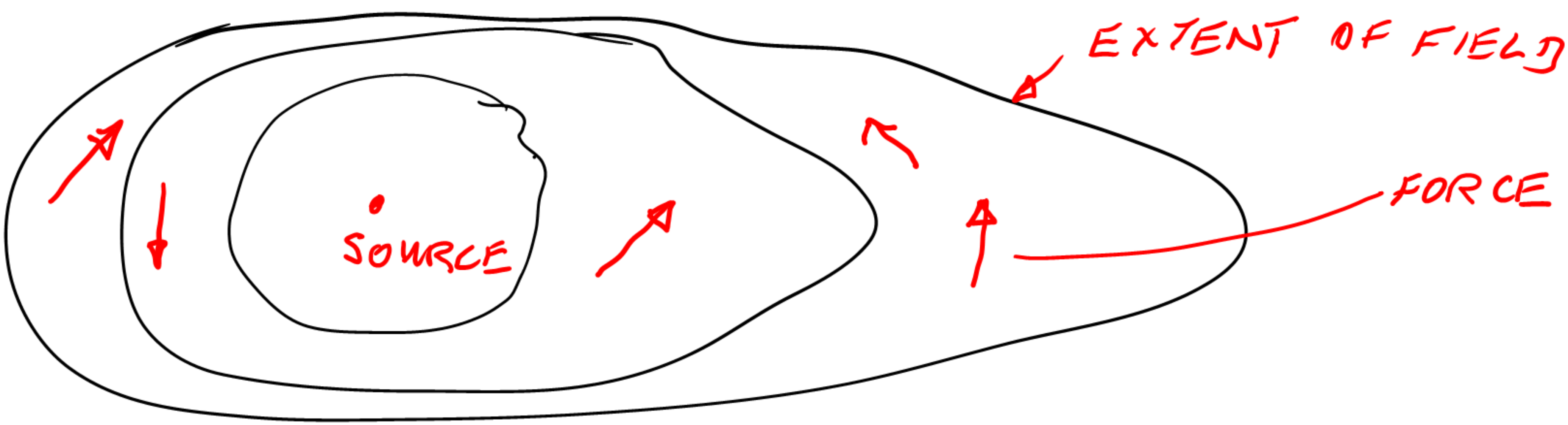
VECTORS

- IN DEVELOPING ELECTROMAGNETISM, WE WILL DEVELOP THE IDEA OF A VECTOR FIELD.
- THE CONCEPT OF A FIELD WAS INVENTED BY MICHAEL FARADAY
- A SOURCE OF SOME FORCE, INDUCES A FORCE OF SOME MAGNITUDE + DIRECTION AT EACH POINT IN SPACE



WE ARE GOING TO UNDERSTAND HOW ELECTRIC CHARGES & CURRENTS INDUCE FORCES ON OTHER CHARGES AND CURRENTS THESE FORCES WILL HAVE MAGNITUDES & DIRECTIONS

- VECTORS ARE EXACTLY THE MATHEMATICL TOOLS THAT WE NEED! :)

