

LINEAR LATTICE DEFECTS

ACCELERATOR IS DESIGNED BY

- CHOOSE DIPOLES TO GIVE CLOSED EQUILIBRIUM ORBIT
- CHOOSE QUADRUPOLES TO GIVE β TRON OSCILLATIONS WHICH FIT IN MACHINE APERTURE.
- CHOOSE SEXTUPOLES TO MAKE MACHINE OPTICS ACHROMATIC.

REAL MAGNETS

— DIPOLES DO NOT HAVE DESIGN FIELD

NOT PERFECT

— DIPOLES HAVE QUADRUPOLE COMPONENT

etc

FIRST CONSIDER A DIPOLE WHICH HAS
A SLIGHTLY INCORRECT FIELD

? WHAT HAPPENS TO THE EQUILIBRIUM
ORBIT ?

SAY ERROR IS $\theta = \frac{\Delta B \cdot l}{B\rho}$ AT $S=0$

ΔB DIPOLE IMPERFECTION OVER LENGTH l
 θ IS RESULTING UNEXPECTED BEND

IS THERE STILL A CLOSED EQUILIBRIUM
ORBIT AROUND THE MACHINE ?

JUST AFTER IMPERFECTION θ ORBIT IS SPECIFIED BY x_0, x_0'

FOR A CLOSED ORBIT, MUST RETURN TO x_0, x_0' AFTER ONE TURN AROUND THE MACHINE

$$M \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

MATRIX FOR SINGLE TURN

ANGLE INTRODUCED BY LATTICE IMPERFECTION

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (\mathbf{I} - M)^{-1} \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

USE $M = e^{J\Delta\phi}$

$$\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

PHASE CHANGE = $2\pi\nu$ ← BETATRON

$$(\mathbf{I} - M)^{-1} = (\mathbf{I} - e^{J2\pi\nu})^{-1}$$

OSCILLATION FREQUENCY

$$(\mathbf{I} - e^{J2\pi v})^{-1} = [e^{J\pi v} (e^{-J\pi v} - e^{+J\pi v})]^{-1}$$

$$= - (2J \sin \pi v)^{-1} (e^{J\pi v})^{-1}$$

$$J^2 = -\mathbf{I} \quad ; \quad J = \frac{-\mathbf{I}}{J} \quad ; \quad J = \sqrt{\mathbf{I}}$$

$$= \frac{J}{2 \sin \pi v} \cdot e^{-J\pi v} = \frac{J}{2 \sin \pi v} \left\{ \mathbf{I} \cos(-v\pi) + J \sin(-\pi v) \right\}$$

$$= \frac{1}{2 \sin \pi v} \left\{ J \mathbf{I} \cos(\pi v) - J^2 \sin(\pi v) \right\}$$

$$= \frac{1}{2 \sin \pi v} \left\{ J \cos(\pi v) + \mathbf{I} \sin(\pi v) \right\}$$

CLOSED ORBIT AT $S=0$

$$\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = (\mathbf{I} - \mathbf{M})^{-1} \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

$$= \frac{1}{2\sin\pi\nu} \left\{ \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \cos\pi\nu + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sin\pi\nu \right\} \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \frac{\theta}{2\sin\pi\nu} \begin{pmatrix} \beta_0 \cos\pi\nu \\ \sin\pi\nu - \alpha_0 \cos\pi\nu \end{pmatrix}$$

$$M_s = \begin{pmatrix} \left(\frac{\beta_2}{\beta_1}\right)^{1/2} (\cos \Delta\phi_c + \alpha_1 \sin \Delta\phi_c) & (\beta_1 \beta_2)^{1/2} \sin \Delta\phi_c \\ -\frac{1 + \alpha_1 \alpha_2}{(\beta_1 \beta_2)^{1/2}} \sin \Delta\phi_c + \frac{\alpha_1 - \alpha_2}{(\beta_1 \beta_2)^{1/2}} \cos \Delta\phi_c & \left(\frac{\beta_1}{\beta_2}\right)^{1/2} (\cos \Delta\phi_c - \alpha_2 \sin \Delta\phi_c) \end{pmatrix}$$

