DIRAC EQUATION

IN NON-RELATIVISTIC QM WE USE SCHRÖDINGER

WHEN WE GO TO RELATIVISTIC SITUATION WE CAN GUESS THAT THERE MUST BE SOME CORRESPONDING RELATIVISTIC WAVE EQUATION

IN TRYING TO CONSTRUCT THIS RELATIVISTIC WAVE EQUATION WE ASSUME THAT WE CAN USE RELATIVISTIC SINGLE PARTICLE WAVE FUNCTIONS

SO WE CANNOT CALCULATE MATRIX ELEMENTS FOR THE CREATION & ANNIHILATION OF PARTICLES

IN QUANTUM FIELD THEORY WAVE FNS $\longrightarrow$ CREATION AND ANNIHILATION OPERATORS

KLEIN - GORDON EQUATION

NEED A LORENTZ INVARIANT EQUATION (COVARIANT)

SCHRÖDINGER → 1ST ORDER TIME DERIVATIVE
→ 2ND ORDER SPACE DERIVATIVE

∴ CANNOT BE COVARIANT.

SCHRÖDINGER CONSTRUCTED FROM $E = \frac{\bar{p}^2}{2m}$ NONRELATIVISTIC

$i\,\partial/\partial t$ $\quad -i\bar{\nabla}$

TRY SAME TRICK FOR $E^2 = \bar{p}^2 + m^2$ RELATIVISTIC

$$\hat{E}^2\psi(\bar{x},t) = \hat{\bar{p}}^2\psi(\bar{x},t) + m^2\psi(\bar{x},t)$$

$$-\frac{\partial^2\psi}{\partial t^2} = -\bar{\nabla}^2\psi + m^2\psi \rightarrow \frac{\partial^2\psi}{\partial t^2} = \bar{\nabla}^2\psi - m^2\psi$$

KLEIN - GORDON
RELATIVISTIC WAVE EQUATION

$$\frac{\partial^2\psi}{\partial t^2} = \bar{\nabla}^2\psi - m^2\psi$$

SECOND ORDER IN SPACE AND TIME

$$\underbrace{\frac{\partial^2\psi}{\partial t^2} - \bar{\nabla}^2\psi}_{\text{4-VECTOR}} + m^2\psi = 0 \rightarrow \left(\partial^\mu\partial_\mu + m^2\right)\psi = 0$$

LORENTZ SCALARS

MANIFESTLY COVARIANT

THE KG HAS FREE PARTICLE PLANE WAVE SOLUTIONS

$$\psi(\bar{x},t) = N e^{i(\bar{p}_x\cdot\bar{x} - E\cdot t)}$$

$$\frac{\partial\psi}{\partial t} = iE\,N e^{i(\bar{p}\cdot\bar{x} - E\cdot t)}$$

$$\frac{\partial^2\psi}{\partial t^2} = -\underset{\text{EIGENVALUE}}{E^2}\,\underbrace{N e^{i(\bar{p}\cdot\bar{x} - E\cdot t)}}_{\text{EIGENSTATE}}$$

SAME THING FOR MOMENTUM:

$$\bar{\nabla}\psi = i\bar{p}\,N e^{i(\bar{p}\cdot\bar{x}-Et)} \rightarrow \bar{\nabla}^2\psi = -p^2 N e^{i(\bar{p}\cdot\bar{x}-Et)}$$

$$\frac{\partial^2\psi}{\partial t^2} - \bar{\nabla}^2\psi + m^2\psi = 0$$

$$-E^2\psi + p^2\psi = -m^2\psi \rightarrow E^2 - p^2 = m^2 \checkmark$$

BY CONSTRUCTING PLANE WAVE SOLUTIONS, THEY SATISFY ENERGY EQUATIONS $E = \pm\sqrt{p^2+m}$

$-$ve E???

