Shell & Collective Nuclear Models

- Consistent Quantum Mechanical Model
  - Nucleons orbit in a common potential due to all other nucleons. $V(\vec{r})$
  - Pauli Exclusion $\rightarrow$ Shell Structure
    cf Atom
  - Spin-orbit Potential $V_{TOT} = V(\vec{r}) - f(\vec{r})\,\vec{L}\cdot\vec{S}$
- Excellent Description
  - Energy levels
  - Spins & magnetic moments
  - Closed shells $\rightarrow$ very stable nuclei

COLLECTIVE MODEL

SHELL MODEL FAILS ON!

- DIPOLE MOMENTS
- QUADRUPOLE MOMENTS
  - SHOULD BE ZERO FOR CLOSED SHELLS → NUCLEI SPHERICAL

EXPERIMENTALLY FIND THAT VERY HEAVY NUCLEI ARE NOT SPHERICAL

- HARD NUCLEAR CORE OF CLOSED SHELLS — LIQUID DROP
- VALENCE NUCLEONS — SPIN/MAGNETIC MOMENTS
- ROTATION OF VALENCE SHELL AROUND CORE

NON CENTRAL POTENTIAL

$$V(\vec{r}) \neq V(|\vec{r}|)$$

MAGIC NUMBERS – EVIDENCE FOR SHELLS

- ATOMS → CLOSED SHELLS → STABLE ELEMENTS
- NUCLEI → PEAKS IN BINDING ENERGY

  $N = 2, 8, 20, 28, 50, 82, 126$

  $Z = 2, 8, 20, 28, 50, 82$

  MAGIC NUCLEI – EXTREMELY STABLE

  $^{4}_{2}He$ $\quad ^{16}_{8}O$ $\quad ^{208}_{82}Pb$ → DOUBLY MAGIC

- MAGIC NUCLEI → MORE STABLE SPECIES

  $N = 20$ → 5 STABLE ISOTONES ← SAME # NEUTRONS

  19 → NONE

  21 → 1, UNSTABLE

- MAGIC NUCLEI → MAGNETIC MOMENTS ZERO

ASSUME NUCLEONS ORBIT IN SOME COMMON POTENTIAL

$$\mathcal{H}\,\psi(\vec{r}) = E\,\psi(\vec{r})$$ → CENTRAL POTENTIAL

IN NON RELATIVISTIC SCHRÖDINGER

$$\left[\nabla^2 + \frac{2m}{\hbar}\left(E - V(r)\right)\right]\psi(\vec{r}) = 0$$

$[H, J] = 0$ → ENERGY & ANGULAR MOMENTUM OPERATORS COMMUTE IN A CENTRAL POTENTIAL

ENERGY EIGENSTATES ARE ANGULAR MOMENTUM EIGENSTATES

CAN LABEL ENERGY STATES WITH ANGULAR MOMENTUM QUANTUM NUMBERS

RECALL FROM HYDROGEN ATOM

$$\bar{J} = \bar{L} + \bar{S}$$

ENERGY LEVEL $\longrightarrow$ $n \; L_J^P$

$P \rightarrow$ PARITY $(-1)^L$

$J \rightarrow$ TOTAL ANGULAR MOMENTUM

$L \leftarrow$ ORBITAL ANGULAR MOMENTUM

SPECTROSCOPIC NOTATION

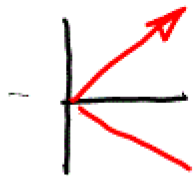
$\ell =$	0	1	2	3	4	5
	S	P	D	F	G	H

eg $1S_{1/2}$ $\quad 1P_{1/2}$ $\quad 1P_{3/2}$ $\quad 1D_{5/2}$ $\quad 1D_{3/2}$

2 - - - - - - - - - - - - - - - - -

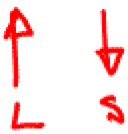
FOR EACH ℓ THERE ARE $2(2\ell+1)$ DEGENERATE STATES