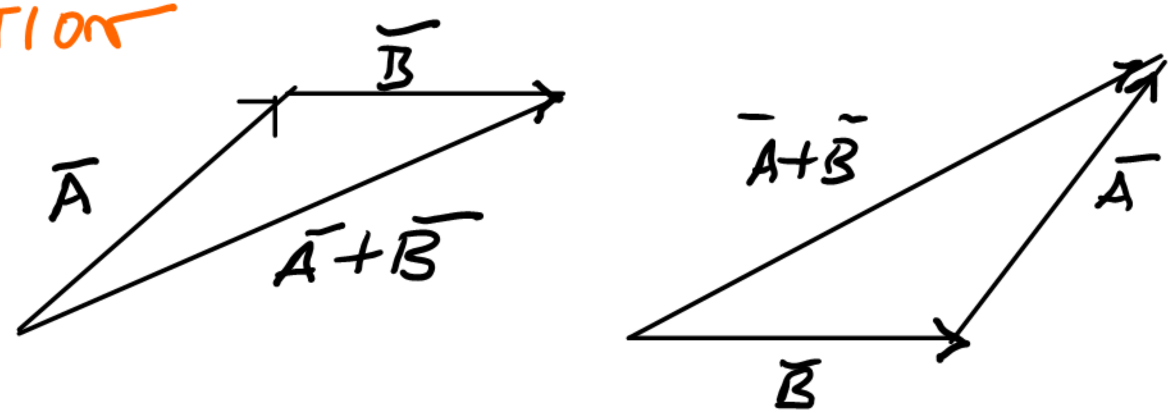
- VARIOUS OF YOU WILL HAVE DIFFERENT LEVELS OF KNOWLEDGE ABOUT VECTORS

BUT MY UNDERSTANDING IS THAT WE WILL HAVE DEVELOP VECTOR OPERATORS ETC.

- LET'S START AT THE BEGINNING — IT ALWAYS MAKES ME FEEL GOOD TO SEE SOMETHING THAT I ALREADY UNDERSTAND.

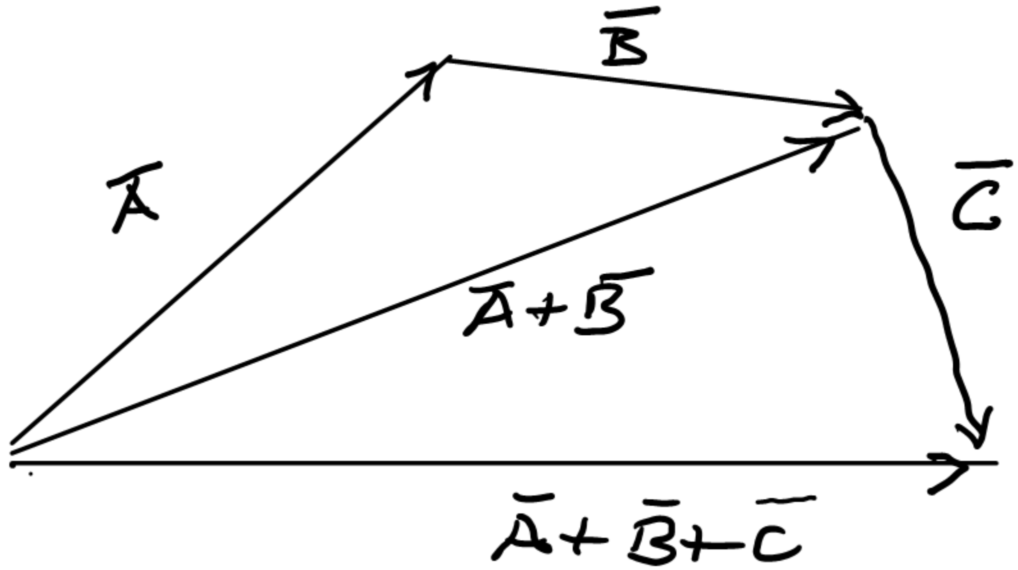
- VECTORS HAVE MAGNITUDE & DIRECTION → BUT NO POSITION IN SPACE
- YOU CAN THINK OF A 3-d VECTOR AS 3 NUMBERS ATTACHED TO THE 3-SPACE COORDINATES
- THERE ARE 4 — ELEMENTARY OPERATIONS ON VECTORS
    - ADDITION
    - 3 KINDS OF MULTIPLICATION
- I GUESS YOU ALREADY KNOW ABOUT THESE OPERATIONS — BUT, LET'S RECAP THEM

