LUMINOSITY

INTERACTION RATE IN A COLLIDER

$$R = \frac{dN_p}{dt} = \sigma_p \mathcal{L} \leftarrow \text{LUMINOSITY}$$

$$\mathcal{L}\left[\frac{1}{cm^2 \cdot s}\right] = \mathcal{L}\left[\frac{10^{33}}{nb \cdot s}\right] = \frac{1}{\sigma_p}\frac{dN_p}{dt}$$

IN MODERN EXPERIMENTS $\sigma_p \ll 1\,nb$

NEED VERY HIGH LUMINOSITY

$$N_p = \sigma_p \int \mathcal{L}\, dt = \sigma_p \mathcal{L}_{INT}$$

INTEGRATED LUMINOSITY

THIS DIAGRAM SAY e^+e^-
BUT IT IS COMPLETELY GENERAL
ASSUME THAT THE DISTRIBUTION OF PARTICLES IS GAUSSIAN IN ALL DIRECTIONS.
SURFACE DESITY OF N_2

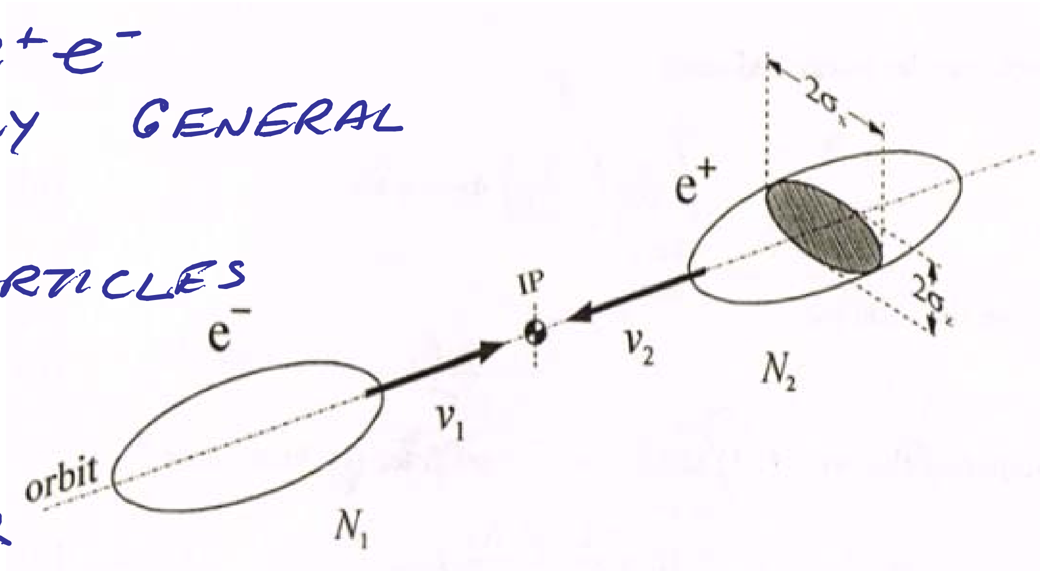


Fig. 7.1 Particle bunches colliding at the interaction point (IP).

THE TWO BUNCHES PASS COMPLETELY THROUGH EACH OTHER. SO CAN PROJECT ONTO xz PLANE – 2d PROBLEM

$$n_2 = \frac{\partial^2 N_2}{\partial x\, \partial z} = \frac{N_2}{2\pi \sigma_x^* \sigma_z^*} \exp\left(\frac{-x^2}{2\sigma_x^{*2}} - \frac{z^2}{2\sigma_z^{*2}} \right)$$

N_2 TOTAL NUMBER OF PARTICLES IN BUNCH
σ_x^* σ_z^* BEAM WIDTH AT INTERACTION POINT IP
CONVENTIONALLY IP QUANTITIES $\rightarrow$ $*$

