# MANIFESTLY COVARIANT FORM OF DIRAC EQUATION

WRITING DE USING $\vec{\alpha}$ $\vec{\beta}$ $\rightarrow$ SEE CONNECTION TO SPIN

$$i\frac{\partial\psi}{\partial t} = \left(-i\alpha_x\frac{\partial\psi}{\partial x} - i\alpha_y\frac{\partial\psi}{\partial y} - i\alpha_z\frac{\partial\psi}{\partial z} + \beta m\right)\psi$$

MULTIPLY THIS BY $\beta$

$$i\beta\alpha_x\frac{\partial\psi}{\partial x} + i\beta\alpha_y\frac{\partial\psi}{\partial y} + i\beta\alpha_z\frac{\partial\psi}{\partial z} + i\beta\frac{\partial\psi}{\partial t} - \beta^2 m\psi = 0$$

DEFINE $\gamma^0 \equiv \beta$ ; $\gamma^1 = \beta\alpha_x$ ; $\gamma^2 = \beta\alpha_y$ ; $\gamma^3 = \beta\alpha_z$

SINCE $\beta^2 = 1$

$$\underbrace{i\gamma^0\frac{\partial\psi}{\partial t} + i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z}}_{\text{LOOKS A BIT LIKE 4-VECTOR, BUT IS NOT.}} - m\psi = 0$$

$$i\gamma^0 \frac{\partial \psi}{\partial t} + i\gamma^1 \frac{\partial \psi}{\partial x} + i\gamma^2 \frac{\partial \psi}{\partial y} + i\gamma^3 \frac{\partial \psi}{\partial z} - m\psi = 0$$

$$\partial_\mu \equiv (\partial_0, \partial_1, \partial_2, \partial_3) \equiv \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

COVARIANT
(NOT OBVIOUS)

NOT COMPONENTS OF A 4-VECTOR
CONSTANT MATRICES INVARIANT UNDER LORENTZ TRANSFORMATION

PROPERTIES OF $\vec{\alpha}$, $\vec{\beta}$ $\rightarrow$ PROPERTIES OF $\gamma^i$

$$\beta^2 = I \,, \quad \alpha_x^2 = I \,, \quad \beta\alpha_x = -\alpha_x\beta$$ (4.9 → 4.11)

So $(\gamma^1)^2 = \beta\alpha_x \cdot \beta\alpha_x = -\alpha_x \beta^2 \alpha_x = -\alpha_x^2 = -I$

$(\gamma^0)^2 = \beta \cdot \beta = I$

$(\gamma^2)^2 = \beta\gamma_y \cdot \beta\gamma_y = -\alpha_y \beta^2 \alpha_y = -I$

$(\gamma^k)^2 = -I \qquad k = 1, 2, 3$

$$\gamma^\mu \gamma^\nu = \beta\alpha_\mu \cdot \beta\alpha_\nu = -\alpha_\mu \beta\beta \alpha_\nu = -\alpha_\mu \alpha_\nu = \alpha_\nu \alpha_\mu = \beta\beta\alpha_\nu\alpha_\mu$$
$$= -\beta\alpha_\nu \beta\alpha_\mu$$
$$= -\gamma^\nu \gamma^\mu$$

ALL OF THIS CAN BE SUMMARIZED

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

ANTI-COMMUTATOR.

# EXPLANATION – READ IF YOU WANT.

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \Rightarrow \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$$

ALL OFF DIAGONAL ELEMENTS $=0$

$$\{\gamma^0, \gamma^0\} = (\gamma^0)^2 + (\gamma^0)^2 = -2$$

$$\{\gamma^k, \gamma^k\} = (\gamma^k)^2 + (\gamma^k)^2 = -2$$

$$2g^{\mu\nu} = 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\beta$ IS HERMITIAN, $\gamma^0 \equiv \beta \rightarrow$ ALSO HERMITIAN

$$(\gamma^1)^\dagger = (\beta\alpha_x)^\dagger = \alpha_x^\dagger \beta^\dagger = \alpha_x \beta = -\beta\alpha_x = -\gamma^1$$

BOTH HERMITIAN

ANTI-HERMITIAN

SUMMARY $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^k)^\dagger = -\gamma^k$

THIS FULLY DEFINES THE ALGEBRA OF $\gamma_s$

IT IS SUFFIENT TO DETERMINE THE PROPERTIES OF DIRAC EQUATION

HOWEVER TO SEE WHAT IS GOING ON $\rightarrow$ AGAIN USE A DEFINITE REPRESENTATION OF THE $\gamma_s$