$n=1 \quad \ell=0$ $\quad 1S_{1/2}$

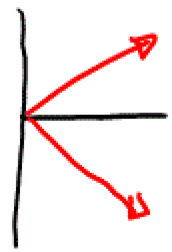


$j_z = +\frac{1}{2}$, $-\frac{1}{2}$ $\quad$ 2 STATES

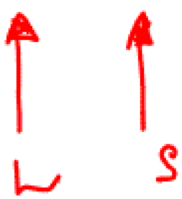
$n=1 \quad \ell=1$



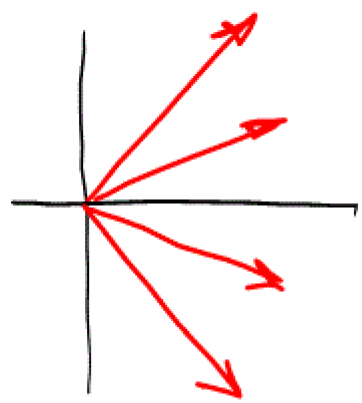
$1P_{1/2}$



$j_z = +\frac{1}{2}$, $-\frac{1}{2}$



$1P_{3/2}$



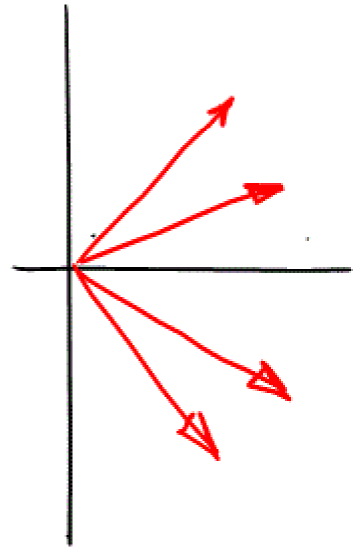
$j_z = +3/2$, $+1/2$, $-1/2$, $-3/2$

6 STATES

$n = 1 \quad \ell = 2$

$\uparrow L \quad \downarrow S$

$^1D_{3/2}$

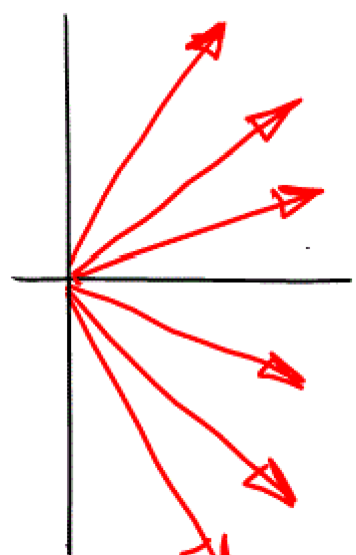


j_z: $+\frac{3}{2}$, $+\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{3}{2}$

$n = 1 \quad \ell = 2$

$\uparrow L \quad \uparrow S$

$^1D_{5/2}$



j_z: $+5/2$, $+3/2$, $+1/2$, $-1/2$, $-3/2$, $-5/2$

$\left.\begin{array}{l} \\ \\ \end{array}\right\}$ 10 STATES

WRITE SCHRÖDINGER IN SPHERICAL COORDS

— SEPARABLE → cf HYDROGEN ATOM

$$\hat{L}^2 Y_{\ell,m_\ell}(\theta,\phi) = \hbar^2 \ell(\ell+1) Y_{\ell,m_\ell}(\theta,\phi)$$

$$L_z Y_{\ell,m_\ell}(\theta,\phi) = \hbar^2 m_\ell Y_{\ell,m_\ell}(\theta,\phi)$$

$$\psi_{n\ell m_\ell}(\vec{r}) = \frac{u_{n\ell}(r)}{r} Y_{\ell,m_\ell}(\theta,\phi)$$

RADIAL WAVE FUNCTION

$$\left(\frac{d^2}{dr^2} + \frac{2m}{\hbar^2}\left(E_{n\ell} - V(r) - \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right)\right) u_{n\ell}(r) = 0$$

n — RADIAL QUANTUM # LABELS ENERGY EIGENSTATES

$$\left(\frac{d^2}{dr^2} + \frac{2m}{\hbar^2}\left(E_{n\ell} - V(r) - \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right)\right) u_{n\ell}(r) = 0$$

