SPIN AND HELICITY STATES

WHEN WE CALCULATE PARTICLE PROCESSES INVOLVING FERMIONS (LEPTONS, QUARKS) THE FACT THAT THEY HAVE SPIN 1/2 IS OF OVERWHELMING IMPORTANCE

FOR A SPIN 1/2 PARTICLE AT REST THE SPINORS $u_1(E,0)$, $u_2(E,0)$ ARE EIGENSTATES OF OF THE Z-COMPONENT OF THE SPIN OPERATOR

$$\hat{S}_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

TAKE $u_1(E,0) = N\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\hat{S}_z u_1(E,0) = \frac{N}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{N}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

EIGENSTATE

$u_1(E,0)$ & $u_2(E,0)$ REPRESENT SPIN UP AND SPIN DOWN D.E. SOLUTIONS OF SAME ENERGY

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix} \qquad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x - i p_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

4-VECTOR

TRY IT!

ARE CLEARLY NOT, IN GENERAL, EIGESTATES OF $\hat{S}_z$

FOR A PARTICLE TRAVELING Z DIRN $\vec{p} = \pm|\vec{p}|\hat{z}$

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{\pm p_z}{E+m} \\ 0 \end{pmatrix} \quad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p_z}{E+m} \end{pmatrix} \quad v_1(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ \frac{\mp p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad v_2(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

4-VECTOR

ARE EIGENSTATES OF $\hat{S}_z$

FOR EXAMPLE

$$\hat{S}_z u_1(E,0,0,\pm p) = \frac{\sqrt{E+m}}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{\pm p}{E+m} \\ 0 \end{pmatrix}$$

$$= \frac{\sqrt{E+m}}{2}\begin{pmatrix} 1 \\ 0 \\ \frac{\pm p}{E+m} \\ 0 \end{pmatrix}$$

EIGENSTATE

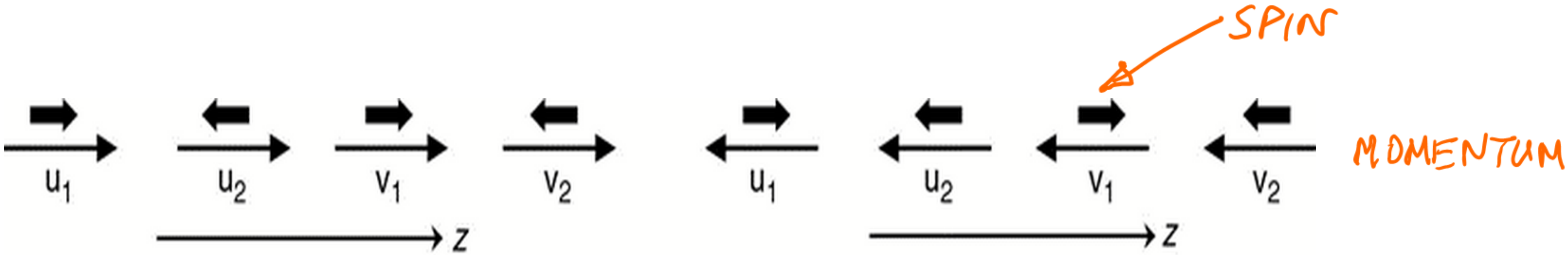
$$\hat{S}_z u_1(E,0,0,\pm p) = +\tfrac{1}{2} u_1(E,0,0,\pm p)$$

$$\hat{S}_z u_2(E,0,0,\pm p) = -\tfrac{1}{2} u_2(E,0,0,\pm p)$$

FOR ANTIPARTICLES PHYSICAL $\hat{S}_z^{(v)} = -\hat{S}_z$

$$\hat{S}_z^{(v)} v_1(E,0,0,\pm p) = -\hat{S}_z v_1 = +\tfrac{1}{2} v_1$$

$$\hat{S}_z^{(v)} v_2(E,0,0,\pm p) = -\hat{S}_z v_2 = -\tfrac{1}{2} v_2$$



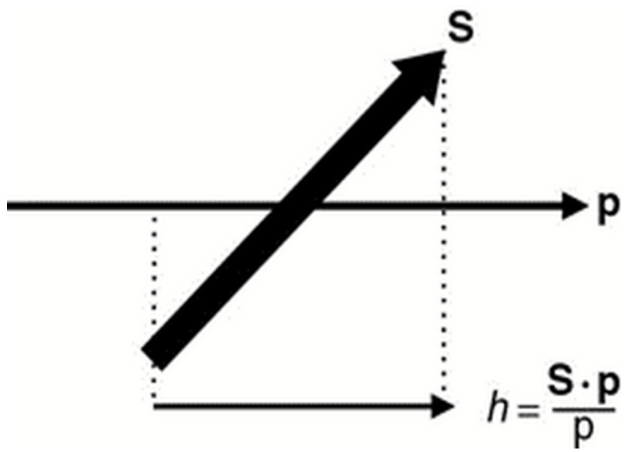
HELICITY

STRUCTURE OF SPIN STATES EXTREMELY IMPORTANT IN ANALYZING INTERACTIONS. BUT ....