# DIRAC–PAULI REPRESENTATION OF $\gamma$s

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \beta\alpha_k = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

# THE ADJOINT SPINOR AND COVARIANT CURRENT

SHOWED PROBABILITY DENSITY $\rho = \psi^\dagger \psi$

PROBABILITY CURRENT $\vec{j} = \psi^\dagger \vec{\alpha} \psi$

4-VECTOR $j^\mu = (\rho, \vec{j}) = \psi^\dagger \gamma^0 \gamma^\mu \psi$

$\rho = \psi^\dagger \psi = \psi^\dagger \gamma^0 \gamma^0 \psi$ ; $(\gamma^0 \gamma^k) = \beta\beta\alpha_k = \alpha_k$

$(\gamma^0)^2 = 1$

$j_k = \psi^\dagger \alpha_k \psi$

$\vec{j}_\mu$ OBEYS CONTINUITY EQUATION

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \longrightarrow \partial_\mu j^\mu = 0$$

$j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi$ CAN BE SIMPLIFIED

DEFINE ADJOINT SPINOR $\bar{\psi} \equiv \psi^\dagger \gamma^0$

$$j^\mu = \bar{\psi} \gamma^\mu \psi$$

DIRAC PAULI REPRESENTATION

$$\bar{\psi} = \psi^\dagger \gamma^0 = (\psi^*)^T \gamma_0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

# SOLUTIONS TO DIRAC EQUATION

CONSTRUCT WAVE FUNCTIONS → CALCULATE PHYSICAL PROCESSES

FREE PARTICLE PLANE WAVE

$$\psi(\bar{x},t) = u(E,\bar{p})\, e^{i(\bar{p}\cdot\bar{x} - Et)}$$

← SPACE, TIME DEPENDENCE

SATISFIES

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

4 COMPONENT SPINOR FUCTION OF $E, \bar{p}$

DERIVATIVES ACT ON EXPONENTS

$$\partial_0 \psi = -iE\psi\,, \quad \partial_1 \psi = i p_x \psi \quad \cdots y \cdots z$$

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)\, u(E,\bar{p})\, e^{i(\bar{p}\cdot\bar{x} - Et)} = 0$$

4 COMPONENT SPINOR $u(E,\bar{p})$ SATISFIES

moi →

$$(\gamma^\mu p_\mu - m)\, u = 0$$

# PARTICLE AT REST

$$\vec{p} = 0 \rightarrow \psi = u(E, 0)\, e^{-iEt}$$

$$(\gamma^\mu p_\mu - m)\, u \longrightarrow E\gamma^0 u = m u$$

$$E \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = m \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

$\gamma^0$ DIAGONAL — 4 ORTHOGONAL SOLUTIONS

$$u_1(E,0) = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_2(E,0) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

+VE EIGENVALUES

$E = +m$

$$u_3(E,0) = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_4(E,0) = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

−VE EIGENVALUES

$E = -m$

$u_1(E,0)\quad u_2(E,0)$ ALSO EIGENSTATES OF $\hat{S}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ SPIN

$u_3(E,0)\quad u_4(E,0)$

$u_1(E,0)\quad u_2(E,0)$ SPIN UP AND DOWN +VE ENERGY

$u_3(E,0)\quad u_4(E,0)$ SPIN UP AND DOWN −VE ENERGY

PUTTING TIME DEPENDENCE IN

$$\psi_1 = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt},\quad \psi_2 = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt}$$ m +ve

$$\psi_3 = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt},\quad \psi_4 = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt}$$ m −ve

# GENERAL FREE PARTICLE SOLUTIONS

DIRECTLY SOLVE DIRAC FOR GENERAL PLANE WAVE SOLN.

$$\psi(\bar{x},t) = u(E,\bar{p})\, e^{i(\bar{p}\cdot\bar{x} - E\cdot t)}$$

FOUND THAT DE FOR SPINOR IS $\quad (\gamma^{\mu} p_{\mu} - m)\, u = 0$

WRITE OUT IN COMPONENTS

$$(E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3 - m)\, u = 0$$

IN DIRAC – PAULI REP:

$$\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \bar{\sigma}\cdot\bar{p} \\ -\bar{\sigma}\cdot\bar{p} & 0 \end{pmatrix} - m\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] u = 0 \qquad (4.4)$$