"CENTRIFUGAL BARRIER"
ACTS AS REPULSIVE POTENTIAL

$u_{n\ell}(r)$ MUST VANISH AT ORIGIN & ∞

$$V'(r) = V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}$$

NO APRIORI WAY OF KNOWING $V(r)$

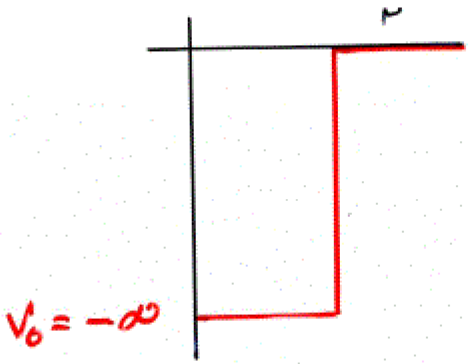
DEDUCE NUCLEAR POTENTIAL

MAKE ASSUMPTION

MODIFY

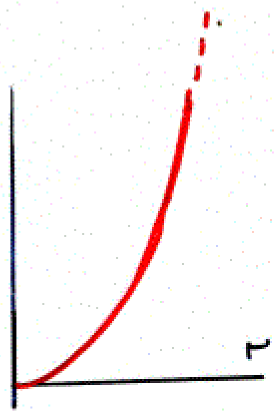
COMPARE WITH EXPERIMENT

GUESSES AT NUCLEAR POTENTIAL



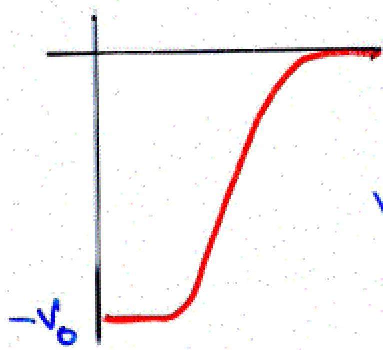
INFINITE SQUARE WELL

$$V(r) = \begin{cases} \infty, & r \geq R \\ 0, & R \geq r \geq 0 \end{cases}$$



HARMONIC OSCILLATOR

$$V(r) = \frac{1}{2} m \omega^2 r^2$$



SAXON – WOODS

$$V(r) = \frac{-V_0}{1 + \exp\left(\frac{r-R}{d}\right)}$$

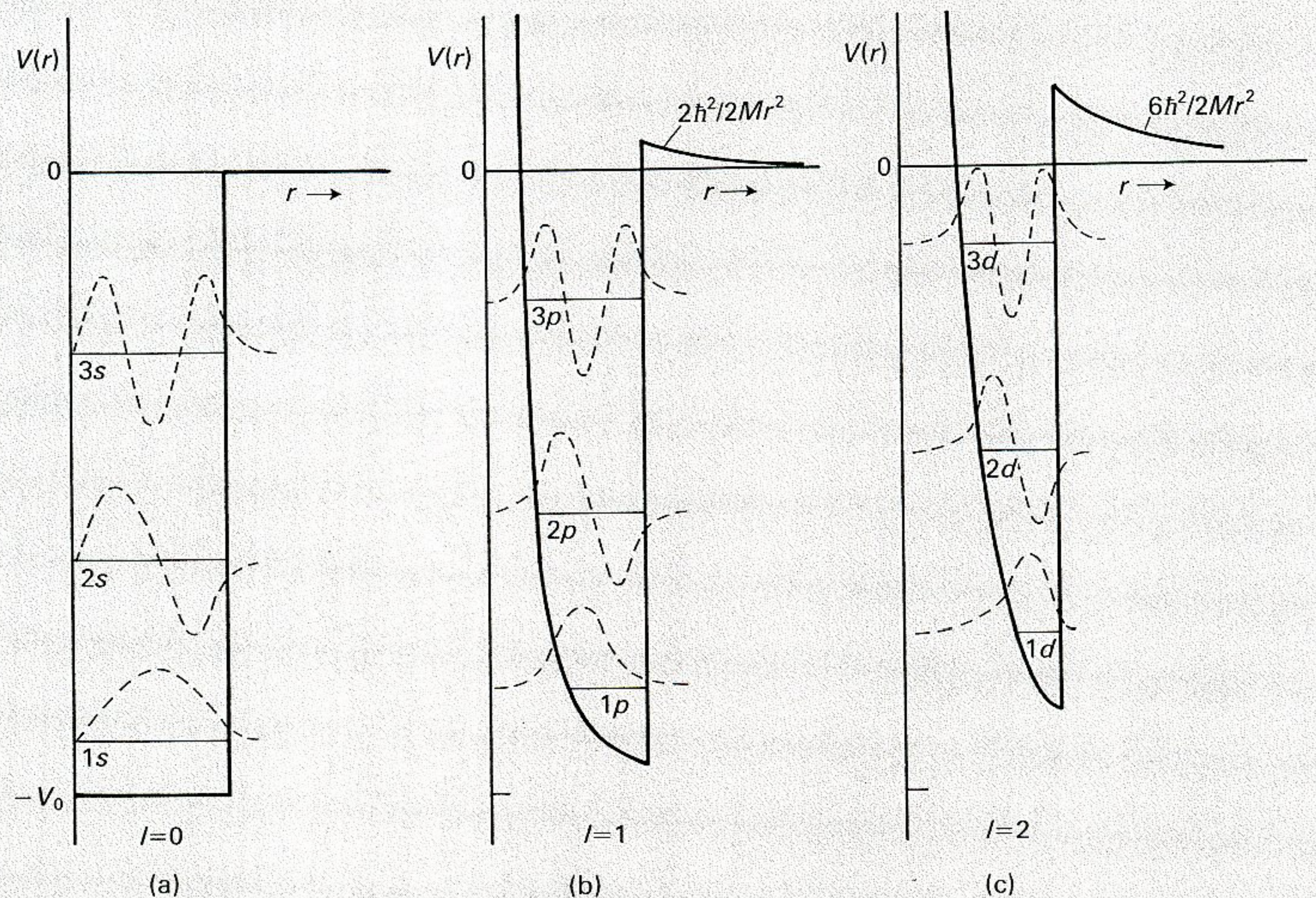


Fig. 8.4 A square-well spherical potential as a function of radius r is shown in (a). It is modified by the angular momentum barrier and the new shapes are given in (b) for $l=1$ and in (c) for $l=2$. The single particle energy levels are shown by horizontal lines for the states nl, $n=1, 2, 3$, and $l=0, 1, 2$ (s, p, d respectively), where n is the principal quantum number and l is the orbital angular momentum quantum number. The broken line shows the form of $rR(r)$ for the appropriate wavefunction, plotted about its energy level line as zero. Remember that the probability density distribution as a function of radius for the particle is proportional to $(rR(r))^2$.

INFINITE SQUARE WELL

SOLUTION OF RADIAL SCHRÖDINGER

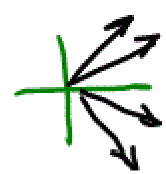
BESSEL

$$u_{n\ell}(r) = j_\ell(k_{n\ell}\, r) \qquad j_\ell(x) = \sqrt{\frac{\pi}{2x}}\, J_{\ell+\frac{1}{2}}(x)$$