$u_1\ u_2\ v_1\ v_2$ ONLY USEFUL FOR PARTICLES TRAVELLING IN THE Z-DIRECTION, WITH THEIR SPINS POINTING IN THAT DIRECTION.

ALSO $[\hat{H}_D, \hat{S}_z] \neq 0$ SIMULTANEOUS EIGENSTATES NOT POSSIBLE

HELICITY $h \equiv \dfrac{\bar{S} \cdot \bar{p}}{|p|}$



FOR DIRAC SPINOR

$$\hat{h} = \frac{\hat{\bar{\Sigma}} \cdot \hat{\bar{p}}}{2|p|} = \frac{1}{2|p|}\begin{pmatrix} \bar{\sigma} \cdot \hat{\bar{p}} & 0 \\ 0 & \bar{\sigma} \cdot \hat{\bar{p}} \end{pmatrix}$$

SHOW THIS! $\longrightarrow$ $[\hat{H}_D, \hat{\bar{\Sigma}} \cdot \hat{\bar{p}}] = 0$

HELICITY IS AN EXTREMELY IMPORTANT CONCEPT IN THE STANDARD MODEL — CENTRAL TO UNDERSTANDING THE WEAK INTERACTION.

CAN IDENTIFY SPINOR STATES WHICH ARE SIMULTANEOUS EIGENSTATES OF FREE $\hat{H}_D$ AND $\hat{h}$

FOR SPIN $\frac{1}{2}$ PARTICLE, COMPONENT OF SPIN MEASURED ALONG ANY AXIS IS $\pm 1/2\ (\hbar)$ $\longrightarrow$ EIGENVALUES OF HELICITY FOR DIRAC PARTICLE $= \pm 1/2$

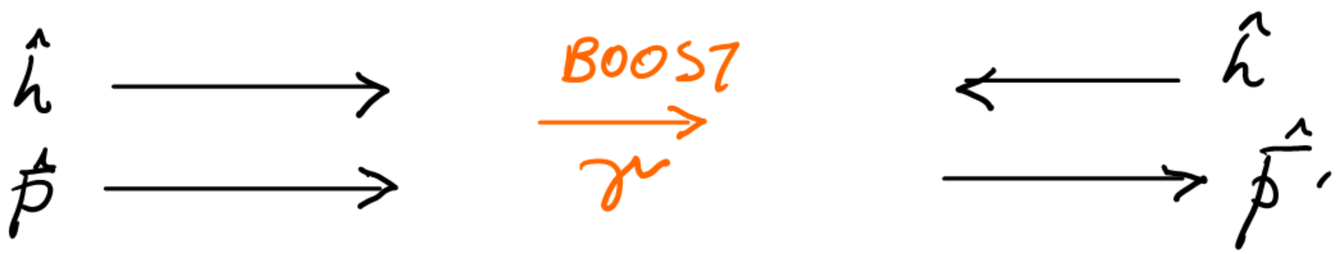
THE TWO HELICITY STATES FOR A FERMION ARE:

RIGHT HANDED	LEFT HANDED
$\longrightarrow \hat{h}$	$\longleftarrow \hat{h}$
$\longrightarrow \hat{p}$	$\longrightarrow \hat{p}$

e^-_{RIGHT} $\qquad \cancel{\nu_{e\,RIGHT}}$ $\qquad \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_{LEFT}$