IN QUANTUM MECHANICS SOLUTIONS OF THE WAVE EQUATION FORM COMPLETE SET OF STATES

CANNOT DISCARD $-$VE SOLUTION

PROBABILITY CURRENT HAS PHYSICAL SIGNIFICANCE IN SCHRÖDINGER WHAT DOES IT LOOK LIKE IN THE KG EQUATION??

$$\frac{\partial^2\psi}{\partial t^2} = \bar{\nabla}^2\psi - m^2\psi \leftarrow (4.1) \quad \text{TAKE } \psi^* \times (4.1) - \psi * (4.1)^*$$

$$\psi^*\frac{\partial^2\psi}{\partial t^2} - \psi\frac{\partial^2\psi^*}{\partial t^2} = \psi^*(\bar{\nabla}^2\psi - m^2\psi) - \psi(\bar{\nabla}^2\psi^* - m^2\psi^*)$$

$$\|$$

$$\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) = \cancel{\frac{\partial\psi^*}{\partial t}\cdot\frac{\partial\psi}{\partial t}} + \psi^*\frac{\partial^2\psi}{\partial t^2} - \cancel{\frac{\partial\psi}{\partial t}\frac{\partial\psi^*}{\partial t}} - \psi\frac{\partial^2\psi^*}{\partial t^2}$$

$$\psi^*(\nabla^2\psi - m^2\psi) - \psi(\nabla^2\psi^* - m^2\psi^*)$$

$$= \bar{\nabla}\cdot(\psi^*\bar{\nabla}\psi) - \cancel{\psi^* m^2\psi} - \bar{\nabla}(\psi\bar{\nabla}\psi^*) + \cancel{m^2\psi\psi^*}$$

$$\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) = \bar{\nabla}\cdot(\psi^*\bar{\nabla}\psi - \psi\bar{\nabla}\psi^*)$$

COMPARE TO CONTINUITY EQUATION

$$\bar{\nabla}\cdot\bar{J} + \frac{\partial\rho}{\partial t} = 0 \rightarrow \quad \rho = i\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) \quad \text{PROB DENSITY}$$

$$\bar{J} = -i(\psi^*\bar{\nabla}\psi - \psi\bar{\nabla}\psi^*) \quad \text{PROB CURRENT}$$

$$\rho = i\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) \quad ; \quad \bar{J} = -i\left(\psi^* \bar{\nabla} \psi - \psi \bar{\nabla} \psi^*\right)$$

$$\psi^* = N e^{-i(\bar{p}\cdot\bar{x} - Et)} \rightarrow \frac{\partial \psi^*}{\partial t} = +i\, E N e^{-i(\bar{p}\cdot x - Et)}$$

$$\psi = N e^{+i(\bar{p}\cdot\bar{x} - Et)} \rightarrow \frac{\partial \psi}{\partial t} = -i\, E N e^{+i(\bar{p}\cdot x - Et)}$$

$$\rho = i\Big(N e^{-i(\bar{p}\cdot x - Et)} \cdot (-i) E N e^{i(\bar{p}\cdot x - Et)}$$

$$- N e^{+i(\bar{p}\cdot x - Et)} \cdot (+i) E N e^{-i(\bar{p}\cdot x - Et}\Big)$$

$$\rho = -i\left(N^2 E(-i) - (+i) N^2 E\right)$$

$$= N^2 E + N^2 E$$

SEE WHY WE WROTE $\rho = i(\ldots\ldots)$?

$$\rho = N^2 E$$

SAME PROCEDURE $\rightarrow$ $\bar{J} = 2N^2 \bar{p}$

WRITE AS 4-VECTOR $J^\mu_{KG} = 2N^2 p^\mu$

$$j^\mu_{KG} = 2N^2 p^\mu$$

PROBABILITY DENSITY $\propto$ E → RELATIVISTIC LENGTH CONTRACTION

REMEMBER E CAN BE -VE → NEGATIVE PROB DENSITY

↕

UNPHYSICAL

KLEIN GORDON CANNOT DESCRIBE SINGLE PARTICLE STATES

IN QFT IT REAPPEARS AS A MULTI-PARTICLE EXCITATION OF SPIN-0 QUANTUM FIELD

COVERED IN QFT!

ANYWAY – DIRAC REALIZED THAT WE NEED A RELATIVISTIC SINGLE PARTICLE WAVE EQUATION.

DIRAC EQUATION

KG EQUATION $\quad E^2 = \bar{p}^2 + m^2 \rightarrow \hat{E}^2\psi = \hat{\bar{p}}^2\psi + m^2\psi$

KG IS COVARIANT BECAUSE POWERS OF E & $\bar{p}$ SAME

-VE ENERGIES

THEY HAVE TO BE, SINCE THEY FORM A 4-VECTOR

DIRAC REASONED THAT $\hat{E}$ & $\hat{\bar{p}}$ MUST BE 1ST ORDER

$\underbrace{E = p + m}_{\text{NOT TRUE}} \rightarrow \hat{E}\psi = \hat{\bar{p}}\psi + m\psi$?

