANALOGY

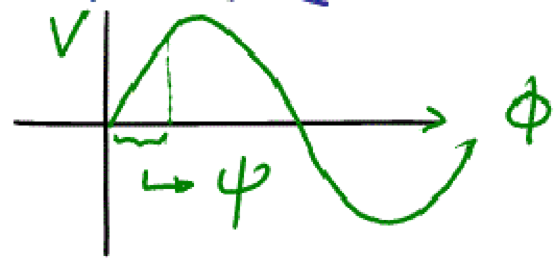
$$\frac{\Delta L}{L} = \frac{1}{\gamma_t^2}\left(\frac{\Delta p}{p}\right)$$

PARAMETER WHICH DEPENDS ON THE GEOMETRY OF THE ACCELERATOR

$$\frac{\Delta\tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta v}{v} \rightarrow \frac{\Delta\tau}{\tau} = \eta \cdot \frac{\Delta p}{p} = \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p}$$

η — SLIP FACTOR

THINK ABOUT ENERGY CHANGE — THIS COULD BE SUCCESSIVE KICKS IN CIRCULAR ACCEL.



E_n ψ_n ← ARBITRARY ARRIVAL PHASE

E_{n+1} ψ_{n+1}

FREQUENCY

IDEAL

PHASES RELATED BY $\psi_{n+1} = \psi_n + \omega(\tau + \Delta\tau)_{n+1}$

NON IDEAL

$$\psi_{n+1} = \psi_n + \omega\tau_{n+1} + \omega\tau_{n+1}\left(\frac{\Delta\tau}{\tau}\right)_{n+1}$$

$$(1)\quad \psi_{n+1} = \psi_n + \omega \tau_{n+1} + \omega \tau_{n+1} \left(\frac{\Delta \tau}{\tau}\right)_{n+1}$$

IDEAL PARTICLE ALWAYS ARRIVE "SAME TIME"

SAME PHASE

REFER ALL PHASES TO CONSTANT PHASE OF IDEAL PARTICLE

$$\psi_{IDEAL} = \omega T_n$$

$\leftarrow$ ARRIVAL TIME AT CAVITY n

FOR ARBITRARY PHASE, SUBTRACT OFF IDEAL PHASE

$$\phi_n = \psi_n - \omega T_n$$

(1) BECOMES

$$\underbrace{\phi_{n+1} + \omega T_{n+1}}_{\psi_{n+1}} = \underbrace{\phi_n + \omega T_n}_{\psi_n} + \omega \tau_{n+1} + \omega \tau_{n+1} \left(\frac{\Delta \tau}{\tau}\right)_{n+1}$$

FOR IDEAL PARTICLE

$$\phi_{n+1} + \omega T_{n+1} = \phi_n + \omega T_n + \omega \tau_{n+1} + \omega \tau_{n+1} \left(\frac{\Delta \tau}{\tau}\right)_{n+1}$$

$$T_n + \tau_{n+1} = T_{n+1}$$

$$\phi_{n+1} + \cancel{\omega T_n} + \cancel{\omega \tau_{n+1}} = \phi_n + \cancel{\omega T_n} + \cancel{\omega \tau_{n+1}} + \omega \tau_{n+1} \left(\frac{\Delta \tau}{\tau}\right)_{n+1}$$

$$\phi_{n+1} = \phi_n + \omega \tau_{n+1} \left(\frac{\Delta \tau}{\tau}\right)_{n+1}$$

IN SUCCESSIVE CAVITIES, THIS TELLS YOU HOW PHASE OF NON-IDEAL PARTICLE VARIE W.R.T TO PHASE OF IDEAL

$$\phi_{n+1} = \phi_n + \omega \tau_{n+1} \left(\frac{\Delta \tau}{\tau}\right)_{n+1}$$

$$\frac{\Delta \tau}{\tau} = \underbrace{\left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)}_{\eta} \frac{\Delta p}{p}$$

$\tau_n \approx \tau_{n+1}$

DIFFERENCE EQUATION OF MOTION

$$\phi_{n+1} = \phi_n + \eta \tau \omega \left(\frac{\Delta p}{p}\right)_{n+1}$$

ASSUME $\omega \tau$ INDEPENDENT OF n

FOR A CIRCULAR ACCELERATOR

$$\omega \tau = h \cdot 2\pi$$

RF

HARMONIC NUMBER

THAT WAS PHASE → WHAT ABOUT ENERGY

$$(E_s)_{n+1} = (E_s)_n + eV \sin\phi_s \qquad (1)$$

IDEAL SYNCHRONOUS PARTICLE

VOLTAGE

IDEAL (SYNCHRONOUS) PHASE

FOR NON-IDEAL PARTICLE

$$E_{n+1} = E_n + eV \sin\phi_n \qquad (2)$$

DIFFERENCE IN ENERGY W.r.t IDEAL $\Delta E = E - E_s$

(2) − (1) → $\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s)$

IN $\phi_{n+1} = \phi_n + \eta \tau \omega \left(\frac{\Delta p}{p}\right)_{n+1}$ $\quad \frac{\Delta p}{p} \to \frac{c^2}{v^2}\frac{\Delta E}{E}$

$$\phi_{n+1} = \phi_n + \frac{\omega \tau \eta c^2}{v^2 E_s} \cdot \Delta E_{n+1}$$