2×2 MATRICES

$$\bar{\sigma}\cdot\bar{p} = \sigma_x p_x + \sigma_y p_y + \sigma_z p_z = \begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & -p_z \end{pmatrix}$$

$u_A$, $u_B$ 2 COMPONENT COLUMN VECTORS $u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$

EXPRESS PREVIOUS MATRIX EQTN 4.4

$$\begin{pmatrix} (E-m)I & -\bar{\sigma}\cdot\bar{p} \\ \bar{\sigma}\cdot\bar{p} & -(E+m)I \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

THIS GIVES 2 COUPLED EQUATIONS!

$$u_A = \frac{\bar{\sigma}\cdot\bar{p}}{E-m} \cdot u_B \; ; \quad u_B = \frac{\bar{\sigma}\cdot\bar{p}}{E+m} \cdot u_A$$

CAN USE THESE EQUATIONS TO CONSTRUCT FOUR ORTHOGONAL SOLUTIONS $u_1$ $u_2$ $u_3$ $u_4$ BY SUCCESSIVELY CHOOSING

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow u_B = \frac{\vec{\sigma} \cdot \vec{p}}{E+m} u_A = \frac{1}{E+m} \begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & -p_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{E+m} \begin{pmatrix} p_z \\ p_x + i p_y \end{pmatrix}$$

$$u_1 = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix}$$

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow u_B = \frac{\bar{\sigma}\cdot\bar{p}}{E+m} u_A = \frac{1}{E+m}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{E+m}\begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$

$$u_2 = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

THEN CHOSE $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

# FOUR ORTHOGONAL SOLUTIONS

$$\psi_i = u_i(E\,\tilde{p})\,e^{i(\tilde{p}\cdot\tilde{x}-Et)}$$

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \dfrac{p_z}{E+m} \\ \dfrac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u_2 = N\begin{pmatrix} 0 \\ 1 \\ \dfrac{p_x-ip_y}{E+m} \\ \dfrac{-p_z}{E+m} \end{pmatrix}$$

$$u_3 = N\begin{pmatrix} \dfrac{p_z}{E-m} \\ \dfrac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = N\begin{pmatrix} \dfrac{p_x-ip_y}{E-m} \\ \dfrac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

IF ANY ONE OF THE SPINORS $u_1$ $u_2$ $u_3$ $u_4$

ARE SUBSTITUTED INTO DE GET $E^2 = p^2 + m^2$

← DO IT!

$\vec{p} = 0$ GET

$$\underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{E\ +ve} \quad \underbrace{\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{E\ -ve}$$

IDENTIFY $u_1$ $u_2$ $E = + \left|\sqrt{p^2+m^2}\right|$ E +ve

$u_3$ $u_4$ $E = - \left|\sqrt{p^2+m^2}\right|$ E -ve

CAN ALL FOUR HAVE $E > 0$ ???

→ EXPONENT OF $\psi_i = u_i e^{i(\vec{p}\cdot\vec{x} - Et)}$ ← SAME FOR ALL

FOUR NO LONGER INDEP → $u_1 = \dfrac{p_z}{E+m} u_3 + \dfrac{p_x + i p_y}{E+m} u_4$