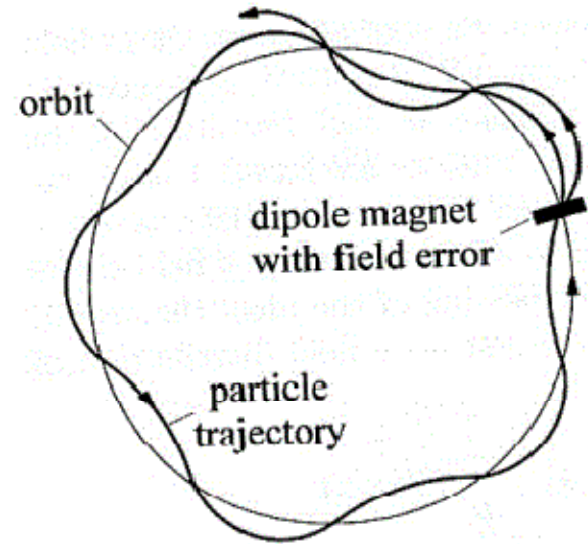
IS THE MATRIX FOR $1 \rightarrow 2$
 POSITION ON PERTURBED ORBIT $\begin{pmatrix} x_s \\ x'_s \end{pmatrix} = M_s \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$

$$\begin{pmatrix} x_s \\ x'_s \end{pmatrix} = M_s \cdot \frac{\theta}{2s\dot{\omega}\pi\nu} \begin{pmatrix} \beta_0 \cos \pi\nu \\ s\dot{\omega}\pi\nu - \alpha_0 \cos \pi\nu \end{pmatrix}$$

$$x(s) = \frac{\theta \beta(s)^{1/2} \cdot \beta_0^{1/2}}{2s\dot{\omega}\pi\nu} \cdot \cos(\underbrace{\phi(s) - \pi\nu}_{\Delta\phi})$$

$$x(s) = \frac{\theta \beta(s)^{1/2} \cdot \beta_0^{1/2}}{2 \sin \pi \nu} \cdot \cos(\phi(s) - \pi \nu)$$

WITH DIPOLE ERROR
STILL HAVE CLOSED ORBIT
WITH A KINK AT ERROR.



NOTE THAT FOR $\nu = \text{INTEGER}$
- INTEGRAL NUMBER OF
BETATRON OSCILLATIONS AROUND
RING $x(s) \rightarrow \infty$

RESONANCE \rightarrow IMPERFECTION GIVES ORBIT
SAME KICK AT SAME $x(s)$ EACH TIME
AROUND. $x(s)$ GROWS UNTIL HIT BEAM
TUBE, AND LOSE BEAM.