- ADDITION

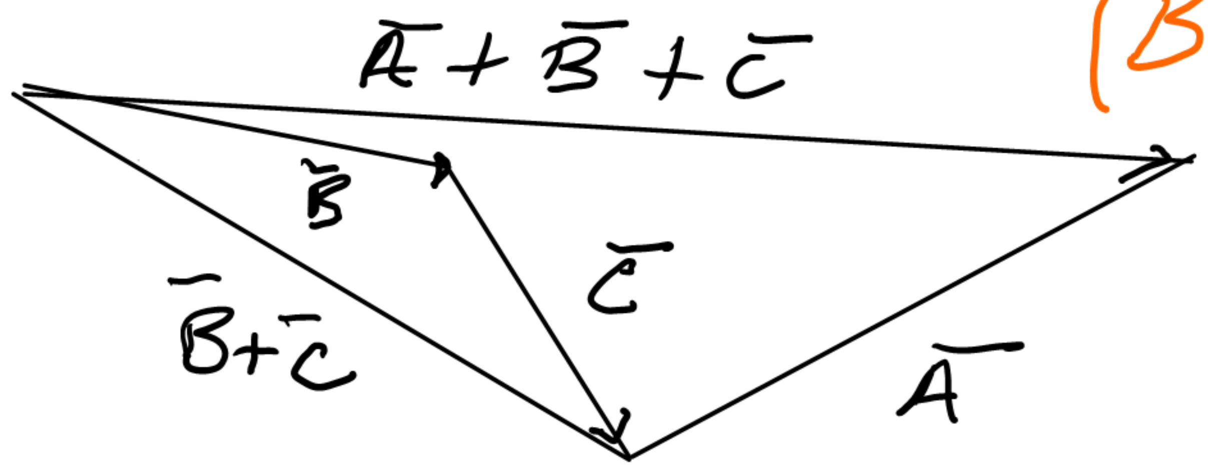


$\bar{A}+\bar{B}=\bar{B}+\bar{A}$

COMMUTATIVE ADDITION



$(\bar{A}+\bar{B}) + \bar{C}$



$(\bar{B}+\bar{C})+\bar{A}$

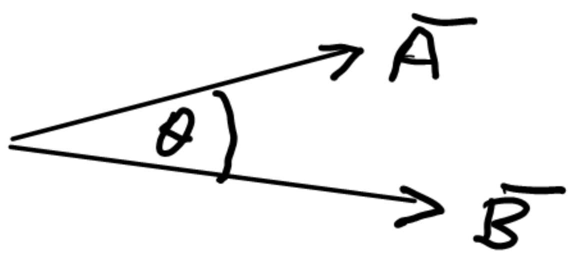
- HOW ABOUT MULTIPLICATION?

- MULTIPLICATION BY A SCALAR → SIMPLE

$$a(\bar{A}+\bar{B}) = a\bar{A} + a\bar{B}$$

THESE ARE DEFINITIONS, BY THE WAY ....

- SCALAR PRODUCT



$$\bar{A}\cdot\bar{B} = A\,B\cos\theta$$

MAGNITUDES → SCALARS

$$\bar{A}\cdot\bar{B} = \bar{B}\cdot\bar{A}$$

OBVIOUS SINCE IT IS A SCALAR

SCALAR PRODUCT:-

$$\bar{A}\cdot(\bar{B}+\bar{C}) = \bar{A}\cdot\bar{B}+\bar{A}\cdot\bar{C}$$ DISTRIBUTIVE

$\bar{A}\cdot\bar{B} = |A|$ TIMES PROJECTION OF $\bar{B}$ ALONG $\bar{A}$

– FOR PARALLEL $\bar{A}\cdot\bar{B} = AB$

$$\bar{A}\cdot\bar{A} = A^2$$

– PERPENDICULAR $\bar{A}\cdot\bar{B} = 0$

– $\bar{C} = \bar{A} - \bar{B}$

$$\bar{C}\cdot\bar{C} = (\bar{A}-\bar{B})(\bar{A}-\bar{B})$$

$$= \bar{A}\cdot\bar{A} - \bar{A}\cdot\bar{B} - \bar{B}\cdot\bar{A} + \bar{B}\cdot\bar{B}$$