$$k_{n\ell} = \left(\frac{2mE_{n\ell}}{\hbar^2}\right)$$

ENERGY EIGENVALUE IS n^{th} ZERO OF ℓ^{TH} BESSEL

n, ℓ NON DEGENERATE

$2(2\ell+1)$ VALUES OF m_ℓ  STILL DEGENERATE

- THIS SIMPLE SHELL MODEL SHOULD REPRODUCE THE MAGIC NUMBERS IF THEY CORRESPOND TO CLOSED SHELLS

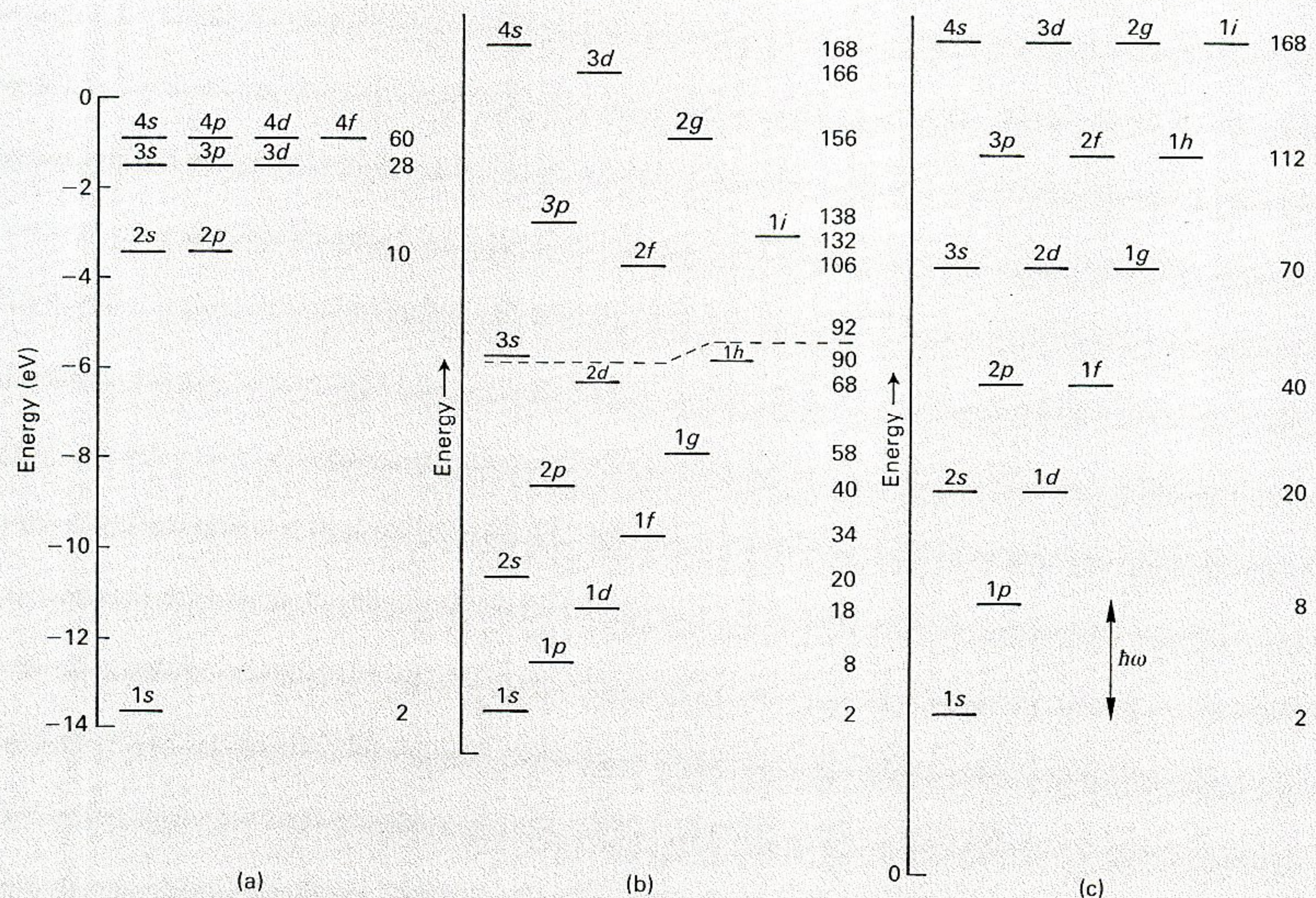
- A CLOSED SHELL CORRESPONDS TO A GIVEN ℓ WHERE ALL j_z STATES FILLED

- FOR $n = 1$

\# STATES

	$\ell = 0$		$\ell = 1$		$\ell = 2$		$\ell = 3$	
	$1S_{1/2}$	2	$1P_{1/2}$	2	$1D_{3/2}$	4	$1F_{5/2}$	6
			$1P_{3/2}$	4	$1D_{5/2}$	6	$1F_{7/2}$	8
MAGIC		2		8		18		32
EXPERIMENT		2		8		20		28

$\rightarrow$ STARTS OUT WELL !



ig. 8.5 The single-particle energy levels in three spherical otential wells: (a) the Coulomb, (b) the infinite square well, nd (c) the harmonic oscillator. In (a) the ordinate gives the ctual energies. In (b) and (c) the energy scale is arbitrary but the relative energies above zero at the well bottom in each case are shown correctly. On the right of each we show the accumulated occupancy starting with the 1*s* level and working upwards in energy. The magic numbers 2, 8, 20 occur for (b) and (c).