HELICITY IN NOT LORENTZ INVARIANT FOR PARTICLES WITH MASS → THEY HAVE TO TRAVEL WITH $v < c$

→ ALWAYS POSSIBLE TO BOOST INTO FRAME WHERE THE HELICITY FLIPS



WHEN WE THOUGHT νs WERE MASSLESS, WE THOUGHT HELICITY OF ν WAS LORENTZ INVARIANT

THESE EIGENSTATES OF $\hat{h}$ ARE ALSO EIGENSTATES OF $\hat{H}_D$

$$\hat{h} u = \lambda u$$

IN FULL

$$\frac{1}{2p}\begin{pmatrix}\vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p}\end{pmatrix}\begin{pmatrix}u_A \\ u_B\end{pmatrix} = \lambda\begin{pmatrix}u_A \\ u_B\end{pmatrix}$$

$$\underbrace{\frac{1}{2p}\begin{pmatrix} \bar{\sigma}\cdot\bar{p} & 0 \\ 0 & \bar{\sigma}\cdot\bar{p} \end{pmatrix}}_{\hat{h}}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \lambda \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

EIGENVALUE

$$(\bar{\sigma}\cdot\bar{p})u_A = 2p\lambda u_A \quad ; \quad (\bar{\sigma}\cdot\bar{p})u_B = 2p\lambda u_B$$

$$(\bar{\sigma}\cdot\bar{p})^2 u_A = 2p\lambda(\bar{\sigma}\cdot\bar{p})u_A \qquad (\bar{\sigma}\cdot\bar{p})^2 = p^2$$

PROB 4.10

$$p^2 u_A = 2p\lambda p\, u_A = 2p^2\lambda u_A$$

$$u_A = 2\lambda u_A \longrightarrow \lambda = \pm\frac{1}{2}$$

PREVIOUSLY $u_A(\bar{\sigma}\cdot\bar{p}) = (E+m)u_B$

$$2p\lambda u_A = (E+m)u_B$$

$$u_B = \frac{2p\lambda u_A}{(E+m)}$$

u_B & u_A, ONCE SOLVED
$(\bar{\sigma}\cdot\bar{p})u_B = 2p\lambda u_B$
AUTOMATICALLY SATISFIED

CAN SOLVE $(\bar{\sigma}\cdot\bar{p})u_A = 2p\lambda u_A$ BY WRITING $\bar{p}$ IN SPHERICAL COORDINATES

$$\bar{p} = (p\sin\theta\cos\phi,\ p\sin\theta\sin\phi,\ p\cos\theta)$$

THE HELICITY OPERATOR IS:

$$\frac{1}{2p}(\bar{\sigma}\cdot\bar{p}) = \frac{1}{2p}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$= \frac{1}{2p}\begin{pmatrix} p\cos\theta & p\sin\theta\cos\phi - ip\sin\theta\sin\phi \\ p\sin\theta\cos\phi - ip\sin\theta\sin\phi & -p\cos\theta \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

WRITE $u_A = \begin{pmatrix} a \\ b \end{pmatrix}$

$$(\bar{\sigma}\cdot\bar{p})u_A = 2p\lambda u_A \rightarrow \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\lambda\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

MULTIPLY OUT FOR a, b

$$\frac{b}{a} = \frac{2\lambda - \cos\theta}{\sin\theta} e^{i\phi} \xrightarrow[\lambda = +1/2]{} \frac{b}{a} = \frac{1-\cos\theta}{\sin\theta} e^{i\phi}$$

$$= 2 \frac{\sin^2(\theta/2)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} = e^{i\phi}\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$$

USING $u_B = 2\lambda \left(\frac{p}{E+m}\right) u_A$

RIGHT HANDED HELICITY SPINOR $u_\uparrow$

$$u_\uparrow = N \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\theta/2 \\ \frac{p}{E+m}\cos\theta/2 \\ \frac{p}{E+m} e^{i\phi}\sin\theta/2 \end{pmatrix}$$

PUTING CORRECT NORMALIZATION $N = \sqrt{E+m}$

RIGHT-HANDED AND LEFT-HANDED SPINORS ARE:

$$u_\uparrow = \sqrt{E+m} \begin{pmatrix} c \\ s e^{i\phi} \\ \frac{p}{E+m} c \\ \frac{p}{E+m} s e^{i\phi} \end{pmatrix} \qquad u_\downarrow = \sqrt{E+m} \begin{pmatrix} -s \\ c e^{i\phi} \\ \frac{p}{E+m} s \\ -\frac{p}{E+m} c e^{i\phi} \end{pmatrix}$$

PARTICLES

FOR ANTIPARTICLES $\hat{S}^{v} = -\hat{S}$, FOR $h = +\frac{1}{2}$ $\left(\frac{\vec{\Sigma}\cdot\vec{p}}{2p}\right) v_\uparrow = -\frac{1}{2} v_\uparrow$

GIVING (DO IT !)