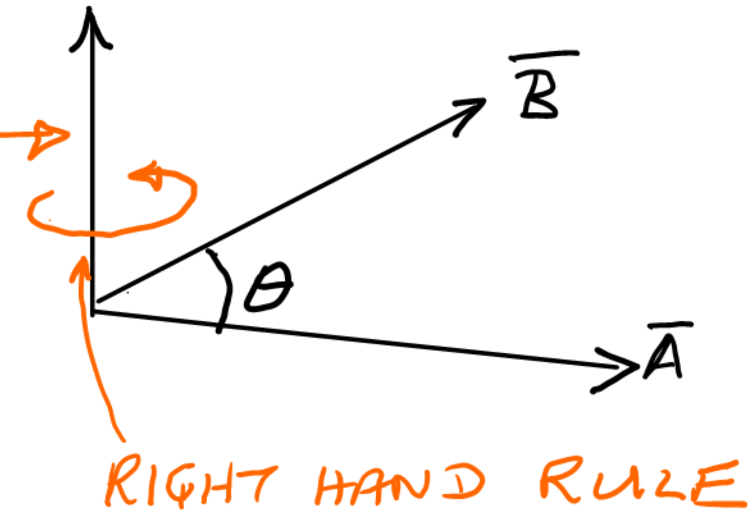
$$= A^2 + B^2 - 2AB\cos\theta$$

└ LAW OF COSINES

CROSS PRODUCT — VECTOR PRODUCT — DEFINITION

$$\bar{A} \times \bar{B} = AB \sin\theta \cdot \hat{n}$$

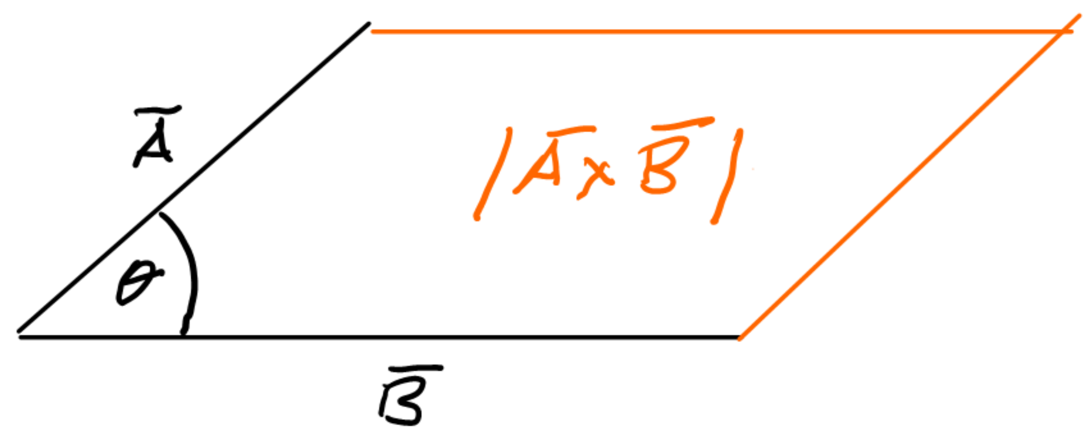
UNIT VECTOR PERP TO PLANE OF $\bar{A}$ & $\bar{B}$



$$\bar{A} \times (\bar{B} + \bar{C}) = \bar{A} \times \bar{B} + \bar{A} \times \bar{C}$$ DISTRIBUTIVE