IF THE BEAMS HAVE THE SAME CROSS SECTION AND OVERLAP COMPLETELY

PROBABILITY OF A PARTICLE IN BUNCH 1 IN A SURFACE ELEMENT $dA = dx\,dz$ INTERACTING WITH A PARTICLE IN BUNCH 2

$$dW = \sigma_p \frac{n_2\,dx\,dz}{dA} = \sigma_p n_2$$

DEFN OF σ

RATE OF BUNCH 1 PARTICLES CROSSING dA

$$\frac{dN_1}{dt} = \frac{b\,f_{REV}\,N_1}{2\pi\sigma_x^*\sigma_z^*}\exp\left(-\frac{x^2}{2\sigma_x^{*2}} - \frac{z^2}{2\sigma_z^{*2}}\right)dx\,dz$$

b EQUALLY SPACED BUNCHES, FREQUENCY f_{REV}

TOTAL INTERACTION RATE

$$\frac{dN_p}{dt} = \sigma_p \frac{dN_1}{dt} n_2 = \sigma_p \frac{b\,f_{REV}\,N_1 N_2}{(2\pi)^2\sigma_x^{*2}\sigma_z^{*2}}\exp\left(-\frac{x^2}{\sigma_x^{*2}} - \frac{z^2}{\sigma_z^{*2}}\right)dx\,dz$$

$$= dW \cdot \frac{dN_1}{dt}$$

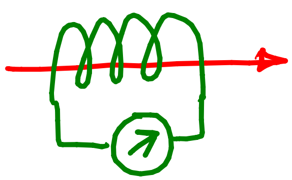
$$\frac{dN_p}{dt} = \sigma_p \frac{dN_1}{dt} n_2 = \sigma_p \frac{b\, f_{REV}\, N_1 N_2}{(2\pi)^2 \sigma_x^{*2} \sigma_z^{*2}} \exp\left(-\frac{x^2}{\sigma_x^{*2}} - \frac{z^2}{\sigma_z^{*2}} \right) dx\, dz$$

CAN USE $\int_{-\infty}^{+\infty} \exp(-y^2/\sigma^2)\, dy = \sqrt{\pi}\, \sigma$

TOTAL INTERACTION RATE

$$\frac{dN_p}{dt} = \sigma_p \frac{b\, f_{REV}\, N_1 N_2}{4\pi \sigma_x^* \sigma_z^*} \rightarrow \mathcal{L} = \frac{b}{4\pi} \frac{N_1 N_2}{\sigma_x^* \sigma_z^*} \cdot f_{REV}$$

EASY TO MEASURE AVERAGE BEAM CURRENT



$$I = N e f_{REV}\, b$$

LARGE

$$\mathcal{L} = \frac{1}{4\pi e^2 f_{REV} \cdot b} \frac{I_1 I_2}{\sigma_x^* \sigma_z^*}$$

SMALL

BEAM-BEAM INTERACTIONS & LUMINOSITY

HAVE SPACE CHARGE INTERACTION BETWEEN BEAMS

GO INTO CENTRE OF MASS OF BUNCH 1 — K'

BUNCH 1 ONLY HAS ELECTRIC FIELD $\bar{E}'$

IN LAB FRAME $\bar{B} + \bar{E}$

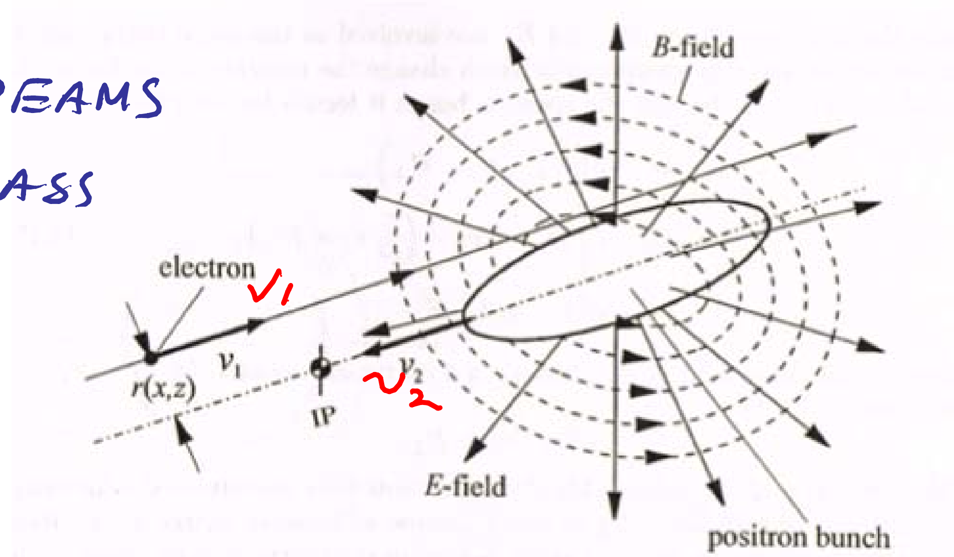


Fig. 7.2 Deflection of an electron due to the space charge of an oncoming bunch.