WHAT ABOUT $\hat{E}\psi = (\underbrace{\bar{\alpha}\cdot\hat{\bar{p}}}_{\text{SCALAR}} + \underbrace{\beta m}_{\text{SCALAR}})\psi$??

WHAT VALUES OF $\bar{\alpha}$, β WOULD MAKE THIS TRUE?

$$\hat{E}\psi = (\underline{\alpha} \cdot \hat{\underline{p}} + \beta m)\psi$$ ← WRITE IN TERMS OF OPERATORS

$$i\frac{\partial \psi}{\partial t} = \left(-i\alpha_x \frac{\partial}{\partial x} - i\alpha_y \frac{\partial}{\partial y} - i\alpha_z \frac{\partial}{\partial z} + \beta m\right)\psi \qquad (4.7)$$

IF SOLUTIONS OF THIS REPRESENT RELATIVISTIC PARTICLES — SOLNS. MUST ALSO SATISFY

$$E^2 = \underline{p}^2 + m^2$$ SQUARE 4.7

$$\frac{\partial^2 \psi}{\partial t^2} = \alpha_x^2 \frac{\partial^2 \psi}{\partial x^2} + \alpha_y^2 \frac{\partial^2 \psi}{\partial y^2} + \alpha_z^2 \frac{\partial^2 \psi}{\partial z^2} - \beta^2 m^2 \psi + \cdots$$

$$+ (\alpha_x\alpha_y + \alpha_y\alpha_x)\frac{\partial^2 \psi}{\partial x \partial y} + (\alpha_y\alpha_z + \alpha_z\alpha_y)\frac{\partial^2 \psi}{\partial y \partial z} + (\alpha_z\alpha_x + \alpha_x\alpha_z)\frac{\partial^2 \psi}{\partial z \partial x}$$

$$+ i(\alpha_x\beta + \beta\alpha_x)\frac{\partial \psi}{\partial x} \cdot m + i(\alpha_y\beta + \beta\alpha_y) m \frac{\partial \psi}{\partial y}$$

$$+ i(\alpha_z\beta + \beta\alpha_z)\frac{\partial \psi}{\partial z} \cdot m$$

FOR ALL THIS TO REDUCE TO $\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - m^2 \psi$

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I$$ ← IDENTITY MATRIX

$$(\alpha_j \beta + \beta \alpha_j) = 0 \quad ; \quad (\alpha_j \alpha_k + \alpha_k \alpha_j) = 0 \quad j \neq k$$

THESE ARE COMMUTATORS AND CANNOT BE SATISFIED BY REAL OR IMAGINARY NUMBERS

→ MATRICES $\quad Tr(ABC) = Tr(BCA) = Tr(CAB)$

FROM ABOVE $\quad \beta^2 = I \quad ; \quad \alpha_i \beta = -\beta \alpha_i$

$$Tr(\alpha_i) = Tr(\alpha_i \beta\beta) = Tr(\beta \alpha_i \beta) = -Tr(\alpha_i \beta\beta)$$

$$Tr(\alpha_i) = -Tr(\alpha_i \beta\beta) = -Tr(\alpha_i)$$

SO α_i MUST HAVE ZERO TRACE

EIGEN EQUATION $\alpha_i \cdot X = \lambda X \rightarrow \alpha_i^2 X = \alpha_i \lambda X$

$\hookrightarrow = I$

$$X = \lambda \alpha_i X \longrightarrow X = \lambda^2 X \rightarrow \lambda = \pm 1$$

YOU KNOW THAT $\sum$ EIGENVALUES = TRACE

$n(+1) + m(-1) = 0$ $n+m$ = NO. OF DIAGONAL ELEMENTS

$n = m = 0$? TRIVAL

$n = 1, m = 0$? NUMBER $Tr \neq 0$

$n = m = 1$? 2x2 MATRIX $\longrightarrow$ OK!