RESONANCES & CHAOTIC MOTION

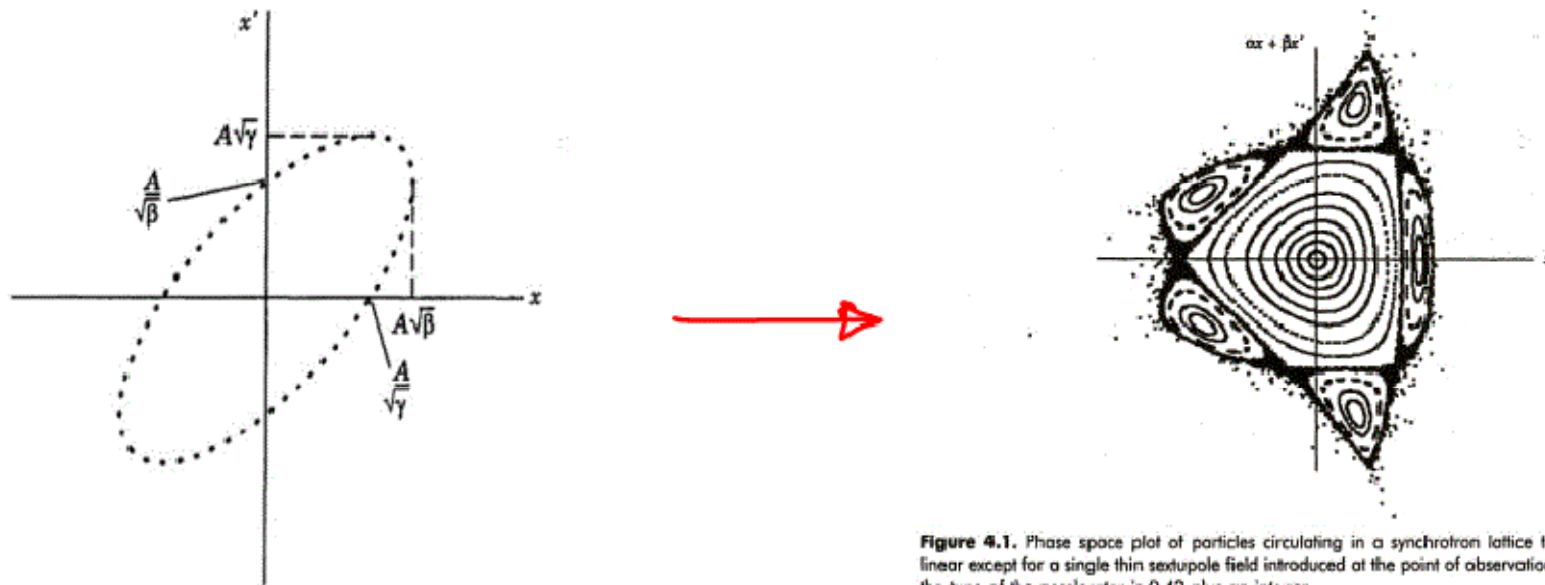


Figure 4.1. Phase space plot of particles circulating in a synchrotron lattice that is perfectly linear except for a single thin sextupole field introduced at the point of observation. For this plot, the tune of the accelerator is 0.42 plus an integer.

FIELD IMPERFECTIONS INTRODUCE
NON LINEARITIES
THEY GROW WITH AMPLITUDE → CHAOTIC
MAKE RUNNING A REAL ACCELERATOR
A COMPLEX TASK.

DRIVEN OSCILLATOR - RESONANCES

SOLUTION TO HILL'S EQUATION

$$x(s) = A \beta^{\frac{1}{2}} \cos[\psi + \delta]$$

DEFINE REDUCED PHASE $\phi = \psi/\nu$

ϕ INCREASES BY 2π EACH TIME AROUND THE RING

$$\nu = \frac{1}{2\pi} \int \frac{ds}{\beta(s)} \longrightarrow \phi(s) = \frac{\psi(s)}{\nu} = \frac{1}{\nu} \oint \frac{ds}{\beta(s)}$$

REPLACE TRANSVERSE COORDINATE $x(s)$

BY $\eta(s) = x(s)/\beta^{\frac{1}{2}}$

$$\eta(s) = A \cos(\nu \phi + \delta)$$

$$\frac{d^2 \eta}{d\phi^2} + \nu^2 \eta = 0$$

SIMPLE HARMONIC
MOTION IN
NEW COORDINATES
FLOQUET'S
TRANSFORMATION

$$\frac{dm}{d\phi} = \frac{dm}{ds} \cdot \frac{ds}{d\phi} = \frac{d}{ds} \left(\frac{x(s)}{\sqrt{\beta(s)}} \right) v \beta(s)$$

$$= \left(\frac{\alpha(s)}{\sqrt{\beta(s)}} \cdot x(s) + \sqrt{\beta(s)} \cdot x'(s) \right) \cdot v$$

AND

$$\frac{d^2 m}{d\phi^2} = \frac{d}{d\phi} \left(\frac{dm}{d\phi} \right) = \frac{d}{ds} \left(\frac{dm}{d\phi} \right) v \beta(s)$$

$$= \left\{ \beta^{3/2}(s) x''(s) + \frac{\alpha^2(s)}{\sqrt{\beta}} \cdot x(s) + \alpha'(s) \sqrt{\beta} x(s) \right\} v^2$$