$$\left.\begin{aligned} \bar{E}_\perp &= \gamma \bar{E}_\perp' & \bar{E}_{\parallel} &= \bar{E}_{\parallel}' \\ \bar{B}_\perp &= \frac{\gamma}{c^2}\, \vec{v}_2 \times \bar{E}_\perp' & B_{\parallel} &= 0 \end{aligned}\right\}$$

ONLY $\bar{B}_\perp$ $\bar{E}_\perp$ GIVE FORCE ON BUNCH 1 PASSING THRU BUNCH 2

FORCE ON 1 $= \bar{F}_\perp = -e\left(\bar{E}_\perp + \bar{v}_1 \times B_\perp\right)$ IN LAB

$$= -e\left[\gamma E_\perp' + \bar{v}_1 \times \bar{B}_\perp\right] = -e\left[\gamma E_\perp' + \bar{v}_1\left(\frac{\gamma}{c^2}\, \bar{v}_2 \times E_\perp'\right)\right]$$

$$= -e\left(1 + \frac{v_1 v_2}{c^2}\right) E_\perp \qquad \frac{v}{c} \to 1 \qquad \boxed{\bar{F}_\perp = -2e E_\perp}$$

THIS IS FOR OPPOSITELY CHARGED BEAMS — FOCUS

ASSUME BEAMS GAUSSIAN IN ALL 3 DIRECTIONS IN CM FRAME OF ①, CHARGE DENSITY IN ② SEEN BY ① $\rho(x', z', s') = \rho'(x, z, s')$

BOOST IS ALONG S

$$\rho'(x, z\, s') = \frac{eN_2}{(2\pi)^{3/2}\sigma_x \sigma_z \sigma_s'} \exp\left\{ \frac{-x^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2} - \frac{(s' - s_0')^2}{2\sigma_s'^2} \right\}$$

$\sigma_{x,z}' = \sigma_{x,z}$ $\sigma_s' = \gamma\sigma_s$, s_0' IS ARBITRARY REFERENCE POINT

$\sigma_s' = \gamma\sigma_s \gg\gg \sigma_x, \sigma_z \rightarrow$ only LONG CHARGE DISTRIB

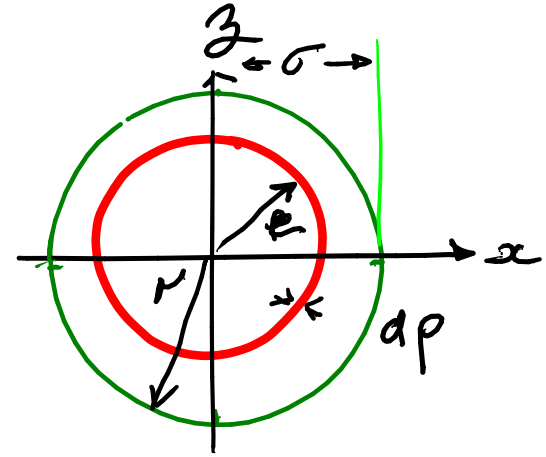
$$\rho'(x, z, s') = A(s') \exp\left(\frac{-x^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2} \right)$$

WITH $$A(s') = \frac{eN_2}{(2\pi)^{3/2}\sigma_x \sigma_z \sigma_s'} \exp\left(- \frac{(s' - s_0')}{2\sigma_s'^2} \right)$$

USE THIS CHARGE DISTRIBUTION TO CALC FORCE ON ① SIMPLIFY $\sigma = \sigma_x = \sigma_z$ $r^2 = x^2 + z^2$

SIMPLIFIED CHARGE DENSITY, SYMMETRICAL BEAMS

$$\rho'(r,s') = A(s')\exp\left(\frac{-r^3}{2\sigma^2}\right)$$



CHARGE ON SHELL $\rho\, d\rho$

$$dq = \rho'(r,s')\, 2\pi\rho\, d\rho\, ds'$$

CYLINDER LENGTH $\Delta s'$

$$\Delta q(s') = 2\pi A(s')\,\Delta s' \int_0^r \exp\left(\frac{-\rho^2}{2\sigma^2}\right)\rho\, d\rho$$