DIFFERENCE EQUATION

$$\phi_{n+1} = \phi_n + \frac{\omega \tau \eta c^2}{v^2 E_s} \Delta E_{n+1}$$

$$\Delta E_{n+1} = \Delta E_n + eV\left(\sin\phi_n - \sin\phi_s\right)$$

IF PHASE & ENERGY CONTINUOUS VARIABLES

$$\left.\begin{aligned} \frac{d\phi}{dn} &= \eta \frac{\omega \tau c^2}{v^2 E_s} \Delta E \\ \frac{d\Delta E}{dn} &= eV\left(\sin\phi - \sin\phi_s\right) \end{aligned}\right\}$$

TURN NUMBER

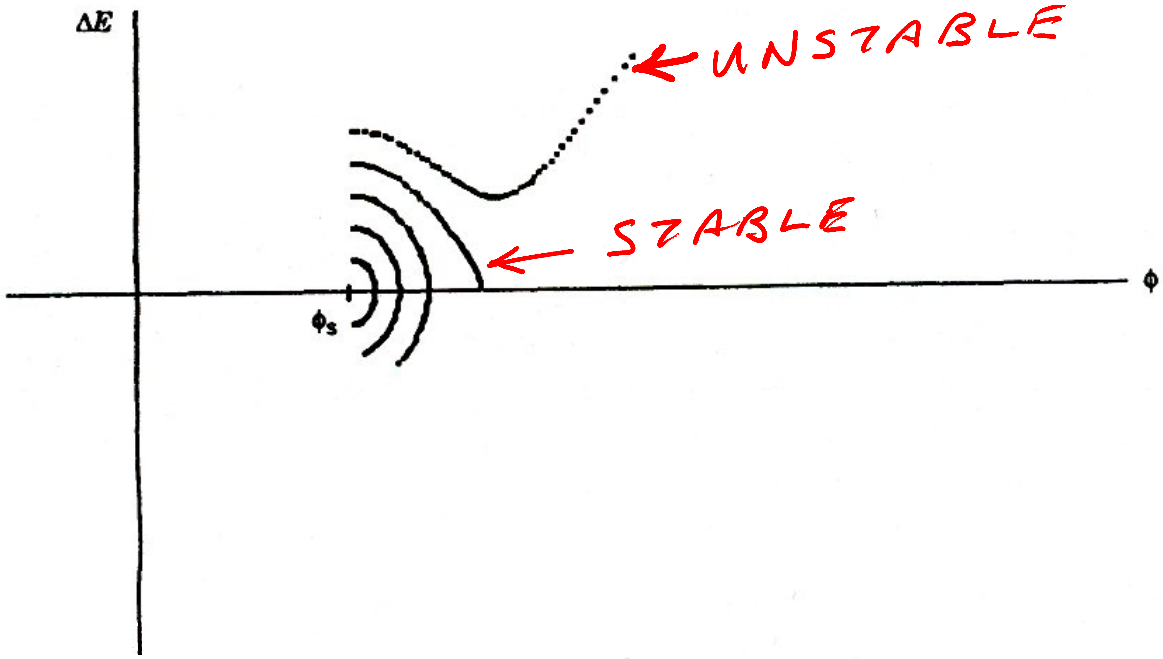


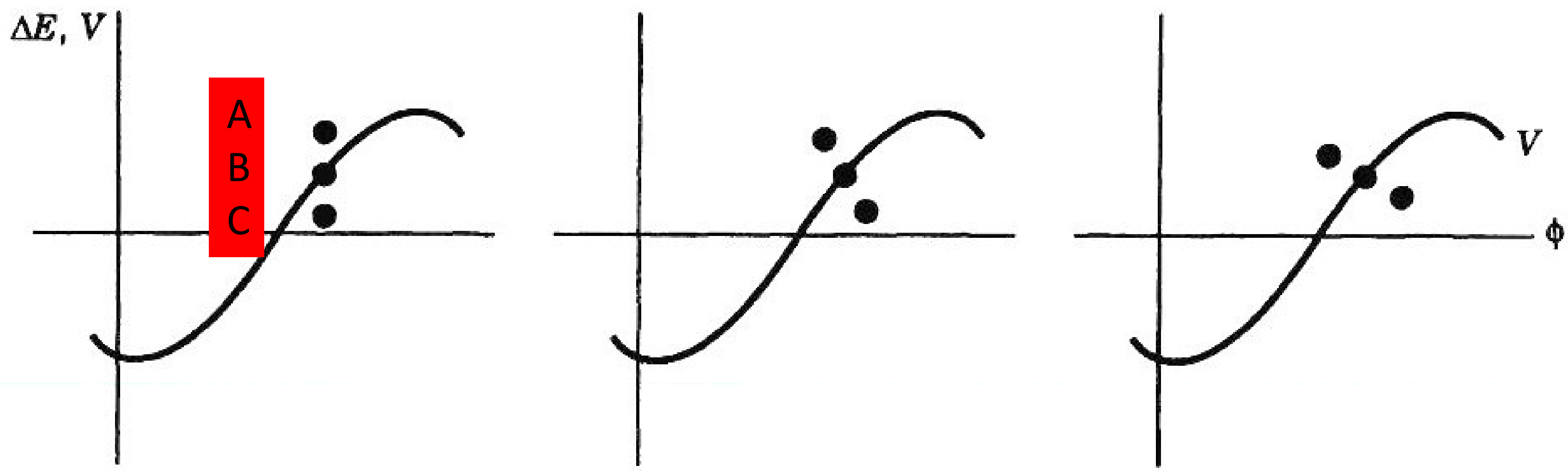
Figure 2.15. Application of the difference equations for synchrotron motion for five initial conditions. In each case, the starting value of the phase is equal to the synchronous phase.