FROM DEFINITIONS OF α, β, γ

$$\alpha'(s) = \beta(s) K(s) - \gamma(s)$$

$$\beta'(s) = -2\alpha(s)$$

$$\gamma'(s) = 2\alpha(s) K(s)$$

$$x''(s) + K(s)x(s) = 0$$

$$\frac{d^2 \eta}{d\phi^2} = \left\{ \beta^{3/2} x'' + \alpha^2 \frac{x}{\sqrt{\beta}} + \left[K\beta - \frac{1+\alpha^2}{\beta} \right] \sqrt{\beta} \cdot x \right\} v^2$$

$$= \left\{ \beta^{3/2} [x'' + K \cdot x] - \frac{x}{\sqrt{\beta}} \right\} v^2$$

MULT $x''(s) + K(s)x(s) = 0$ BY $\beta^{3/2} v^2$

$$[x''(s) + K(s)x(s)] \beta^{3/2} v^2 = 0$$

$$= \underbrace{\left\{ \beta^{3/2} [x'' + K \cdot x] - \frac{x}{\sqrt{\beta}} \right\}}_{d^2 \eta / d\phi^2} v^2 + \frac{x}{\sqrt{\beta}} v^2$$

$$\frac{d^2 \eta}{d\phi^2} + v^2 \eta = 0$$

FLOQUET'S
TRANSFORMATION

IN AN IDEAL LINEAR MACHINE
MAGNETIC FIELD

$$\begin{aligned}\frac{e}{p} B_y(x, s) &= \frac{e}{p} B_{y_0}(s) + \frac{e}{p} \frac{dB_y}{dx} \cdot x(s) \\ &= \frac{1}{\rho} + k(s) \cdot x(s)\end{aligned}$$

LOOK BACK A
FEW LECTURES

▽
•

THIS IS EXACTLY THE FIELD THAT GAVE
US

$$\ddot{x}(s) + \left[\frac{1}{\rho^2} + \frac{1}{B\rho} k(s) \right] x(s) = 0$$

$\frac{dB_y}{dx}$

AND THE HOMOGENEOUS HILL'S EQUATION

$$\ddot{x}(s) + K(s) \cdot x(s) = 0$$

NOW WE ADD A DIPOLE IMPERFECTION

$$\begin{aligned}\frac{e}{p} \tilde{B}_y(x, s) &= \frac{1}{\rho} + k(s)x(s) + \frac{e}{p} \Delta B(x, s) \\ &= \frac{1}{\rho} + k(s)x(s) + \frac{\Delta B}{\rho B_{y0}}\end{aligned}$$

THIS FIELD GIVES US INHOMOGENEOUS HILB'S

$$x''(s) + K(s)x(s) = \frac{\Delta B}{\rho B_{y0}} \xrightarrow{\text{FLOWMET}} \frac{d^2 \eta}{d\phi^2} + \nu^2 \eta = \beta^{3/2} \frac{\nu^2 \Delta B}{\rho \beta}$$

PERIODIC
SOLUTION

$$\eta(\phi) = \frac{\nu}{2 \sin \pi \nu} \int_{\phi}^{\phi+2\pi} \beta^{3/2} \frac{\Delta B}{\rho \beta} \cos[\nu(\pi + \phi + \theta)] d\theta$$

RESONANCE FOR $\nu = \text{INTEGER}$

EXPAND FIELD ERROR IN MULTIPOLES

$$\Delta B(x) = \Delta B_0 + \frac{d\Delta B}{dx} \cdot x + \frac{1}{2} \frac{d^2\Delta B}{dx^2} + \frac{1}{3} \frac{d^3\Delta B}{dx^3} + \dots$$

RECALL $\frac{d}{d\eta} = \frac{d}{dx} \frac{dx}{d\eta}$ AND $\frac{dx}{d\eta} = \sqrt{\beta}$

$$\Delta B(\eta) = \Delta B_0 + \beta^{\frac{1}{2}} \frac{d\Delta B}{d\eta} + \frac{1}{2} \beta^{\frac{3}{2}} \frac{d^2\Delta B}{d\eta^2} \eta^2 + \dots$$