$n = 1, m = 2$? 3x3 MATRIX

$$\begin{pmatrix} -1 & & \\ & +1 & \\ & & -1 \end{pmatrix} \rightarrow Tr \neq 0$$

α, β MATRICES $\longleftarrow$

EVEN DIMENSION

$$\hat{E}\psi = (\bar{\alpha}\cdot\bar{p} + \beta m)\psi \;, \quad \hat{H}_D = (\bar{\alpha}\cdot\bar{p} + \beta m)$$

$\longrightarrow \hat{H}_D \psi = E\psi$

SINCE E IS AN OBSERVABLE $\rightarrow \hat{H}_D$ MUST BE HERMITIAN

$\rightarrow \bar{\alpha}$, β MATRICES HAVE TO BE HERMITIAN

$$\alpha_x = \alpha_x^{+} \;, \quad \alpha_y = \alpha_y^{+} \;, \quad \alpha_z = \alpha_z^{+} \;, \quad \beta = \beta^{+}$$

$\alpha_x, \alpha_y, \alpha_z, \beta$ ARE 4 MUTUALLY ANTICOMMUTING MATRICES OF EVEN DIMENSION

$\rightarrow$ FOR 2×2 TRACELESS MATRICES, ONLY 3 INDEPENDENT

HAVE ALREADY RULED OUT 3×3 MATRICES

NEXT UP IS 4×4 $\longleftarrow$ THIS IS IT

$$\hat{H}_D \psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi = E\psi$$

DIRAC HAMILTONIAN MUST BE 4×4 MATRIX OF OPERATORS, ACTING ON A 4-COMPONENT WAVE FUNCTION

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \rightarrow \text{DIRAC SPINOR}$$

FOR WAVE FUNCTION TO SATISFY DIRAC EQTN AND ALSO KLEIN GORDON, WE END UP WITH FOUR DEGREES OF FREEDOM, IN ADDITION TO $\vec{x}$ AND t

→ WHAT DO THESE NEW DEGREES OF FREEDOM CORRESPOND TO??

$$\alpha_z^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I \leftarrow \text{IDENTITY MATRIX}$$

$$(\alpha_j \beta + \beta \alpha_j) = 0 \quad ; \quad (\alpha_j \alpha_k + \alpha_k \alpha_j) = 0 \quad j \neq k$$

THIS ALGEBRA COMPLETELY DEFINES DIRAC EQUATION

EASIER TO UNDERSTAND WHAT IS GOING ON IF WE INTRODUCE EXPLICIT FORM OF $\underline{\alpha}$, β

CONVENTIONALLY → DIRAC-PAULI REPRESENTATION

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

→ PAULI SPIN MATRICES

PROPERTIES OF DIRAC EQTN COME FROM $\underline{\alpha}$, β ALGEBRA NOT ANY SPECIFIC REPRESENTATION.

COULD CHOOSE ANY OTHER SET OF 4x4, UNITARY HERMITIAN MATRICES

PROBABILITY DENSITY AND PROBABILITY CURRENT

KG WAS PATHOLOGICAL $\longrightarrow$ −VE PROBABILITIES.

DIRAC? $\longrightarrow$ WAVE FUNCTIONS $\longrightarrow$ 4-COMP SPINORS

COMPLEX CONJUGATE $\longrightarrow$ HERMITIAN CONJUGATE

$$\psi^* \longrightarrow \psi^\dagger = (\psi^*)^T \longrightarrow \text{TRANSPOSE}$$

DIRAC $$-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + m\beta\psi = i\frac{\partial \psi}{\partial t} \qquad (4.14)$$

HERMITIAN CONJUGATE IS:

$$+i\frac{\partial \psi^\dagger}{\partial x}\alpha_x^\dagger + i\frac{\partial \psi^\dagger}{\partial y}\alpha_y^\dagger + i\frac{\partial \psi^\dagger}{\partial z}\alpha_z^\dagger + m\psi^\dagger\beta^\dagger = -i\frac{\partial \psi^\dagger}{\partial t}$$

$\psi^\dagger \times 4.14$, $\psi \times$ HERMITIAN CONJUGATE

(a) $\psi^{+}\left(-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + \beta m \psi\right) = +i\psi^{+}\frac{\partial \psi}{\partial t}$

(b) $\psi\left(i\frac{\partial \psi^{+}}{\partial x}\alpha_x + i\frac{\partial \psi^{+}}{\partial y}\alpha_y + i\frac{\partial \psi^{+}}{\partial z}\alpha_z + m\psi^{+}\beta^{+}\right) = i\psi\frac{\partial \psi^{+}}{\partial t}$

$\hookrightarrow \alpha_x^{+} = \alpha_x$ etc ......