ITERATION OF DIFFERENCE EQUATIONS
OF MOTION

→ START FROM DIFFERENT ΔE

- UNSTABLE — PARTICLE DRIFTS AWAY FROM SYNCHRONOUS PHASE
- STABLE — PARTICLE ORBITS IN PHASE SPACE AROUND SYNCHRONOUS PHASE

Successive turns around accelerator lattice



- B is synchronous with RF phase
- A too energetic to be in phase
- C not energetic enough to be in phase

Closed Oscillations in Phase
(non relativistic)

$\gamma < \gamma_\tau \rightarrow \Delta v$ MORE IMPORTANT THAN ΔL

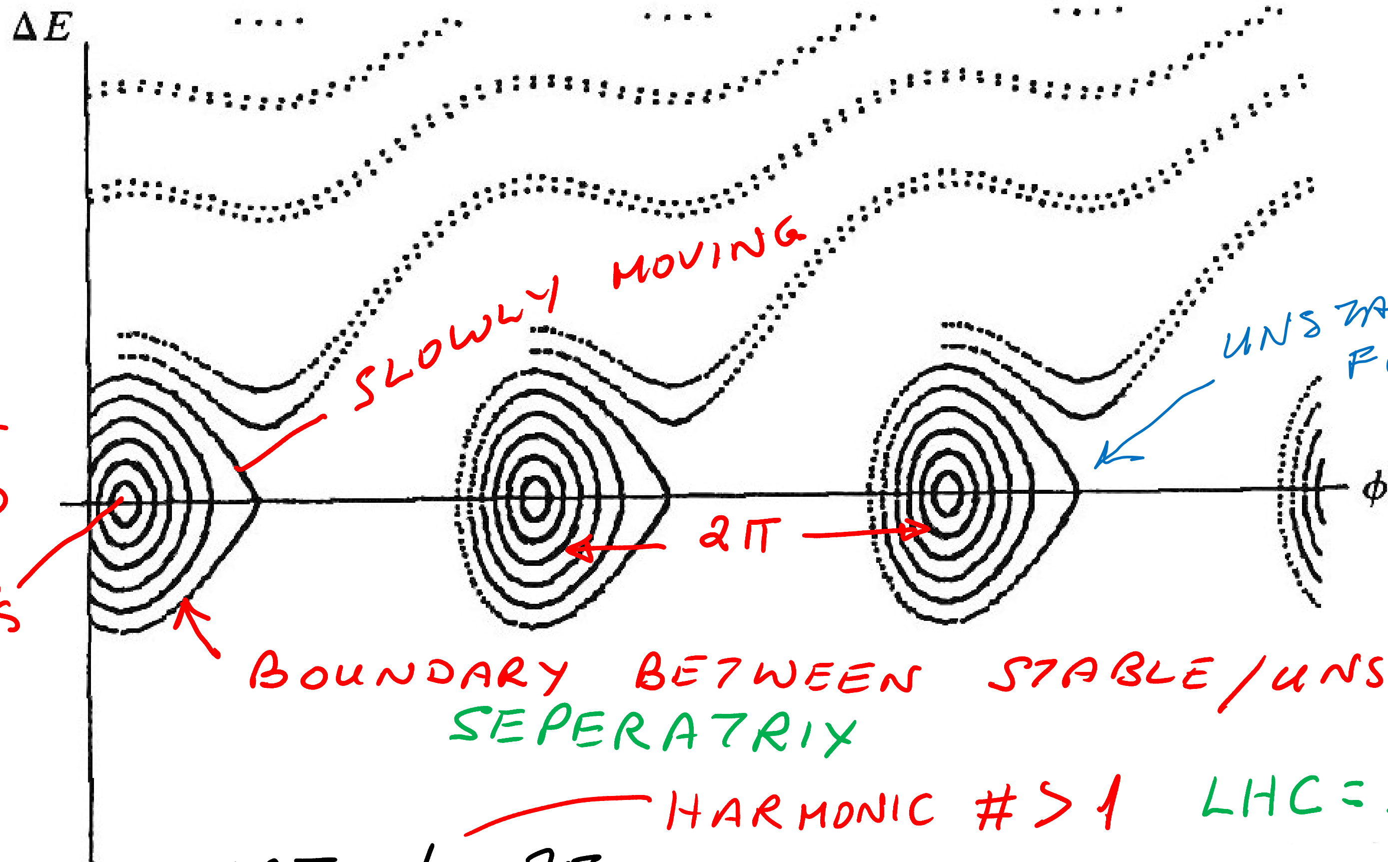
$$(E_s)_{n+1} = (E_s)_n + eV\sin\phi_s$$