SPIN ORBIT COUPLING

SUGGEST $$V_{TOT} = V(r) - f(r)\,\bar{L}\cdot\bar{S}$$

CENTRAL POTENTIAL $\quad$ ORBIT $\quad$ SPIN

THIS WILL SPLIT DEGENERATE $\quad j = \ell \pm \frac{1}{2}$

cf PROBLEM SET $$\langle \bar{L}\cdot\bar{S}\rangle = \langle \tfrac{1}{2}(\bar{J}^2 - \bar{L}^2 - \bar{S}^2)\rangle$$

$$= \frac{\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) - s(s+1)\right]$$

$$= \frac{\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) - 3/4\right]$$

$$= \begin{cases} \frac{\hbar^2}{2}\cdot \ell & \text{FOR } j = \ell + \frac{1}{2} \\ -\frac{\hbar^2}{2}(\ell+1) & \text{FOR } j = \ell - 1/2 \end{cases}$$

ENERGY SHIFT DUE TO SPIN ORBIT COUPLING

$$\mathcal{H}\psi_n = E\psi_n \rightarrow \langle \mathcal{H} \rangle = \int d^3r\, \mathcal{H}\, \psi_n^* \psi = \int d^3r\, E_n \psi_n^* \psi_n$$

GENERAL EIGENVALUE EQUATION

$$= E_n \int d^3r\, |\psi|^2$$

$$\Delta E_{n\ell}\left(j = \ell + 1/2\right) = \int d^3r\, f(r) \langle L \cdot S \rangle |\psi_{n\ell}|^2$$

$$= \frac{\hbar^2 \ell}{2} \int d^3r\, |\psi_{n\ell}|^2 f(r)$$

$$\Delta E_{n\ell}\left(j = \ell - 1/2\right) = -\frac{\hbar^2(\ell+1)}{2} \int d^3r\, |\psi_{n\ell}|^2 f(r)$$

TOTAL LEVEL SPLITTING

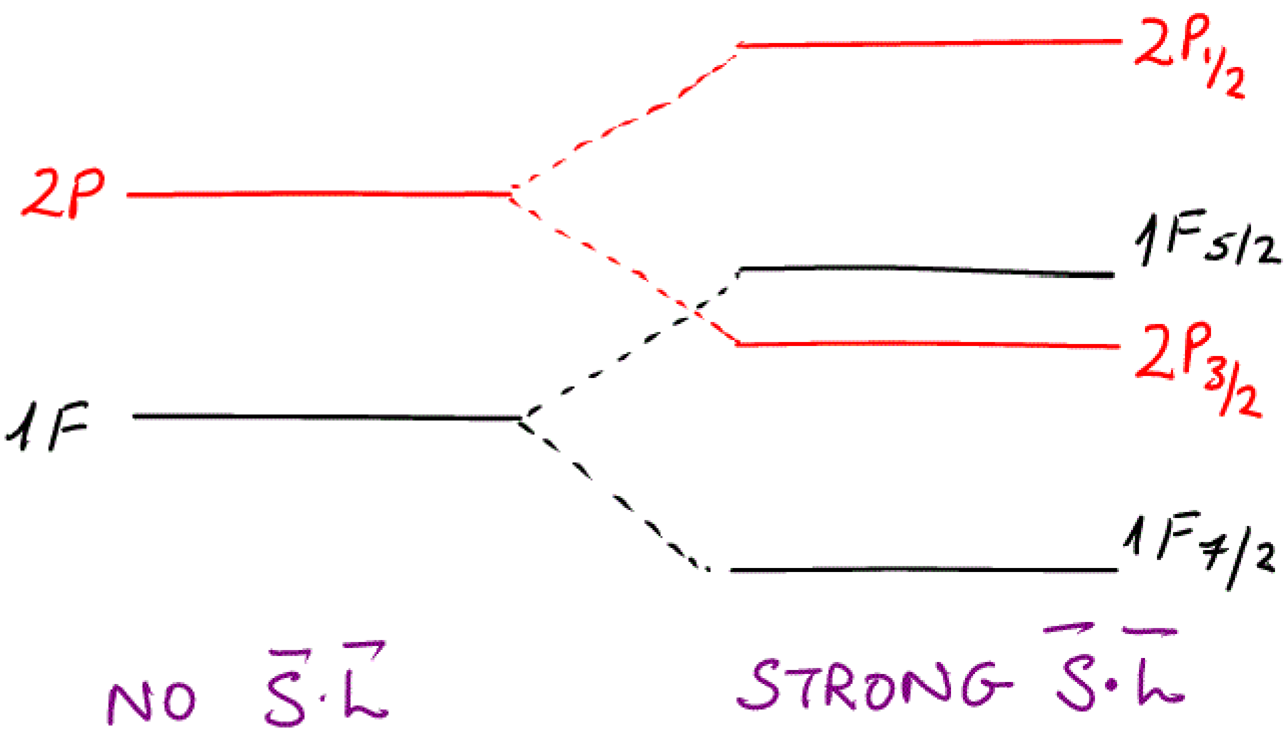
$\ell + \frac{1}{2} \rightarrow \ell - 1/2$

$$\Delta = \hbar^2\left(\ell + \tfrac{1}{2}\right) \int d^3r\, |\psi_{n\ell}|^2 f(r)$$

LEVEL SPLITTING $\Delta = \hbar^2 (\ell + \frac{1}{2}) \int d^3r \, |\psi_{n\ell}|^2 \, f(r)$