$\hookrightarrow = \beta$

(a) − (b) $= i\psi^{+}\frac{\partial \psi}{\partial t} - i\psi\frac{\partial \psi^{+}}{\partial t} = (\ ) - (\ )$

CAN SIMPLIFY THINGS BY WRITING:

$$\psi^{+}\alpha_x \frac{\partial \psi}{\partial x} + \frac{\partial \psi^{+}}{\partial x}\alpha_x \psi = \frac{\partial(\psi^{+}\alpha_x\psi)}{\partial x}$$

AND $$\psi^{+}\frac{\partial \psi}{\partial t} + \frac{\partial \psi^{+}}{\partial t}\psi = \frac{\partial(\psi^{+}\psi)}{\partial t}$$

$$-i\,\bar{\nabla}\cdot(\psi^{+}\bar{\alpha}\psi) = i\frac{\partial(\psi^{+}\psi)}{\partial t} \rightarrow \bar{\nabla}\cdot(\psi^{+}\bar{\alpha}\psi) + \frac{\partial(\psi^{+}\psi)}{\partial t} = 0$$

$$\bar{\nabla}\cdot(\psi^{+}\bar{\alpha}\psi)+\frac{\partial(\psi^{+}\psi)}{\partial t}=0$$

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix},\quad \psi^{+}=(\psi_1^{*},\psi_2^{*},\psi_3^{*},\psi_4^{*})$$

COMPARE TO CONTINUITY EQUATION

$$\bar{\nabla}\cdot\bar{j}+\frac{\partial\rho}{\partial t}=0$$

$$\bar{j}=\psi^{+}\bar{\alpha}\psi\ ,\quad \rho=\psi^{+}\psi$$

PROBABILITY DENSITY $\rho=\psi^{+}\psi=(\psi_1^{*}\ \psi_2^{*}\ \psi_3^{*}\ \psi_4^{*})\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$

$$\rho=\psi_1^{*}\psi_1+\psi_2^{*}\psi_2+\psi_3^{*}\psi_3+\psi_4^{*}\psi_4$$

$$=|\psi_1|^2+|\psi_2|^2+|\psi_3|^2+|\psi_4|^2$$

ALL SOLUTIONS OF DIRAC EQUATION HAVE POSITIVE PROBABILITY DENSITY

REQUIRE WAVE FUNCTIONS TO SATISFY A WAVE EQUATION LINEAR IN BOTH SPACE & TIME DERIVATIVES

AND BE SOLUTIONS OF KLEIN GORDON

HAVE SOLVED PROBLEM OF NEGATIVE PROBABILITY DENSITIES

ADDITIONAL DEGREES OF FREEDOM IN 4-COMPONENT SPINOR

→ INTRINSIC ANGULAR MOMENTUM OF SPIN 1/2 PARTICLES

ANTIPARTICLES

SPIN AND THE DIRAC EQUATION

TIME DEPENDENCE OF OBSERVABLE

$$\frac{dO}{dt} = \frac{d}{dt}(\hat{O}) = i\langle\psi|[\hat{H},\hat{O}]|\psi\rangle$$

FOR A FREE NONRELATIVISTIC PARTICLE $\hat{H}_{SE} = \dfrac{\hat{\vec{p}}^{\,2}}{2m}$

THIS COMMUTES WITH $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$ → PROB SET

ORBITAL ANGULAR MOMENTUM → CONSERVED

FREE PARTICLE HAMILTONIAN FOR DIRAC

$$\hat{H}_D = \vec{\alpha}\cdot\vec{p} + \beta m$$

COMMUTATION RELATION IS

$$[\hat{H}_D, \hat{\vec{L}}] = [\vec{\alpha}\cdot\vec{p} + \beta m, \hat{\vec{r}} \times \hat{\vec{p}}] = [\vec{\alpha}\cdot\vec{p}, \hat{\vec{r}} \times \hat{\vec{p}}]$$

$$\hookrightarrow \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \rightarrow \text{COMMUTES}$$