MAPLE → $$\Delta q(s') = 2\pi A(s')\,\sigma^2\left[1-\exp\left(\frac{-r^2}{2\sigma^2}\right)\right]\Delta s'$$

GAUSS'S THEOREM ON CYLINDRICAL SURFACE

$$E_\perp'(r)\, 2\pi r\, \Delta s' = \Delta q(s')/\varepsilon_0$$

$$E_\perp'(r) = \frac{\Delta q(s')}{2\pi\varepsilon_0 r\,\Delta s'}$$ ← PUT IN $\Delta q(s')$ AND $A(s')$

$$E'_\perp(r) = \frac{e N_2}{(2\pi)^{3/2}\varepsilon_0 r \sigma_s'} \exp\left(-\frac{(s'-s_0')^2}{2\sigma_s'^2}\right)\left[1-\exp\left(\frac{-r^2}{2\sigma^2}\right)\right]$$

LAB FRAME

$s' = \gamma s \qquad \sigma_s' = \gamma \sigma_s$

$$E_\perp(r,s) = \gamma E'_\perp(r,s) = \frac{e N_2}{(2\pi)^{3/2}\varepsilon_0 \sigma_s} \exp\left(-\frac{(s-s_0)^2}{2\sigma_s^2}\right)\frac{1}{r}\left[1-\exp\left(\frac{-r^2}{2\sigma^2}\right)\right]$$

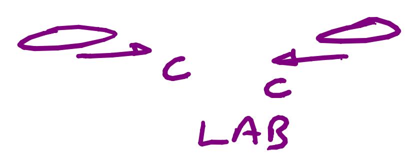
ONLY CONSIDER 1 AT VERY SMALL DISTANCE FROM BEAM AXIS PASSING THROUGH 2

$$\exp\left(\frac{-r^2}{2\sigma^2}\right) = 1 - \frac{r^2}{2\sigma^2} + \frac{1}{2!}\left(\frac{r^2}{2\sigma^2}\right)^2 \cdots\cdots \approx 1 - \frac{r^2}{2\sigma^2}$$

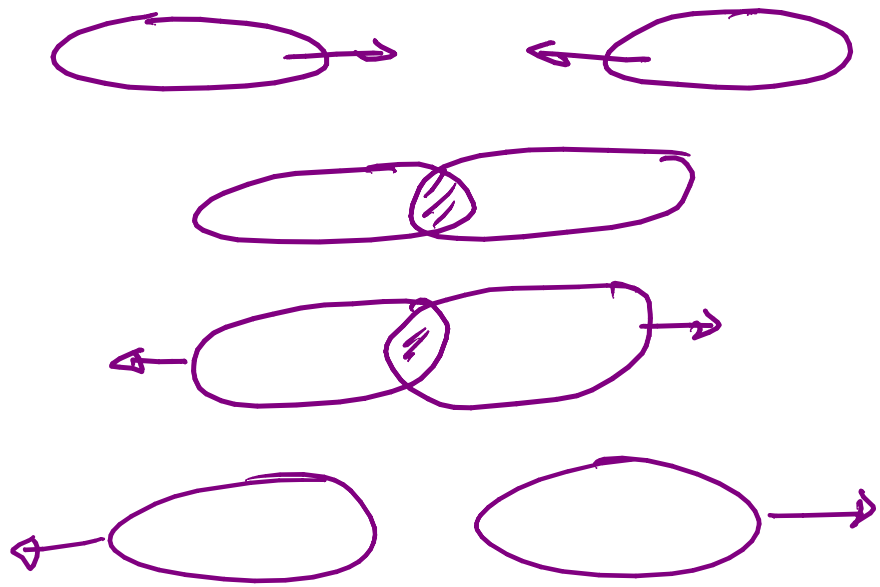
$$E_\perp(r,s) = \frac{e N_2}{(2\pi)^{3/2}\varepsilon_0 \sigma_s} \exp\left(-\frac{(s-s_0)^2}{2\sigma_s^2}\right)\frac{r}{2\sigma^2}$$

PARTICLE 1 IS ACCELERATED BY $E_\perp$ CHANGES TRANSVERSE MOMENTUM

$$dp_\perp = F_\perp \, dt = F_\perp \frac{ds}{2c}$$



WHERE DOES THE FACTOR OF $\frac{1}{2}$ REALLY COME FROM?