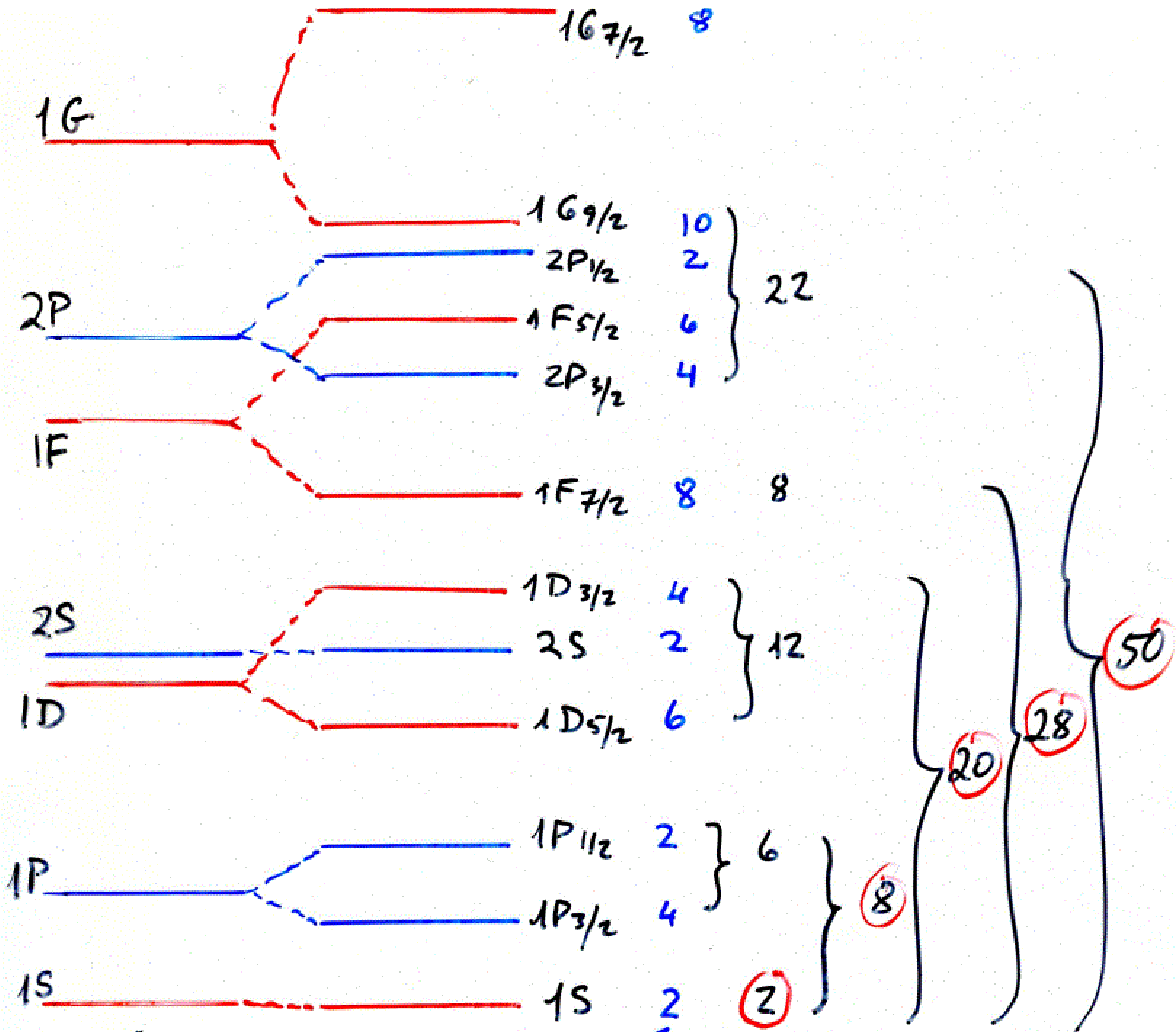
FIT TO MATCH DATA

- SPLITTING GETS LARGER FOR LARGE ORBITAL ANGULAR MOMENTUM
- THIS LEADS TO LEVEL CROSSING



NUCLEONS FILL STATES ACCORDING TO INCREASING ENERGY

GET OBSERVED MAGIC NUMBERS



1G
1G 7/2 8
1G 9/2 10
2P
2P 1/2 2
1F 5/2 6
2P 3/2 4
22
1F
1F 7/2 8
8
2S
1D 3/2 4
2S 2
1D
1D 5/2 6
12
50
28
20
1P
1P 1/2 2
1P 3/2 4
6
8
1S
1S 2
2