$$(\bar{B} \times \bar{A}) = -(\bar{A} \times \bar{B})$$ NON COMMUTATIVE

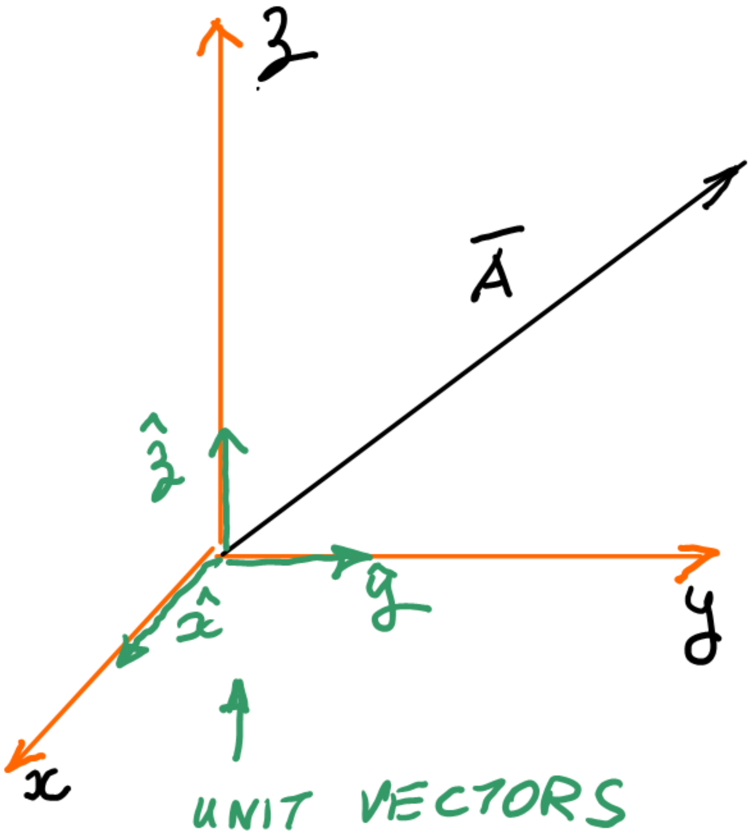
$$\bar{A} \times \bar{A} = 0$$



FOLLOWING THESE RULES – CAN MANIPULATE VECTORS JUST LIKE ANY ALGEBRAIC SYMBOLS

↳ NO REFERENCE TO COORDINATE SYSTEM

WHEN WANT TO REFER TO COORDINATES, USE COMPONENTS REFERED TO COORDINATE SYSTEM



$$\bar{A} = A_x \cdot \hat{x} + A_y \cdot \hat{y} + A_z \hat{z}$$

A_x, A_y, A_z ARE PROJECTIONS ON AXES.

$$\bar{A} + \bar{B} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} + B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

$$= (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}$$

SINCE $\hat{x}$ $\hat{y}$ $\hat{z}$ ARE MUTUALLY PERPENDICULAR UNIT VECTORS

→ FROM PROPERTIES OF DOT PRODUCT

$$\left.\begin{aligned} \hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1 \\ \hat{x}\cdot\hat{y} = \hat{x}\cdot\hat{z} = \hat{y}\cdot\hat{z} = 0 \end{aligned}\right\} \text{USE IN SCALAR PRODUCTS}$$

$$\bar{A}\cdot\bar{B} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$$

$$= A_xB_x\,\underline{\hat{x}\cdot\hat{x}} + A_xB_y\,\hat{x}\cdot\hat{y} + A_xB_z\,\hat{x}\cdot\hat{z}$$
$$+ A_yB_x\,\hat{y}\cdot\hat{x} + A_yB_y\,\underline{\hat{y}\cdot\hat{y}} + A_yB_y\,\hat{y}\cdot\hat{z}$$
$$+ A_zB_x\,\hat{z}\cdot\hat{x} + A_zB_y\,\hat{z}\cdot\hat{y} + A_zB_z\,\underline{\hat{z}\cdot\hat{z}}$$

$$= A_xB_x + A_yB_y + A_zB_z \rightarrow \bar{A}\cdot\bar{A} = A_x^2 + A_y^2 + A_z^2$$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

CAN ALSO LOOK AT HOW THIS WORKS FOR CROSS PRODUCT.

$$\hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0$$

$$\hat{x}\times\hat{y} = -\hat{y}\times\hat{x} = \hat{z}$$

$$\hat{y}\times\hat{z} = -\hat{z}\times\hat{y} = \hat{x}$$

$$\hat{z}\times\hat{x} = -\hat{x}\times\hat{z} = \hat{y}$$

$$\bar{A}\times\bar{B} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\times(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$$

$$= A_xB_x\,\hat{x}\times\hat{x} + A_xB_y\,\hat{x}\times\hat{y} + A_xB_z\,\hat{x}\times\hat{z}$$
$$+ A_yB_x\,\hat{y}\times\hat{x} + A_yB_y\,\hat{y}\times\hat{y} + A_yB_z\,\hat{y}\times\hat{z}$$
$$+ A_zB_x\,\hat{z}\times\hat{x} + A_zB_y\,\hat{z}\times\hat{y} + A_zB_y\,\hat{z}\times\hat{z}$$