FORCE ONLY FELT WHILE BUNCHES COLLIDING AS SOON A "TEST PARTICLE" HAS TRAVELED $\frac{1}{2}$ BUNCH LENGTH → THE ONCOMING BUNCH HAS GONE PAST.

$$dp_\perp = F_\perp \frac{ds}{2c} = -2eE_\perp \frac{ds}{2c}$$

$$dp_\perp = -\frac{e^2 N_2 r}{4\pi\varepsilon_0 c\sigma^2} \frac{1}{\sqrt{2\pi}\,\sigma_s} \cdot \exp\left(-\frac{(s-s_0)^2}{2\sigma_s^2}\right) ds$$

GAUSSIAN IN S DIRECTION DIES AWAY FOR $|s-s_0|$ LARGE $\therefore \int -\infty \rightarrow \infty$

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{(s-s_0)^2}{2\sigma_s^2}\right) ds = \sqrt{2\pi}\,\sigma_s$$

$$\Delta p_\perp(r) = -\frac{e^2 N_2}{2\pi\varepsilon_0 c} \cdot \frac{r}{2\sigma^2} \rightarrow \quad \Delta x' = \frac{\Delta p_x}{p} = \frac{-e^2 N_2}{2\pi p c\varepsilon_0} \frac{x}{2\sigma^2}$$

TOTAL CHANGE IN $P_\perp$

$$\Delta z' = \frac{\Delta p_z}{p} = \frac{-e^2 N_2}{2\pi p c\varepsilon_0} \frac{z}{2\sigma^2}$$

CHANGE IN ANGLES

$$\Delta x' = -\frac{e^2 N_2}{2\pi p c \varepsilon_0} \frac{x}{2\sigma^2}$$

FOR A MAGNETIC QUADRUPOLE LENS

$$\Delta x' = k \cdot l \cdot x$$

SO BUNCH 2 ACTS ON 1 AS QUADRUPOLE AND VICE VERSA – FOCUS/DEFOCUS DEPENDS ON WHETHER BEAMS ARE SAME/OPPOSITE SIGN.

QUADRUPOLE STRENGTH

EFFECT OF QUAD Δk

$$k_r \cdot l = -\frac{e^2}{2\pi} \frac{N_2}{p c \varepsilon_0} \frac{1}{2\sigma^2}$$

BUT TUNE SHIFT

$$\Delta \nu = \frac{1}{4\pi} \int_{s_0}^{s_0+l} \Delta k \, \beta(s) \, ds$$

β fn

$$\Delta \nu_{x,z} = \frac{\beta^*_{x,z}}{4\pi} k_r \cdot l$$

$$\Delta \nu_{x,z} = \frac{e^2 N_2}{8\pi^2 p c \varepsilon_0} \cdot \frac{\beta^*_{x,z}}{2\sigma^2}$$

WILLE PAGE 121. eqtn: 3.291

$$\Delta \nu_{x,z} = \frac{e^2 N_2}{8\pi^2 p c \varepsilon_0} \frac{\beta^*_{x,y}}{2\sigma^2}$$

β FN AT INTERACTION POINT

BEAM – BEAM TUNE SHIFT

AS NUMBER OF PARTICLES IN A BUNCH INCREASES TUNE SHIFT INCREASES UNTIL BEAM MOVES ONTO RESONANCE AND IS LOST

FOR ELLIPTICAL CROSS SECTION

$$2\sigma^2 \rightarrow \begin{array}{l} \sigma_x^*(\sigma_x^* + \sigma_z^*) \\ \sigma_z^*(\sigma_x^* + \sigma_z^*) \end{array}$$

$$\Delta \nu_x = \frac{e^2 N_2}{8\pi^2 p c \varepsilon_0} \frac{\beta_x^*}{\sigma_x^*(\sigma_x^* + \sigma_z^*)}$$

$$\Delta \nu_z = \frac{e^2 N_2}{8\pi^2 p c \varepsilon_0} \frac{\beta_z^*}{\sigma_z^*(\sigma_x^* + \sigma_z^*)}$$

$$N_2 = \frac{I_2}{b f_{rev} e} \qquad \sigma^*_{x,z} = \sqrt{\varepsilon_{x,z}\, \beta^*_{x,z}}$$