$\hat{H}_D$ $4\times4$, $Tr = 0$ ⇒ EQUAL NUMBERS OF +ve AND -VE ENERGY SOLUTIONS

# ANTIPARTICLES

QUANTUM MECHANICS → COMPLETE SET OF BASIS STATES

DIRAC REALIZED WHAT THE PROBLEM WAS WITH RELATIVISTIC QUANTUM MECHANICS

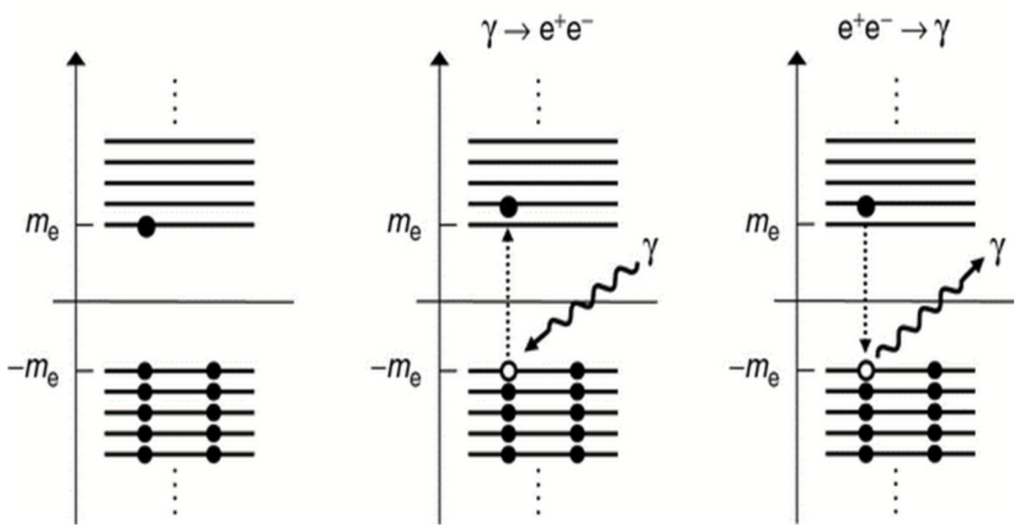
→ SINGLE PARTICLE THEORY

→ NEED THEORY FOR CREATION + ANNIHILATION OF PARTICLES

↳ QUANTUM FIELD THEORY

EXPERIMENTALLY ANTIPARTICLES HAVE SAME MASS, SPIN, BUT OPPOSITE ELECTRIC CHARGE, AND OPPOSITE OF SOME OTHER INTERNAL QUANTUM NUMBERS

# DIRAC "SEA"




WHAT DO THE NEGATIVE ENERGY SOLUTIONS CORRESPOND TO?