0

$$
\begin{aligned}
= \; & A_x B_x \cdot 0 + A_x B_y \hat{z} + A_x B_z (-\hat{y}) \\
& + A_y B_x (-\hat{z}) + A_y B_y \, 0 + A_y B_z \hat{x} \\
& + A_z B_x \hat{y} + A_z B_y -\hat{x} + A_z B_y \, 0
\end{aligned}
$$

$$
\begin{aligned}
= \; & A_x B_y \hat{z} - A_x B_z \hat{y} \\
& - A_y B_x \hat{z} + A_y B_z \hat{x} \\
& + A_z B_x \hat{y} - A_z B_y \hat{x}
\end{aligned}
$$

$$
\left.
\begin{aligned}
= \; & (A_y B_z - A_z B_y)\hat{x} \\
& + (A_z B_x - A_x B_z)\hat{y} \\
& + (A_x B_y - A_y B_x)\hat{z}
\end{aligned}
\right\}
=
\begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
A_x & A_y & A_z \\
B_x & B_y & B_z
\end{vmatrix}
$$

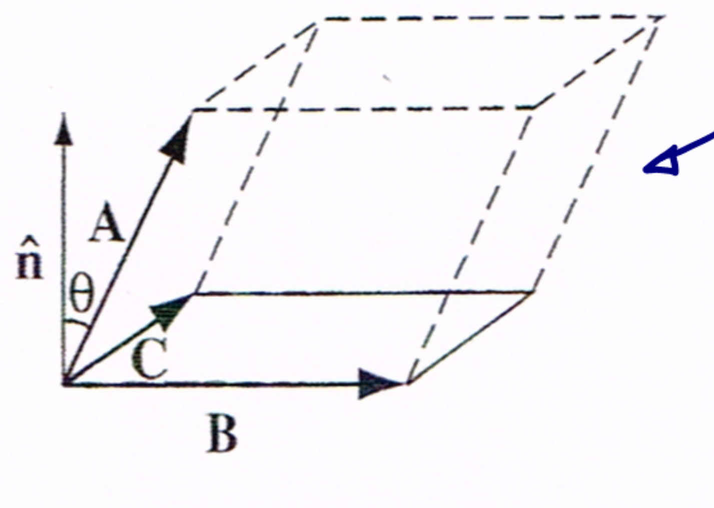
AIDE - MÉMOIRE - DETERMINANTS

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - gf) + c(dh - eg)$$

SCALAR TRIPLE PRODUCT: $\bar{A}\cdot(\bar{B}\times\bar{C})$



$|\bar{A}\cdot(\bar{B}\times\bar{C})|$ IS VOLUME OF

$|\bar{B}\times\bar{C}|$ AREA OF BASE

$|A\cos\theta|$ IS ALTITUDE

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = \bar{B}\cdot(\bar{C}\times\bar{A}) = \bar{C}\cdot(\bar{A}\times\bar{B})$$ ALL SAME VOLUME

SINCE $(\bar{B}\times\bar{A}) = -(\bar{A}\times\bar{B})$ NON ALPHABET HAVE OPPOSITE SIGN

$$\bar{A}\cdot(\bar{C}\times\bar{B}) = \bar{B}\cdot(\bar{A}\times\bar{C}) = \bar{C}\cdot(\bar{B}\times\bar{A})$$

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = (\bar{A}\times\bar{B})\cdot\bar{C}$$

VECTOR TRIPLE PRODUCT $\bar{A}\times(\bar{B}\times\bar{C})$

CAN BE SIMPLIFIED

$$\bar{A}\times(\bar{B}\times\bar{C}) = \bar{B}\,(\bar{A}\cdot\bar{C}) - \bar{C}\,\underbrace{(\bar{A}\cdot\bar{B})}_{\text{SCALAR}}$$

$$(\bar{A}\times\bar{B})\times\bar{C} = -\bar{C}\times(\bar{A}\times\bar{B}) = -\bar{A}\,(\bar{B}\cdot\bar{C}) + \bar{B}\,(\bar{A}\cdot\bar{C})$$