REMEMBER HILL'S EQUATION !

$$E = pc \; , \quad \varepsilon_0 = 1/\mu_0 c^2$$

$$\Delta\nu_x = \frac{\mu_0 e c^2 I_2}{8\pi^2 b f_{rev} E} \cdot \frac{\sqrt{\beta^*_x}}{\sqrt{\varepsilon_x}\left(\sqrt{\varepsilon_x \beta^*_x} + \sqrt{\varepsilon_z \beta^*_z}\right)}$$

$$\Delta\nu_z = \frac{\mu_0 e c^2 I_2}{8\pi^2 b f_{rev} E} \cdot \frac{\sqrt{\beta^*_z}}{\sqrt{\varepsilon_z}\left(\sqrt{\varepsilon_x \beta^*_x} + \sqrt{\varepsilon_z \beta^*_z}\right)}$$

TYPICALLY $\Delta\nu \approx 0.2$

SO TUNE SHIFT RESTRICTS MAXIMUM CURRENT (AND HENCE LUMINOSITY)

$$I_{MAX,x} = \frac{8\pi^2 b f_{REV} E \sqrt{\varepsilon_x}\left(\sqrt{\varepsilon_x \beta_x^*} + \sqrt{\varepsilon_z \beta_z^*}\right)}{\mu_0 e c^2 \sqrt{\beta_x^*}} \cdot \Delta\nu_x^{MAX}$$

$$I_{MAX,z} = \frac{8\pi^2 b f_{REV} E \sqrt{\varepsilon_z}\left(\sqrt{\varepsilon_x \beta_x^*} + \sqrt{\varepsilon_z \beta_z^*}\right)}{\mu_0 e c^2 \sqrt{\beta_z^*}} \cdot \Delta\nu_z^{MAX}$$

HORIZONTAL – VERTICAL COUPLING OF BTRON OSC

$$k = \frac{\varepsilon_z}{\varepsilon_x} \rightarrow \quad \varepsilon_x = \frac{\varepsilon_{x_0}}{1+k}, \quad \varepsilon_z = \frac{k\,\varepsilon_{x_0}}{1+k}$$

IDEAL MACHINE $k = 0$ BUT COUPLING DRIVES VERTICAL OSCILLATION

IDEAL MACHINE

$\varepsilon_z = 0$

$\varepsilon_x = \varepsilon_{x_0}$

ONLY TRUE FOR AN ELECTRON MACHINE

$$I_{MAX,x} = \frac{8\pi^2 b f_{REV} E \varepsilon_{x_0} \left(\sqrt{\beta_x^*} + \sqrt{k\beta_z^*}\right)}{\mu_0 e c^2 (1+k)\sqrt{\beta_x^*}} \cdot \Delta\nu_x^{MAX}$$

$$I_{MAX,z} = \frac{8\pi^2 b f_{REV} E \varepsilon_{x_0} \sqrt{k} \left(\sqrt{\beta_x^*} + \sqrt{k\beta_z^*}\right)}{\mu_0 e c^2 (1+k)\sqrt{\beta_z^*}} \Delta\nu_z^{MAX}$$

THESE ARE THE MAXIMUM CURRENTS ALLOWED BY x OR z TUNE SHIFT. THE MAXIMUM CURRENT IN THE MACHINE DEPENDS OF WHICH OF x OR z CAN HAVE MAXIMUM TUNE SHIFT

MAXIMUM CURRENT DEPENDS ON ENERGY

EMITTANCE ALSO ENERGY DEPENDENT

NORMALIZED EMITTANCE

$$\varepsilon_{x,z} = \gamma^2 \tilde{\varepsilon}_{x,z}$$

$\gamma = \frac{E}{E_0}$ $\rightarrow m_0 c^2$

emittance at $E_0 = m_0 c^2$

WILLE 6.48

$$I_{MAX} = \frac{8\pi^2 b\, m_0 f_{REV}\, \gamma^3 \tilde{\varepsilon}_{x_0} \sqrt{k} \left(\sqrt{\beta_x^*} + \sqrt{k\beta_z^*}\right)}{\mu_0 e (1+k)\sqrt{\beta_z^*}} \Delta\nu_{MAX}$$