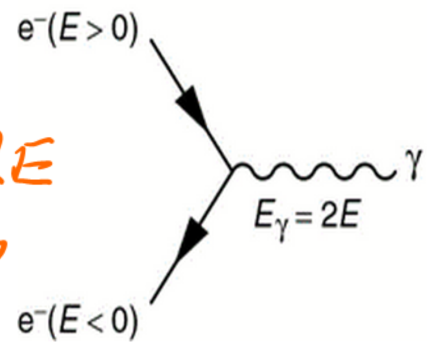
ANTIPARTICLE → "HOLE" IN NEGATIVE SEA

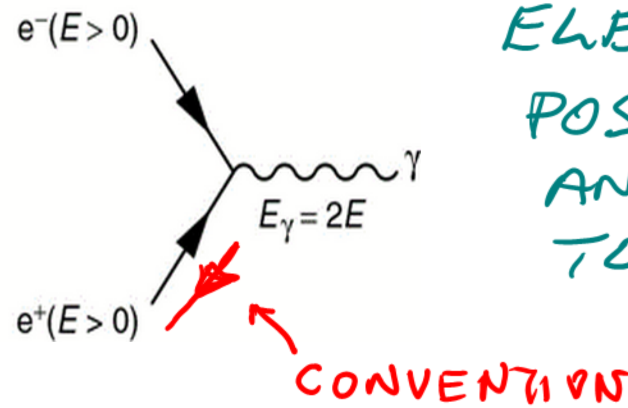
ONLY OF HISTORICAL INTEREST

# FEYNMAN – STÜCKELBERG

ELECTRON EMITS $E_\gamma = 2E$

–VE ENERGY MOVES BACK IN TIME



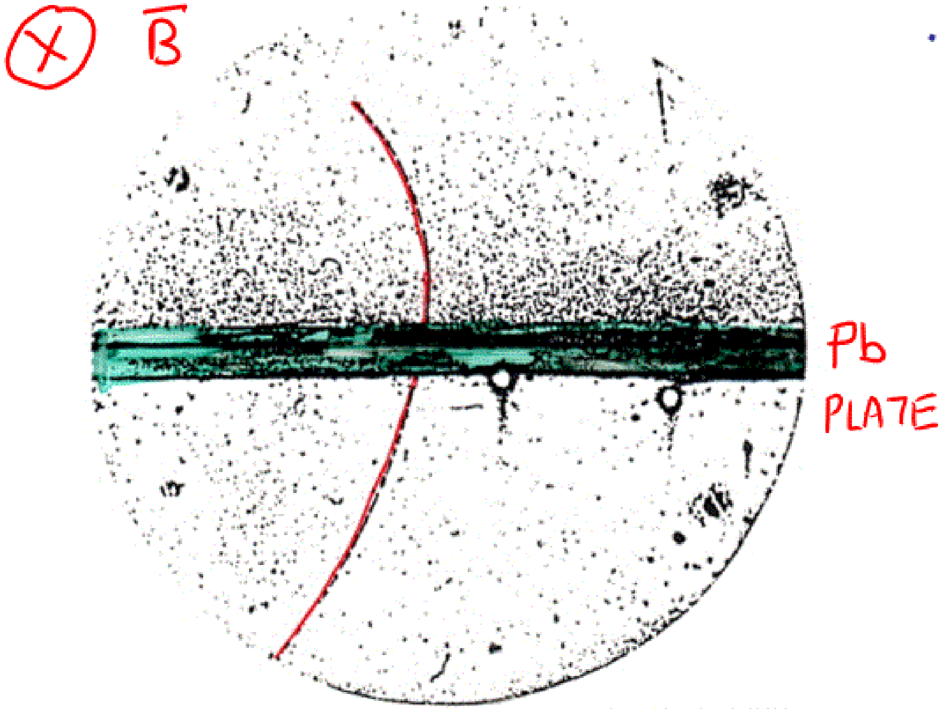

ELECTRON AND POSITRON ANNIHILATE TO $E_\gamma = 2E$

$E < 0$ SOLUTIONS ARE NEGATIVE PARTICLES MOVING BACKWARDS IN TIME

COMPLETELY EQUIVALENT TO POSITIVE CHARGE MOVING FORWARDS IN TIME

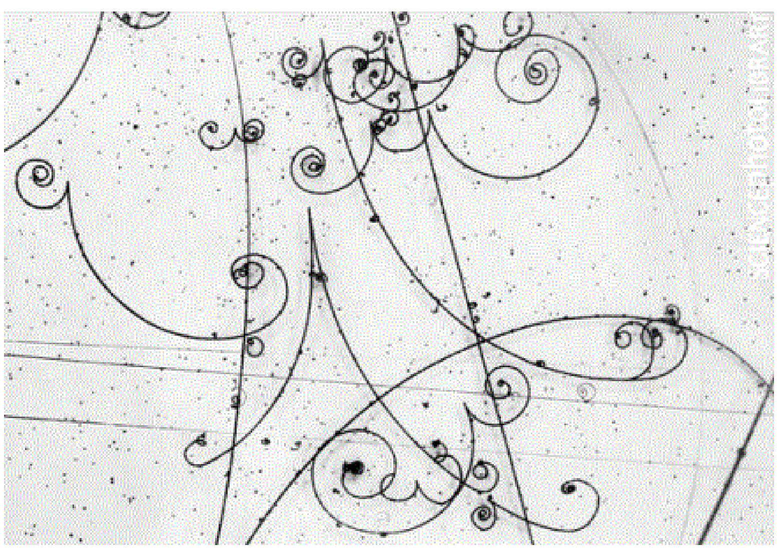
$$\exp(-iEt) \equiv \exp(-i(-E)(-t))$$

"ONLY THE MATHEMATICS COUNTS"

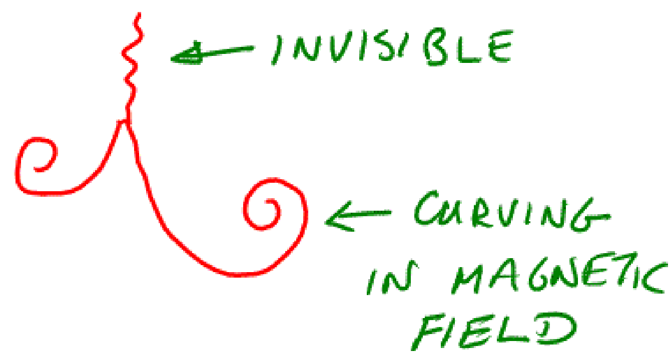



A 63 million volt positron ($H\rho = 2.1 \times 10^5$ gauss-cm) passing through a 6 mm lead plate and emerging as a 23 million volt positron ($H\rho = 7.5 \times 10^4$ gauss-cm). The length of this latter path is at least ten times greater than the possible length of a proton path of this curvature.

- DISCOVERY OF POSITRON BY CARL ANDERSON 1933
- PARTICLE SLOWS DOWN IN Pb PLATE ∴ COMING FROM BELOW
- FROM SIGN OF MAGNETIC FIELD MUST B +VE
- 10x RANGE OF PROTON (ONLY +VE PARTICLE KNOWN)



$\gamma \rightarrow e^+e^-$ IN A LARGE MODERN BUBBLE CHAMBER



# ANTIPARTICLE SPINORS

COULD CALCULATE USING NEGATIVE ENERGY SPINORS → BUT THEN WOULD HAVE TO KEEP IN MIND ENERGY IN SPINOR IS NEGATIVE OF PHYSICAL ENERGY

ALSO SINCE $u_3$ $u_4$ BACKWARDS IN TIME, SPINOR MOMENTUM NEGATIVE OF PHYSICAL MOMENTUM.