PUT THIS INTO $\frac{d^2\eta}{d\phi^2} + \nu^2 \eta = \beta^{3/2} \nu^2 \frac{\Delta B}{\rho \beta}$

$$\frac{d^2\eta}{d\phi^2} + \nu^2 \eta = \frac{\nu^2}{\rho \beta} \left(\beta^{3/2} \Delta B + \beta^{4/2} \frac{d\Delta B}{d\eta} \eta + \frac{1}{2} \beta^{5/2} \frac{d^2\Delta B}{d\eta^2} \eta^2 + \dots \right)$$

OSCILLATOR DRIVEN BY MULTIPOLES

$$\frac{d^2 \eta}{d\phi^2} + \nu^2 \eta = \frac{\nu^2}{\rho B} \left(\underbrace{\beta^{3/2} \Delta B}_{\text{green}} + \beta^{4/2} \frac{d\Delta B}{d\eta} \eta + \frac{1}{2} \beta^{5/2} \frac{d^2 \Delta B}{d\eta^2} \eta^2 + \dots \right)$$

↳ THESE TERMS CAN BE WRITTEN AS FOURIER SERIES

$$\left(\beta^{(n+3)/2} \frac{d^n \Delta B}{d\eta^n} \right) = \sum_k c_k e^{\pm i k \phi}$$

AND SOLUTION OF EQUATION OF MOTION

$$\eta(\phi) = \eta_0 e^{\pm i \nu \phi}$$

$k = \nu$ INTEGER RESONANCE

$k - \nu = \nu \rightarrow k = 2\nu$ $\frac{1}{2}$ INTEGER RESONANCE

$k - 2\nu = \nu \rightarrow k = 3\nu$ $\frac{1}{3}$ INTEGER RESONANCE

DRIVING FIELD

CONDITION

DIPOLE

$$\nu = p$$

QUADRUPOLE

$$2\nu = p$$

SEXTUPOLE

$$3\nu = p$$

HAVE RESONANCES IN BOTH PLANES
IN A REAL ACCELERATOR \rightarrow COUPLED

RESONANCE CONDITION IS!

$$M\nu_x + N\nu_y = P$$

EG SEXTUPOLE

$$3\nu_x = p$$

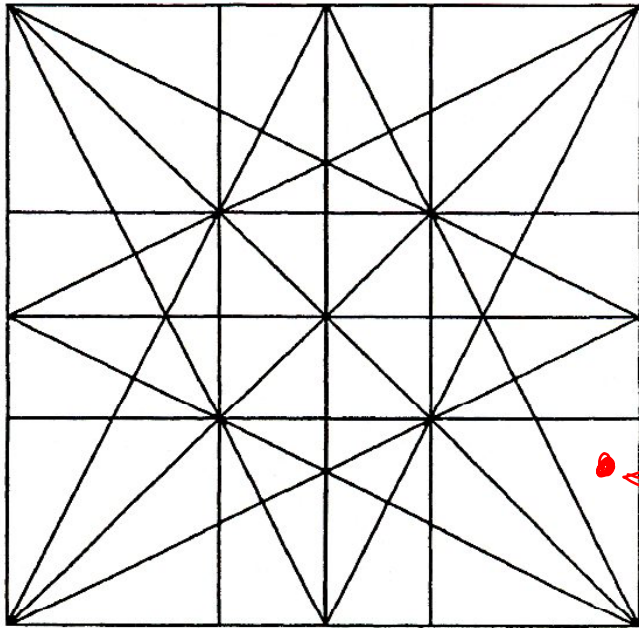
$$2\nu_x + \nu_y = p$$

$$\nu_x + 2\nu_y = p$$

$$3\nu_y = p$$

GIVE UNSTABLE
RESONANCES

MACHINE DOES NOT
WORK!



NO STABLE BEAM IF
LAND ON RESONANCE
LINE

POSSIBLE TUNE
GIVING STABLE BEAM

Figure 5.7. Resonance lines in a unit square of the tune plane for third integer and below.



Figure 5.8. Resonance lines in the tune plane for ninth integer and below.