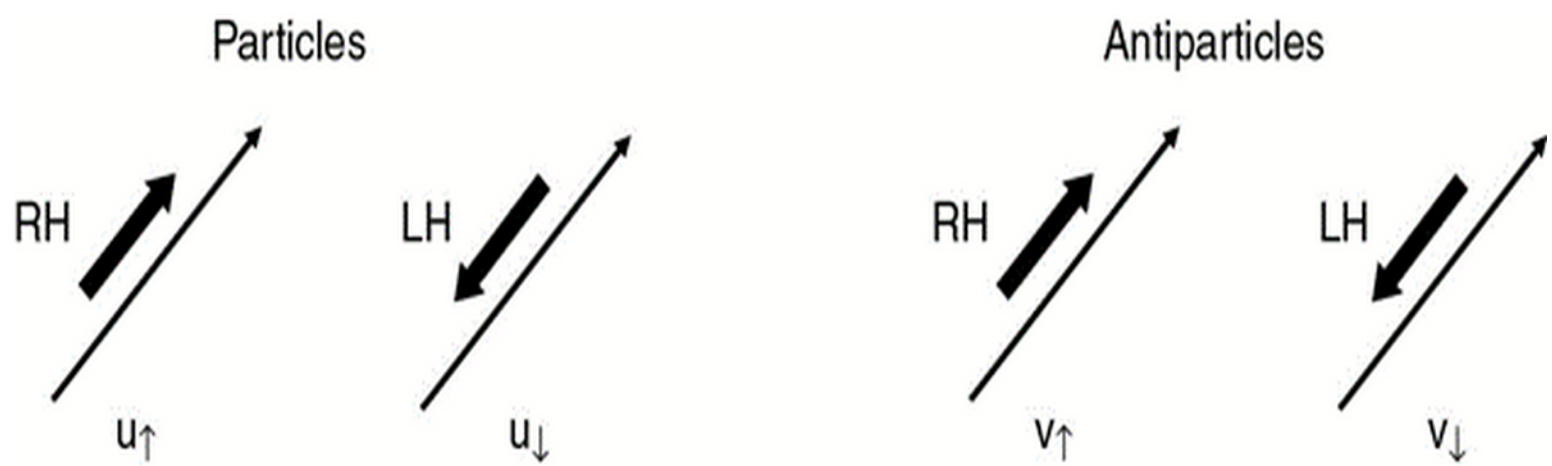
$$v_\uparrow = \sqrt{E+m} \begin{pmatrix} \frac{p}{E+m} s \\ -\frac{p}{E+m} c e^{i\phi} \\ -s \\ c e^{i\phi} \end{pmatrix} \qquad v_\downarrow = \sqrt{E+m} \begin{pmatrix} \frac{p}{E+m} c \\ \frac{p}{E+m} s e^{i\phi} \\ c \\ s e^{i\phi} \end{pmatrix}$$

ANTI PARTICLES

WELL, THIS COURSE IS "INTRODUCTION TO HIGH ENERGY PHYSICS" $\longrightarrow E \gg m$

SO RIGHT HANDED AND LEFT HANDED SPINORS FOR PARTICLES AND ANTIPARTICLES ARE:

$$u_\uparrow \approx \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}, \quad u_\downarrow \approx \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix} \quad v_\uparrow \approx \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix} \quad v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$



SUMMARY OF DIRAC EQUATION

$$\hat{H}_D \psi = (\bar{\alpha} \cdot \bar{p} + \beta m)\psi = i \frac{\partial \psi}{\partial t}$$

IS HOW ELECTRONS BEHAVE, $\psi \rightarrow$ 4 COMPONENT

EXTRA DEGREES OF FREEDOM $\rightarrow$ SPIN

$-ve$ E t $-ve$ $\equiv$ $+ve$ E, t $+ve$ $\longrightarrow$ ANTIPARTICLES

COVARIANT FORM $\quad (i\gamma^\mu \partial_\mu - m)\psi = 0$

4-VECTOR PROB CURRENT $\quad j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi = \bar{\psi} \gamma^\mu \psi$

ADJOINT SPINOR

SOLVED $\quad u_\uparrow \; u_\downarrow \; v_\uparrow \; v_\downarrow$

CHARGE CONJUGATION

$$\psi \rightarrow \hat{C}\psi = i\gamma^2 \psi^*$$

PARITY

$$\psi \rightarrow \hat{P}\psi = \gamma^0 \psi$$

READ ABOUT THESE TWO