USE SPINORS WITH PHYSICAL ENERGY AND MOMENTUM

NEGATIVE ENERGY SPINORS CAN BE REWRITTEN BY JUST REVERSING SIGNS ON $E$ & $\bar{p}$

$$v_1(E,\bar{p})\,e^{-i(\bar{p}\cdot\bar{x}-Et)} \rightarrow u_4(-E,-\bar{p})\,e^{+i(-\bar{p}\cdot\bar{x}-(-E)t)}$$

$$v_2(E,\bar{p})\,e^{-i(\bar{p}\cdot\bar{x}-Et)} \rightarrow u_3(-E,-\bar{p})\,e^{+i(-\bar{p}\cdot\bar{x}-(-E)t)}$$

IDENTIFY ANTIPARTICLE SPINORS BY LOOKING FOR DIRAC EQUATION SOLUTIONS

$$\psi(\bar{x},t) = v(E,\bar{p})e^{-i(\bar{p}\cdot\bar{x}-Et)}$$

FOR PHYSICAL $E>0$, STILL "-VE ENERGY SOLN"

$$i\frac{\partial}{\partial t}\cdot\psi = -E\psi$$

INTO $(i\gamma^\mu\partial_\mu - m)\psi = 0$

$$(-\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z - m)v = 0$$

$$(\gamma^\mu p_\mu + m)v = 0$$

DIRAC EQUATION FOR ANTIPARTICLE SPINORS

$v$

AS BEFORE WRITE $v = \begin{pmatrix} v_A \\ v_B \end{pmatrix}$ AND GET:

$$v_A = \frac{\bar{\sigma}\cdot\bar{p}}{E+m}\, v_B \quad ; \quad v_B = \frac{\bar{\sigma}\cdot\bar{p}}{E-m}\, v_A$$

$$v_1 = N\begin{pmatrix} \frac{p_x - i p_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = N\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \qquad E = +\left|\sqrt{\bar{p}^2 + m^2}\right|$$

$$E = -\left|\sqrt{\bar{p}^2 + m^2}\right| \qquad v_3 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix} \qquad v_4 = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x - i p_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

# WAVE FUNCTION NORMALIZATION

HAVE TO DETERMINE NORMALIZATION $N$

$$\rho = \psi^\dagger \psi = (\psi^*)^T \psi = u_1^\dagger u_1$$

$$u_1^\dagger u_1 = N^2 \left(1, 0, \frac{p_z}{E+m}, \frac{p_x - ip_y}{E+m}\right) \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}$$

$$= N^2 \left( 1 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2 + p_y^2}{E+m^2} \right)$$

$$= N^2 \left( \frac{(E+m)^2 + p^2}{(E+m)^2} \right)$$

$$= N^2 \left( \frac{(E+m)^2 + E^2 - m^2}{(E+m)^2} \right) \rightarrow$$

$uu^\dagger = 2E$ PARTICLES

$$2E = \frac{N^2 2E}{E+m}$$

$$N = \sqrt{E+m}$$

HAVE 8 SOLUTIONS, ONLY 4 INDEPENDENT

USE +VE PHYSICAL ENERGY $u_1$ $u_2$ $v_1$ $v_2$

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

4-VECTOR

$$v_1(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

# SUBTLE POINT FOR ANTIPARTICLES

$$\psi = v(E,\bar{p})\, e^{-i(\bar{p}\cdot\bar{x} - Et)}$$

SO

$$\hat{H}\psi = i\frac{\partial\psi}{\partial t} = -E \quad , \quad \hat{\bar{p}}\psi = -i\bar{\nabla}\psi = -\bar{p}\psi$$

MINUS → $v$ ARE STILL −VE ENERGY SOLUTIONS

REWRITTEN IN TERMS OF PHYSICAL $E, \bar{p}$

RELEVANT OPERATORS ARE:

$$\hat{H}^{(v)} = -i\,\partial/\partial t \quad , \quad \hat{\bar{p}}^{(v)} = +i\bar{\nabla}$$

$$(E,\bar{p}) \rightarrow (-E,-\bar{p}) \qquad \bar{L} = \bar{r}\times\bar{p} \longrightarrow -\bar{L}$$

$$\hat{\bar{S}}^{(v)} = -\bar{S}$$