CONSIDER ONE COMPONENT

$$[\hat{H}_D, \hat{L}_x] = [\underline{\alpha}\cdot\underline{p}, (\hat{\underline{r}}\times\hat{\underline{p}})_x]$$

$$= [\alpha_x \hat{p}_x + \alpha_y \hat{p}_y + \alpha_z \hat{p}_z, \; \hat{y}\hat{p}_z - \hat{z}\hat{p}_y]$$

$$= \alpha_x \hat{p}_x \cdot \hat{y}\hat{p}_z + \alpha_y \hat{p}_y \cdot \hat{y}\hat{p}_z \; ① + \alpha_z \hat{p}_z \cdot \hat{y}\hat{p}_z$$

$$- \alpha_x \hat{p}_x \cdot \hat{z}\hat{p}_y - \alpha_y \hat{p}_y \cdot \hat{z}\hat{p}_y - \alpha_z \hat{p}_z \cdot \hat{z}\hat{p}_y \; ③$$

$$- \hat{y}\hat{p}_z \alpha_x \hat{p}_x - \hat{y}\hat{p}_z \alpha_y \hat{p}_y \; ② - \hat{y}\hat{p}_z \alpha_z \hat{p}_z$$

$$+ \hat{z}\hat{p}_y \cdot \alpha_x \hat{p}_x + \hat{z}\hat{p}_z \alpha_y \hat{p}_y + \hat{z}\hat{p}_y \alpha_z \hat{p}_z \; ④$$

$\hat{y}\hat{p}_y - \hat{p}_y\hat{y} = i$ THINGS LIKE $\hat{p}_x\hat{p}_z$ COMMUTE $\hat{p}_x\hat{p}_z - \hat{p}_z\hat{p}_x = 0$

$\hat{z}\hat{p}_z - \hat{p}_z\hat{z} = i$ FROM ① + ② $\alpha_y \hat{p}_y \cdot \hat{y}\hat{p}_z - \hat{y}\hat{p}_z \alpha_y \hat{p}_y$

$$= \alpha_y [\hat{p}_y, \hat{y}] \hat{p}_z$$

FROM ③ + ④ $\quad -\alpha_z \hat{p}_z \cdot \hat{z}\hat{p}_y + \hat{z}\hat{p}_y \cdot \alpha_z \hat{p}_z$

$$= -\alpha_z [p_z, \hat{z}]\hat{p}_y$$

SO $\quad [\hat{H}_D, \hat{L}_x] = \alpha_y [\hat{p}_y, \hat{y}]\hat{p}_z - \alpha_z [\hat{p}_z, \hat{z}]\hat{p}_y$

$$[\hat{p}_y, \hat{y}] \cdot i = -[\hat{y}, \hat{p}_y] = -i$$

$$= -i\alpha_y \hat{p}_z + i\alpha_z \hat{p}_y = -i(\alpha_y \hat{p}_z - \alpha_z \hat{p}_y)$$

$$[\hat{H}_D, \hat{L}_x] = -i(\bar{\alpha} \times \bar{p})_x$$

BY INSPECTION $\quad [\hat{H}_D, \hat{\bar{L}}] = -i\bar{\alpha} \times \bar{p}$

ORBITAL ANGULAR MOMENTUM DOES NOT COMMUTE WITH $\hat{H}_D$ $\longrightarrow$ IS NOT CONSERVED

FORM 4×4 MATRIX FROM PAULI SPIN MATRICES

$$\hat{\vec{S}} \equiv \frac{1}{2}\hat{\vec{\Sigma}} \equiv \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

$$\hat{\Sigma}_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \hat{\Sigma}_y \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \hat{\Sigma}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\vec{\alpha}$, $\hat{\vec{\Sigma}}$ BOTH DEFINED BY PAULI SPIN MATRICES

$\rightarrow$ WELL DEFINED COMMUTATORS

WRITE MATRICES AS BLOCK MATRICES

$$\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sigma_i & 0 \\ 0 & 0 & 0 & \sigma_i \\ \sigma_i & 0 & 0 & 0 \\ 0 & \sigma_i & 0 & 0 \end{pmatrix}$$

$$[\alpha_i,\ \hat{\Sigma}_x] = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

x, y, z

$$= \begin{pmatrix} 0 & \sigma_i\sigma_x \\ \sigma_i\sigma_x & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_x\sigma_i \\ \sigma_x\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & [\sigma_i, \sigma_x] \\ [\sigma_i, \sigma_x] & \end{pmatrix} \quad (4.23)$$