MAXIMUM LUMINOSITY IS WHEN BOTH BEAMS ARE AT SPACE-CHARGE TUNE SHIFT LIMIT

$$\mathcal{L}_{MAX} = \frac{16\pi^3 b\, f_{REV}\, m_0^2 \gamma^4 \tilde{\varepsilon}_{x_0} \sqrt{k} \left(\sqrt{\beta_x^*} + \sqrt{k\beta_z^*}\right)^2 \Delta\nu_{MAX}^2}{\mu_0^2 e^4 (1+k)\sqrt{\beta_x^*}\; \beta_z^{*\,3/2}}$$

MAXIMUM LUMINOSITY $\beta_z \rightarrow 0$

$\varepsilon_{x_0} \rightarrow$ BEAM PIPE APERTURE

DEFINE $\varepsilon_{x_0} = E^2 \hat{\varepsilon}_{x0}$

IN GeV

NORMALIZED TO 1 GeV

COLLECT ALL PHYSICAL CONSTANTS:

$$I_{MAX}(mA) = 699.06 \frac{b f_{REV} E^3 \hat{\varepsilon}_{x0} \sqrt{k} \left(\sqrt{\beta_x^*} + \sqrt{k \beta_y^*}\right)}{(1+k)\sqrt{\beta_y^*}} \Delta V_{MAX}$$

E in GeV

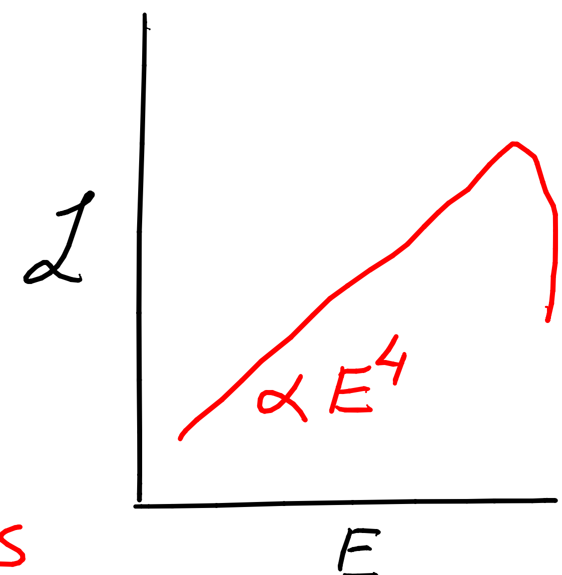
$$\mathcal{L}_{MAX}\left[cm^{-2}s^{-1}\right] = 1.515 \times 10^{32} \frac{b f_{REV} E^4 \hat{\varepsilon}_{x0} \sqrt{k} \left(\sqrt{\beta_x^*} + \sqrt{k \beta_y^*}\right)^2 \Delta V_{MAX}^2}{(1+k)\sqrt{\beta_x^*}\, {\beta_y^*}^{3/2}}$$

$\mathcal{L} \propto E^4$

$I \propto E^3$ ← NEEDS RF @ MAX ENERGY

RUN OUT OF RF

$SR \propto E^4$ ∴ ANY INCREASE IN $E \to I$ DROPS $\to \mathcal{L}$ DROPS



ASSUME MAXIMUM TUNE SHIFT IS THE SAME IN BOTH PLANES

$$\Delta\nu_{MAX,x} = \Delta\nu_{MAX,y}$$

$$k = \beta_y^* / \beta_x^*$$

$$I_{MAX} = \frac{8\pi^2 b f_{REV}\, m_0 \gamma^3 \tilde{\varepsilon}_{x0}}{\mu_0 e} \Delta\nu_{MAX}$$

$$\sigma_x^* \sigma_y^* = \frac{\beta_x^* \beta_y^*}{\beta_x^* + \beta_y^*} \cdot \varepsilon_{x0}$$

$$\mathcal{L}_{MAX} = \frac{16\pi^2 b\, f_{REV}\, m_0^2 \gamma^4 \tilde{\varepsilon}_{x0} (\beta_x^* + \beta_y^*) \Delta\nu^2}{\mu_0^2 e^4 \beta_x^* \beta_y^*}$$