Synchronous particle

$$\frac{\Delta\tau}{\tau} = \left(\frac{1}{\gamma_\tau^2} - \frac{1}{\gamma^2}\right)$$

Change in transit time around lattice

$\gamma < \gamma_T$



IDEAL

$\Delta E = 0$

$\phi = \phi_s$

BOUNDARY BETWEEN STABLE/UNSTABLE

SEPERATRIX

HARMONIC # > 1 LHC = 2808

$\omega \tau = h \cdot 2\pi$

MANY STABLE FIXED POINTS AROUND RING

SPACED BY 2π

PHASE SPACE WITHIN SEPERATRIX → RF BUCKET

UNSTABLE PARTICLES UNDULATE IN ENERGY

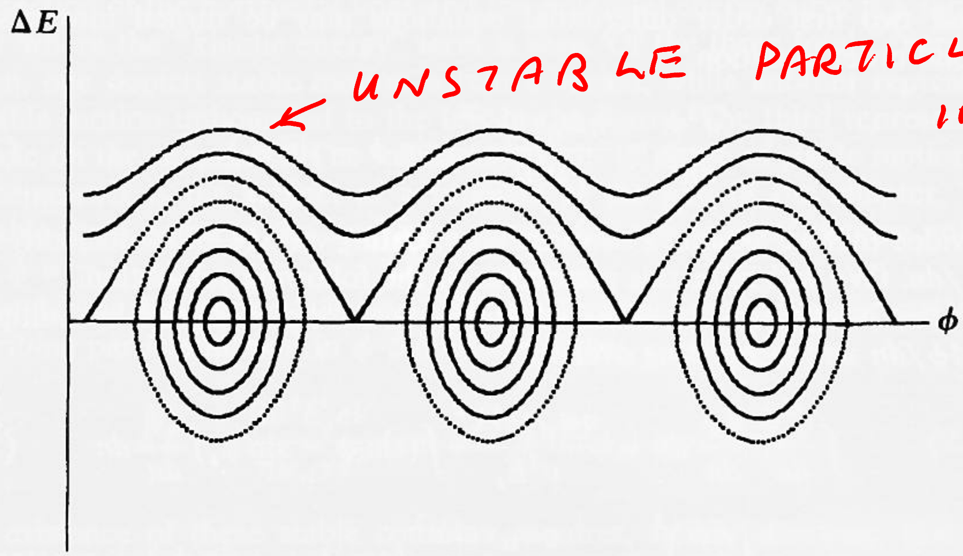


Figure 2.18. Application of the difference equations to a number of initial conditions for $\phi_s = 0$ or π. The regions within the separatrices are called stationary buckets.

- STORAGE RING → IDEAL PARTICLE UNACCELERATED

$$\phi_s = 0 \text{ OR } \pi$$

- THIS IS A STORAGE RING WITH NO ENERGY LOSS
  RF BUNCHES BEAMS

- FOR ACCELERATION, OR MAKING UP SYNCHROTRON RADIATION ENERGY LOSS $\phi_s \neq 0$

$$\frac{d\phi}{dn} = \frac{\eta \omega \tau c^2}{v^2 E_s} \cdot \Delta E \qquad \frac{d\Delta E}{dn} = eV(\sin\phi - \sin\phi_s)$$

YIELDS
$$\frac{d^2\phi}{dn^2} = \frac{\eta \omega \tau e V c^2}{v^2 E_s}(\sin\phi - \sin\phi_s)$$

$V \to$ CONSTANT $\qquad dE_s/dn$ SMALL

CONSTANT

$$\int \frac{d^2\phi}{dn^2} \cdot \frac{d\phi}{dn} \cdot dn = \frac{\eta \omega \tau e V c^2}{v^2 E_s} \cdot \int (\sin\phi - \sin\phi_s) \frac{d\phi}{dn} \cdot dn$$

$$\frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 = -\frac{\eta \omega \tau e V c^2}{v^2 E_s}(\cos\phi + \phi\sin\phi_s) + C$$