SINCE $[\sigma_x, \sigma_x] = 0 \quad [\sigma_y, \sigma_x] = -2i\sigma_z \ , \ [\sigma_z, \sigma_x] = 2i\sigma_y$

$$[\alpha_x,\ \Sigma_x] = 0$$

$$[\alpha_y,\ \hat{\Sigma}_x] = \begin{pmatrix} 0 & -2i\sigma_z \\ -2i\sigma_z & 0 \end{pmatrix} = -2i\,\alpha_z \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$[\alpha_z,\ \hat{\Sigma}_x] = \begin{pmatrix} 0 & 2i\sigma_y \\ 2i\sigma_y & 0 \end{pmatrix} = 2i\,\alpha_y$$

CONSIDER COMMUTATOR of $\hat{\Sigma}_x$, $\hat{H}_D$

$$[\hat{H}_D, \hat{\Sigma}_x] = [\bar{\alpha}\cdot\hat{p} + \beta m, \hat{\Sigma}_x]$$

$[\beta, \hat{\Sigma}_x] = 0$ SHOW THIS!

$$= [\bar{\alpha}\cdot\hat{p}, \hat{\Sigma}_x] = [\alpha_x \hat{p}_x + \alpha_y \hat{p}_y + \alpha_z \hat{p}_z, \hat{\Sigma}_x]$$

$$= \hat{p}_x [\alpha_x, \hat{\Sigma}_x] + \hat{p}_y [\alpha_y, \hat{\Sigma}_x] + \hat{p}_z [\alpha_y, \hat{\Sigma}_x]$$

JUST DERIVED THESE

$$[\hat{H}_D, \hat{\Sigma}_x] = -2i\,\hat{p}_y \alpha_z + 2i\,\hat{p}_z \alpha_y = 2i\,(\bar{\alpha} \times \hat{p})_x$$

GENERALIZE TO: $[\hat{H}_D, \hat{\Sigma}] = 2i\,(\bar{\alpha} \times \bar{p})$

DEFINED $\hat{S} = \frac{1}{2}\hat{\Sigma}$ → $\hat{S}$ NOT CONSERVED

$$[\hat{H}_D, \hat{S}] = i\bar{\alpha} \times \hat{p}$$

SO WHAT IS THE POINT OF ALL THIS?

WELL......

HAD $[\hat{H}_D, \hat{\vec{L}}] = -i\vec{\alpha}\times\hat{\vec{p}}$

AND $[\hat{H}_D, \hat{\vec{S}}] = +i\vec{\alpha}\times\hat{\vec{p}}$

$$\left.\begin{matrix} [\hat{H}_D, \hat{\vec{L}}] = -i\vec{\alpha}\times\hat{\vec{p}} \\ [\hat{H}_D, \hat{\vec{S}}] = +i\vec{\alpha}\times\hat{\vec{p}} \end{matrix}\right\} \quad [\hat{H}_D, \hat{\vec{J}}] = 0$$

$\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$

TOTAL ANGULAR MOMENTUM

ORBITAL ANGULAR MOMENTUM NOT CONSERVED

IF WANT TO KEEP ANGULAR MOMENTUM CONSERVATION

↳ NEED TO ASSUME DIRAC PARTICLE HAS INTRINSIC ANGULAR MOMENTUM

LET'S LOOK AT $\hat{\vec{S}}$ A BIT MORE

$$\hat{\vec{S}} = \frac{1}{2}\hat{\vec{\Sigma}} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

→ $\vec{\sigma}$ PAULI SPIN MATRICES

4×4 MATRIX

WHY CAN WE INTERPRET $\hat{\vec{S}}$ AS AN INTRINSIC ANGULAR MOMENTUM ?

$\hat{\vec{S}}$ DEFIND IN TERMS OF PAULI SPIN MATRICES

THESE HAVE SAME COMMUTATOR ALGEBRA AS ORBITAL ANGULAR MOMENTUM

SPIN → BEHAVES LIKE ANGULAR MOMENTUM

$$\hat{S}^2 = \frac{1}{4}\left(\hat{\Sigma}_x^2 + \hat{\Sigma}_y^2 + \hat{\Sigma}_z^2\right) = \frac{3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

EIGENVALUE OF

$$\hat{S}^2 = \frac{3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

THIS IS JUST $\hat{S}^2 |s, m_s\rangle = s(s+1)|s, m_s\rangle$

$s = \frac{1}{2}$ $(\hbar)$

ANY PARTICLE DESCRIBED BY THE DIRAC EQUATION HAS INTRINSIC ANGULAR MOMENTUM OF $\hbar/2$

COMPLETELY UNEXPECTED

↳ PROFOUND RESULT.