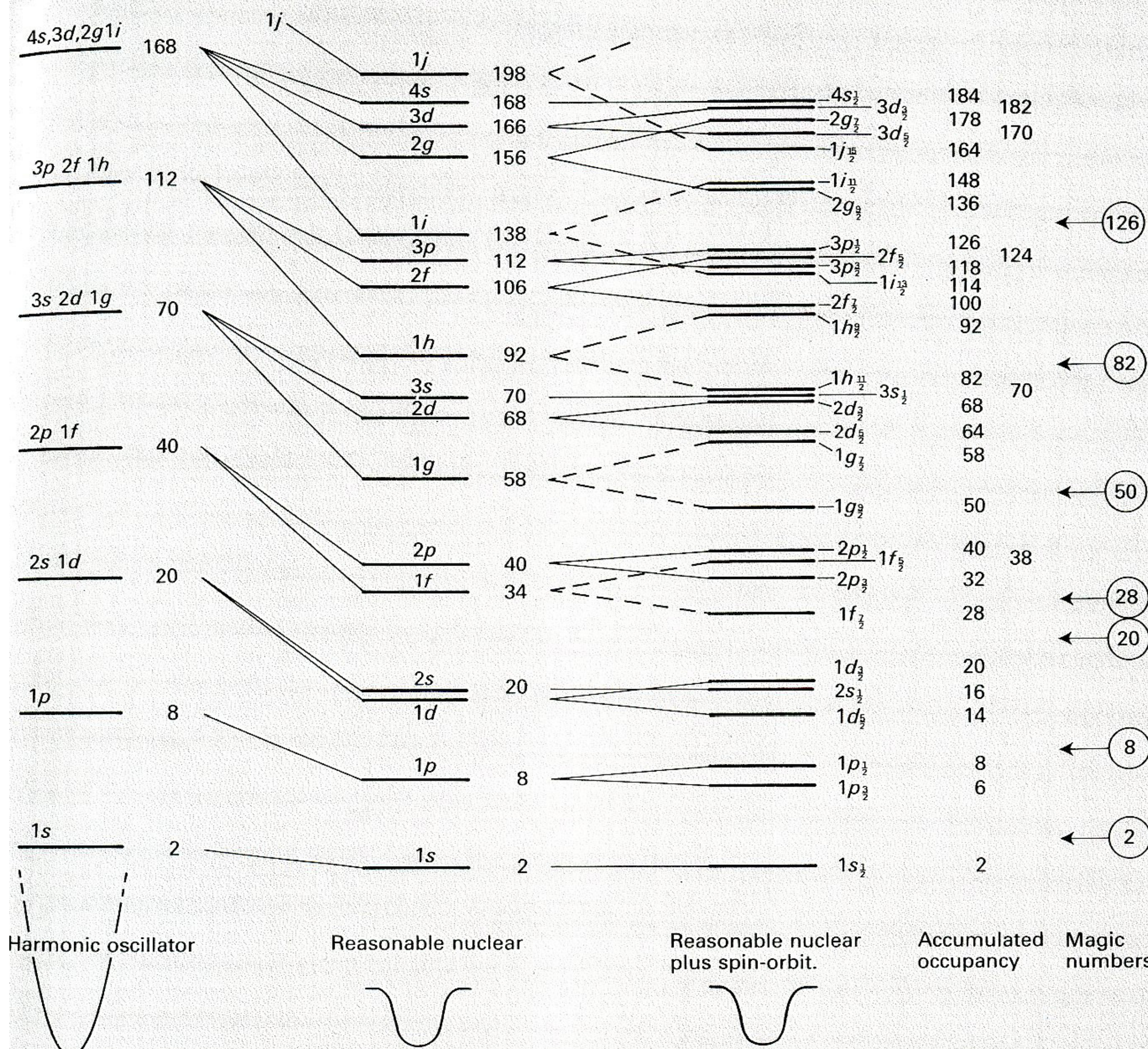


Fig. 8.6 A schematic representation of the change in single-particle level ordering in moving from (a) the harmonic oscillator to (b) a reasonable nuclear potential to (c) the same with an inverted spin-orbit interaction added. On the right of each are shown the accumulated occupancy numbers. An ordering is given in (c) which gives the magic numbers. Between these numbers the residual interactions between nucleons will alter the ordering as the levels are filled. The splittings caused by the spin–orbit interaction which are vital to the explanation of the observed magic numbers are emphasized by heavy broken lines: they are $1f \rightarrow 1f_{7/2} + 1f_{5/2}$, $1g \rightarrow 1g_{9/2} + 1g_{7/2}$, $1h \rightarrow 1h_{11/2} + 1h_{9/2}$, and $1i \rightarrow 1i_{13/2} + 1i_{11/2}$. These are the splittings mainly responsible for the gaps defining the magic numbers.

NUCLEAR MAGNETIC MOMENTS

- NUCLEI WITH SPIN $\geq \frac{1}{2}$ $\rightarrow$ DIPOLE MAGNETIC MOMENTS
- EVEN-EVEN NUCLEI $\rightarrow$ NUCLEON SPINS PAIR TO GIVE ZERO MAGNETIC MOMENT
- ANGULAR MOMENTUM COUPLING IN ODD-ODD IS RATHER COMPLEX
- POSSIBLE TO CHECK PREDICTIONS OF SHELL MODEL FOR A-ODD NUCLEI

REMEMBER INTRINSIC MAGNETIC MOMENTS

PROTON $\mu_s = 2.7928$ NUCLEAR MAGNETONS

NEUTRON $\mu_s = -1.9130$

• GENERALLY $\mu = g \cdot j \cdot \frac{e\hbar}{2M_p}$

MAGNETIC MOMENT

g FACTOR

p = 2.7928
n = −1.9130

ANGULAR MOMENTUM

• IN ODD-A NUCLEI, ALL NUCLEONS ARE PAIRED EXCEPT ONE.