OSCILLATIONS IN PHASE

ACCELERATION

$$\frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 = -\frac{\eta\omega\tau eV}{v^2 E_s}\left(\cos\phi + \phi\sin\phi_s\right) + C$$

$$\left(\frac{\eta\omega\tau c^2}{v^2 E_s}\right)^2 \Delta E^2 = -\frac{2\eta\omega\tau c^2 eV}{v^2 E_s}\left(\cos\phi + \phi\sin\phi_s\right) + C$$

$$\Delta E^2 = -\frac{2 v^2 E_s}{\eta\omega\tau c^2}\cdot eV\left(\cos\phi + \phi\sin\phi_s\right) + C$$

KINETIC ENERGY

$$V(\phi) = \frac{2 v^2 E_s eV}{\underbrace{\eta\omega^2\tau C^2}_{x}}\left(\cos\phi + \phi\sin\phi_s\right)$$

TURNING POINTS IN MOTION FROM

$$\frac{dV}{d\phi} = \chi\left(-\sin\phi + \sin\phi_s\right) = 0$$

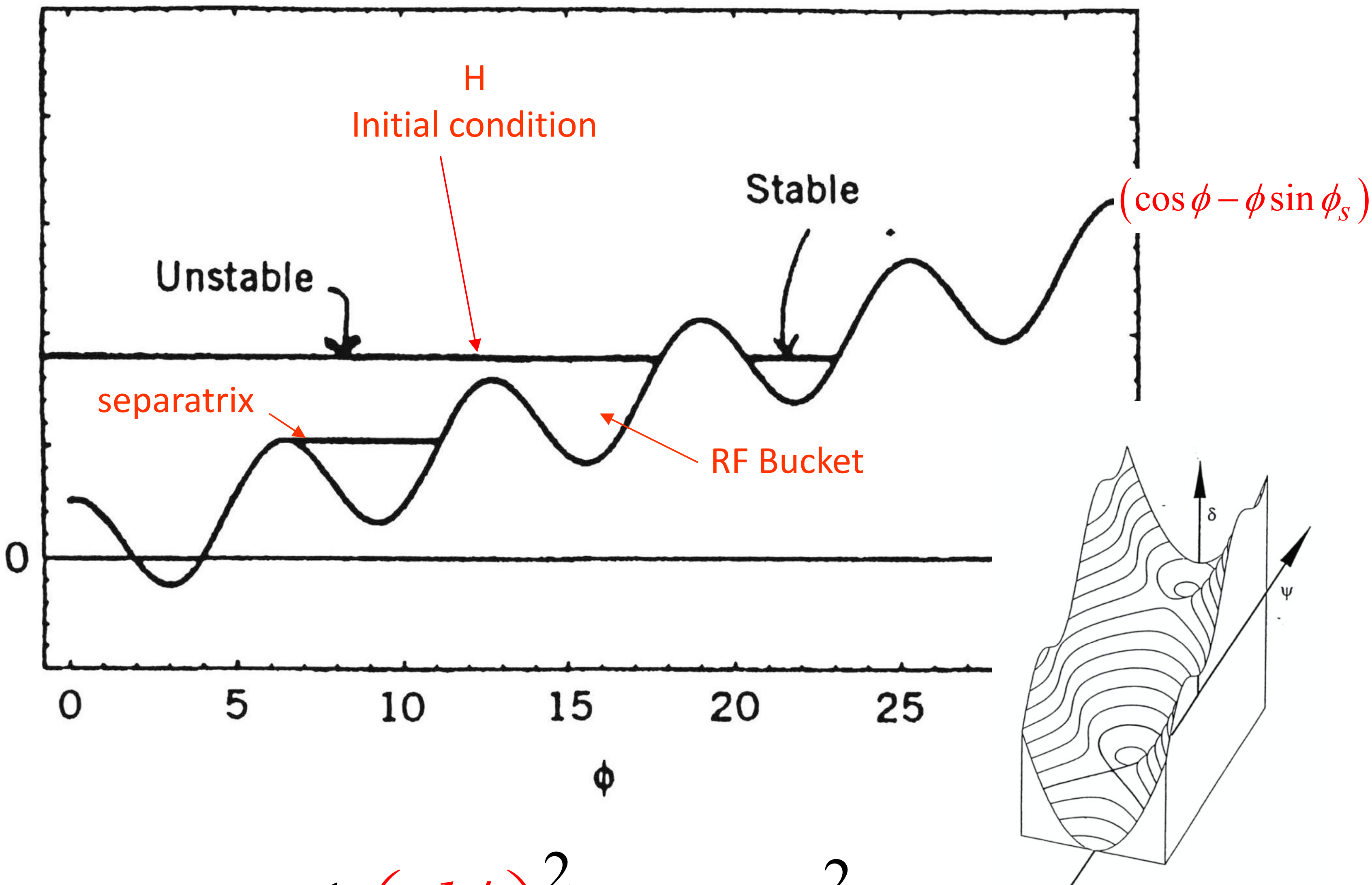
THERE IS A MINIMUM EVERY

$$\phi = \phi_s + 2\pi$$ ← RF BUCKET

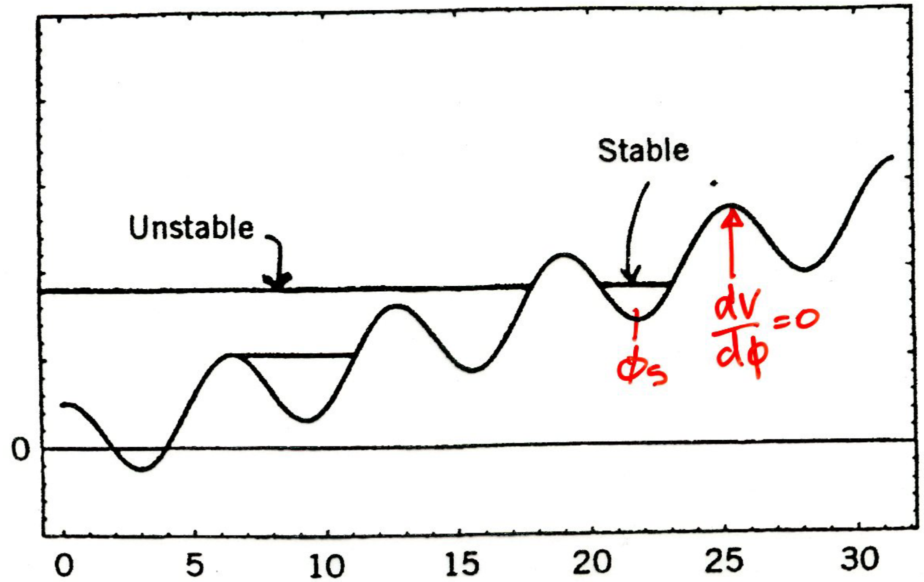
ALSO A TURNING POINT $\phi_{MAX} = \phi_s - \pi$

STABLE MOTION FOR $\phi < \phi_{MAX}$

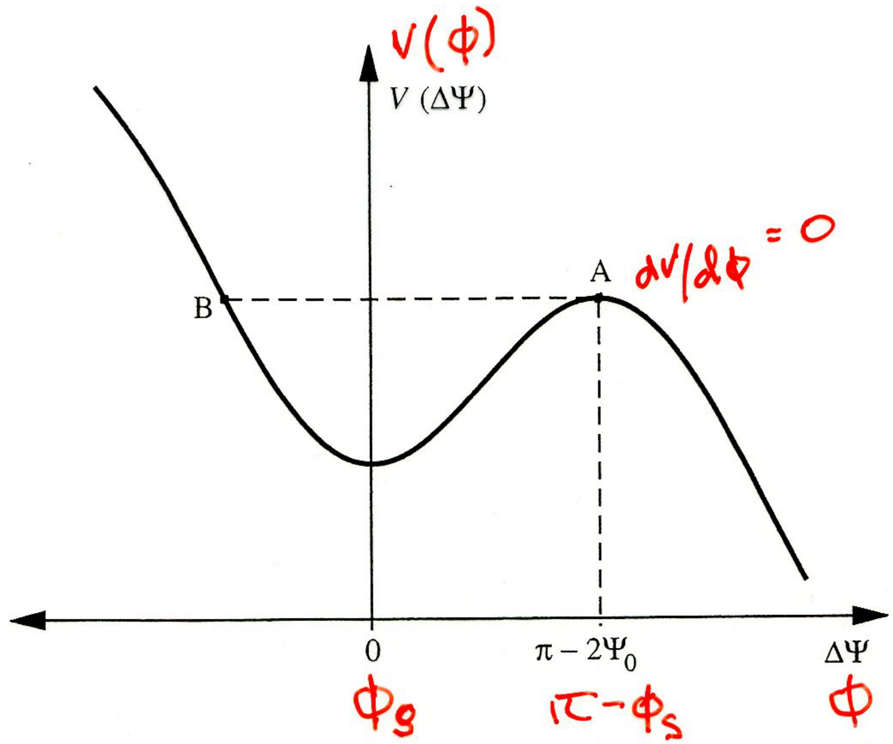
$$V(\phi) \leq V(\phi_{MAX})$$



$$\text{constant} = \mathrm{H} = \frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 + \frac{\eta\omega\tau c^2}{v^2 E_s}\left(\cos\phi - \phi\sin\phi_s\right)$$



Stable
Unstable
$\frac{dV}{d\phi} = 0$
ϕ_s
0
0
5
10
15
20
25
30



$V(\phi)$
$V\,(\Delta\Psi)$
A
B
$dV/d\phi = 0$
0
$\pi - 2\Psi_0$
$\Delta\Psi$
ϕ_s
$\pi - \phi_s$
ϕ

COASTING $\phi_s = \pi$

ACCELERATING $5\pi/6$, $2\pi/3$

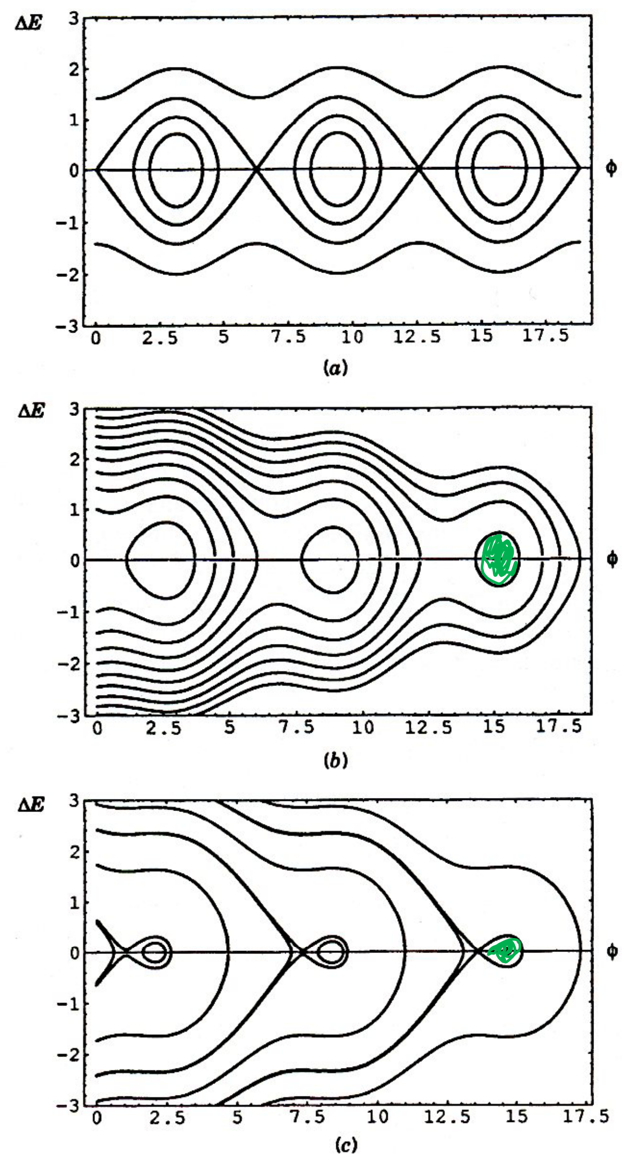


Figure 2.20. Contours of particle motion in longitudinal phase space obtained by solving the differential equations of motion. The curves shown are for $\eta > 0$ and for the cases (a) $\phi_s = \pi$, (b) $\phi_s = 5\pi/6$, and (c) $\phi_s = 2\pi/3$.

HAD $\quad \dfrac{d^2\phi}{dn^2} = \dfrac{\eta \omega \tau e V c^2}{v^2 E_s}\left(\sin\phi - \sin\phi_s\right)$

ASSUME $\quad \Delta\phi = \phi - \phi_s$ SMALL $\qquad \dfrac{d\Delta\phi}{dn} = \dfrac{d\phi}{dn}$

$$\sin\phi - \sin\phi_s \approx \sin(\phi_s + \Delta\phi) - \sin\phi_s$$

$$= \cos\phi_s \sin\Delta\phi + \sin\phi_s \cos\Delta\phi - \sin\phi_s$$

$$\approx \cos\phi_s \, \Delta\phi$$

$$\frac{d^2\Delta\phi}{dn^2} + \left(-\frac{\eta \omega \tau c^2 e V}{v^2 E_s}\cos\phi_s\right)\Delta\phi = 0$$

$$\frac{d^2\Delta\phi}{dn^2} + (2\pi\nu_s)^2\,\Delta\phi = 0$$

SYNCHROTRON OSCILLATIONS IN ϕ FOR $2\pi\nu_s$ REAL

ν_s = FREQUENCY OF SYNCHROTRON OSCILLATIONS

$$\frac{d^2\Delta\phi}{dn^2} + (2\pi\nu_s)^2\Delta\phi = 0$$

$$\nu_s = \left(-\frac{\eta\omega\tau c^2 eV\cos\phi_s}{4\pi^2 v^2 E_2}\right)^{\frac{1}{2}}$$