• GENERALLY TWO CONTRIBUTIONS TO MAGNETIC MOMENT

— INTRINSIC SPIN $\mu_s = g_s \cdot S$ $= 2.7928 \times \frac{e\hbar}{2M}$ PROTON
$= -1.9130 \times e\hbar/2M$ NEUTRON

— FROM ORBITAL MOTION $\mu_\ell = g_\ell \cdot \ell$ $= 1 \times \frac{e\hbar}{2M}$ PROTON
$= 0$ NEUTRON

NO CHARGE

$$\mu_N = \frac{e\hbar}{2M_N}$$

FOR ODD-A NUCLEI – 2 COMPONENTS OF μ

— INTRINSIC SPIN

— ORBITAL

FOR A PROTON $\mu_l = g_l \cdot l$ MAGNETONS

$$g_L = 1$$

$\mu_s = g_s \cdot s$ MAGNETONS

$$2.7928 = g_s \cdot \frac{1}{2}$$

$$g_s = 5.5856$$

SIMILARLY FOR AN UNPAIRED NEUTRON

$g_l = 0$ (UNCHARGED)

$$g_s = 2 \times (-1.9130) = -3.8261$$

FOR UNPAIRED NUCLEON $\mu = g_j \cdot j$

$$\bar{\mu}_s = g_s \bar{S}, \quad \bar{\mu}_\ell = g_\ell \bar{L}, \quad \bar{\mu} = g_j \bar{J}$$

$\bar{J} = \bar{L} + S$ AND $\bar{\mu} = g_j \bar{J} = g_\ell \bar{L} + g_s \bar{S}$

$$g_j \bar{J} \cdot \bar{J} = g_\ell \bar{L} \cdot \bar{J} + g_s \bar{S} \cdot \bar{J}$$

$$= g_\ell \bar{L}(\bar{L} + \bar{S}) + g_s \bar{S}(\bar{S} + \bar{L})$$

$$= g_\ell (\bar{L}^2 + \bar{L} \cdot \bar{S}) + g_s (\bar{S}^2 + \bar{L} \cdot \bar{S})$$

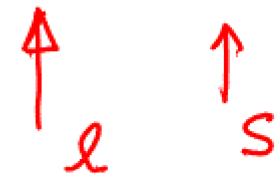
APPLY THIS TO $|j, \ell, s\rangle$

USE PROB SET RESULT

$$g_j = g_\ell \frac{[j(j+1) + \ell(\ell+1) - s(s+1)]}{2j(j+1)} + g_s \frac{[j(j+1) - \ell(\ell+1) + s(s+1)]}{2j(j+1)}$$

$$g_j = g_\ell \frac{[j(j+1) + \ell(\ell+1) - s(s+1)]}{2j(j+1)} + g_s \frac{[j(j+1) - \ell(\ell+1) + s(s+1)]}{2j(j+1)}$$

SPECIAL (SIMPLE) CASE



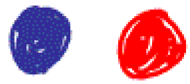
ORBITAL & SPIN ADD TO GIVE MAXIMUM TOTAL ANGULAR MOMENTUM

$$j = \ell + s \quad , \quad s = \frac{1}{2}$$

$$\mu = g_j j = g_\ell \ell + g_s \frac{1}{2} \quad \text{MAGNETONS}$$

LOOK AT SOME EXAMPLES →

DEUTERON



n p

ASSUME n, p ARE IN

$1S_{\frac{1}{2}} \rightarrow \ell = 0$

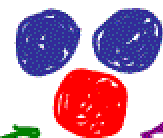
JUST ADD INTRINSIC

$$\mu_d = 2.79\mu_N - 1.91\mu_N = 0.88\mu_N$$

EXPERIMENT $0.86\mu_N$

TRITIUM ^3_1H

n n



CLOSED SHELL $1S_{1/2}$

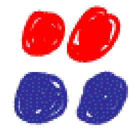
GIVES NUCLEAR SPIN & MAGNETIC MOMENT

$1S_{\frac{1}{2}} \therefore \ell = 0$

$$\mu_{^3_1H} = \mu_p = 2.79\mu_N$$

$2.98\mu_N$ EXPERIMENT

α - PARTICLE $\,{}^{4}_{2}He$



n, p CLOSED SHELLS
NO SPIN
NO MAGNETIC MOMENT

DOUBLY MAGIC

BERYLIUM $\,{}^{10}_{5}B$ $\quad (1S_{1/2})^{2}(1P_{3/2})^{3}$

5 PROTONS + 5 NEUTRON

2 OUT OF 2
STATES FILLED

3 OUT OF 4
STATES FILLED
– 2 NUCLEONS PAIR

ONE UNPAIRED NEUTRON, ONE UNPAIRED PROTON

PROTON IS $\ell = 1$ $\rightarrow$ $\mu = \left(\frac{e\hbar}{2m_p}\right) \cdot \ell = \mu_N$

UNCHARGED NEUTRON ONLY CONTRIBUTES INTRINSIC

$$\mu_{BE} = 2.79\mu_N + \mu_N - 1.91\mu_N = 1.88\mu_N$$

PROTON
INTRINSIC

PROTON
ORBITAL

NEUTRON
INTRINSIC

$1.80\mu_N$
EXPERIMENT.