STABLE OSCILLATIONS $\nu_s \rightarrow$ REAL

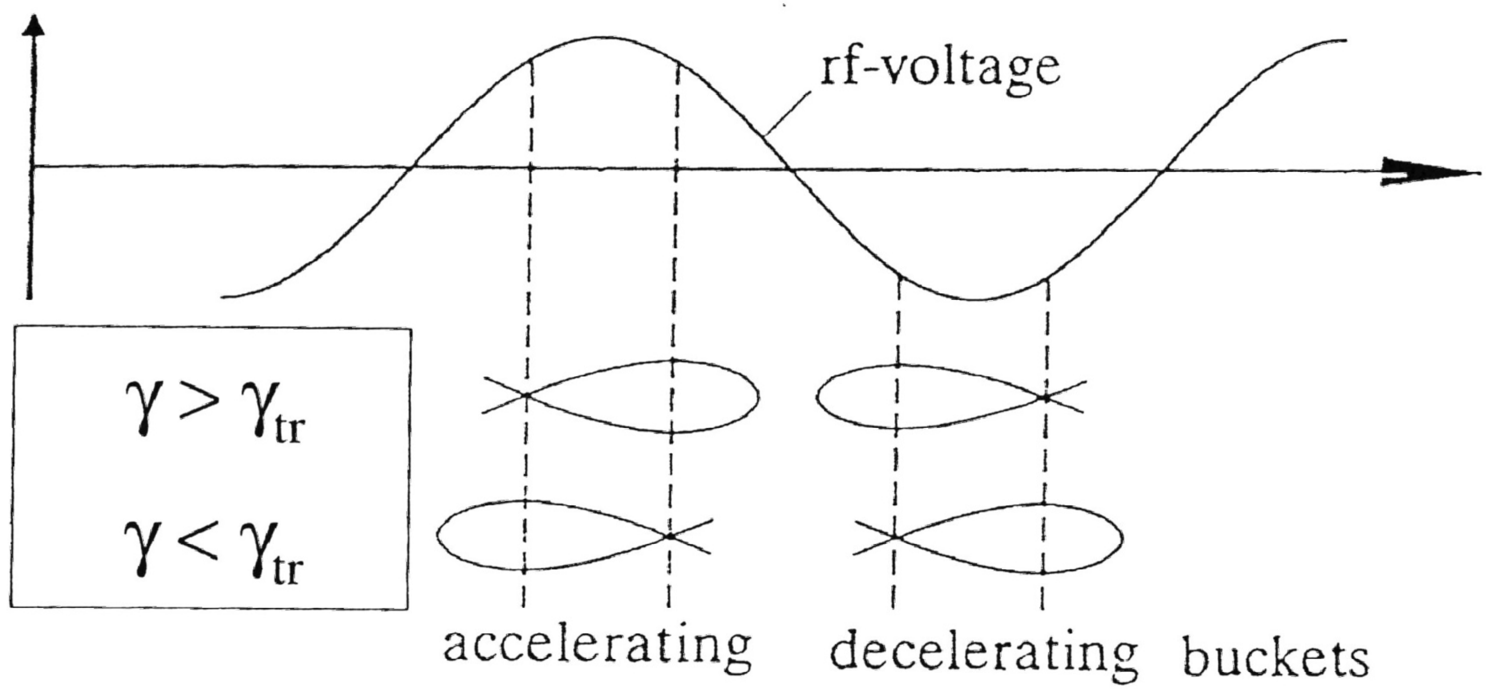
$$\eta = \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)$$

FOR $\eta < 0$ $(\gamma < \gamma_t)$ $\cos\phi_s > 0$

FOR $\eta > 0$ $(\gamma > \gamma_t)$ $\cos\phi_s < 0$

γ_t IS TRANSITION ENERGY

FOR A CIRCULAR ACCELERATOR AS ENERGY PASSES THROUGH $\gamma = \gamma_T$ RF FREQUENCY MUST GO THROUGH PHASE SHIFT.



Synchrotron (phase) oscillations

$$\frac{d^2\Delta\phi}{dn^2}+\left(\frac{-\eta\omega\tau c^2 eV}{v^2E_s}\cos\phi_s\right)\Delta\phi=0$$

$$\frac{d^2\Delta\phi}{dn^2}+\left(2\pi\nu_s\right)\Delta\phi=0$$

$$\eta<0 \quad ; \quad (\gamma<\gamma_t) \quad ; \quad \cos\phi_s>0$$

$$\eta>0 \quad ; \quad (\gamma>\gamma_t) \quad ; \quad \cos\phi_s<0$$

FOR A LINEAR ACCELERATOR

$$\eta = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2} \rightarrow \frac{-1}{\gamma^2}$$

THERE IS NO DIFFERENCE IN LENGTH TRAVERSED BY PARTICLES DUE TO $\Delta\phi$

$$\nu_s \sim \sqrt{\frac{1}{E_s}} \rightarrow 0 \quad \text{AS} \quad E_s \uparrow$$

FOR $\gamma \gg 1 \rightarrow \Delta\phi = \text{CONSTANT.}$