ANY TERM IN ANY EXPRESSION ONLY NEEDS TO HAVE ONE CROSS PRODUCT

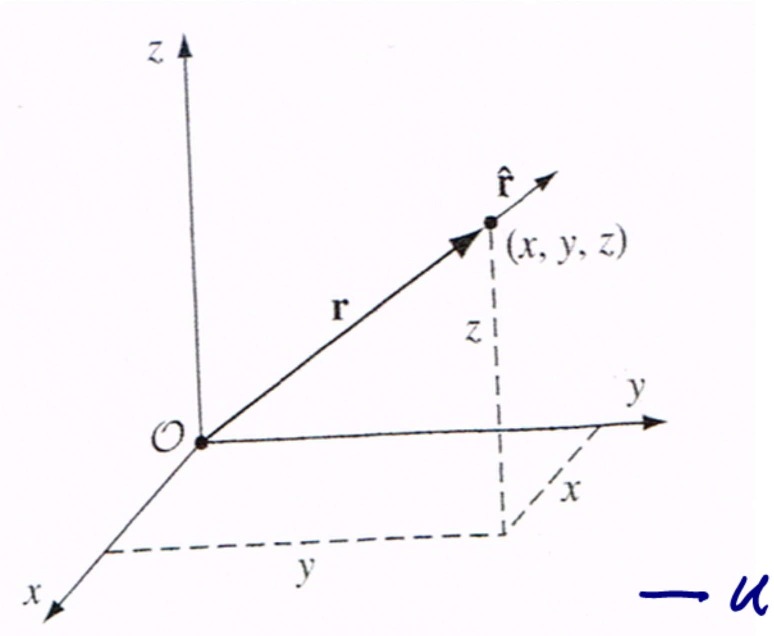
$$(\bar{A}\times\bar{B})\cdot(\bar{C}\times\bar{D}) = (\bar{A}\cdot\bar{C})(\bar{B}\cdot\bar{D}) - (\bar{A}\cdot\bar{D})(\bar{B}\cdot\bar{C})$$

$$\bar{A}\times[\bar{B}\times(\bar{C}\times\bar{D})] = \bar{B}[\bar{A}\cdot(\bar{C}\times\bar{D})] - (\bar{A}\cdot\bar{B})(\bar{C}\times\bar{D})$$

POSITION, DISPLACEMENT, SEPARATION VECTORS

- LOCATION OF POINT IN 3-D → CARTESIAN COORDS (x, y, z)
- VECTOR TO THAT POINT IS

POSITION VECTOR $\vec{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$



MAGNITUDE $r = \sqrt{x^2 + y^2 + z^2}$

DISTANCE FROM ORIGIN

IN THE TEXT WE ARE USING

$\vec{r}$ ALWAYS POINTS TO ORIGIN

- UNIT VECTOR POINTING RADIALLY, OUT

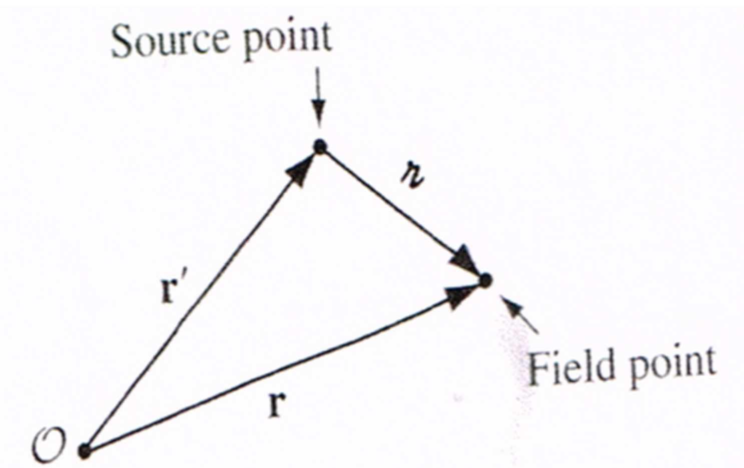
$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x \cdot \hat{x} + y \cdot \hat{y} + z \cdot \hat{z}}{\sqrt{x^2 + y^2 + z^2}}$$

INFINITESIMAL DISPLACEMENT VECTOR

USE FREQUENTLY IN INTEGRALS

$$(x, y, z) \longrightarrow (x+dx,\ y+dy,\ z+dz)$$

$$d\bar{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$



SPECIAL TO OUR TEXT

SYMBOL FOR SOURCE $\longrightarrow$ FIELD

$$\bar{\mathscr{r}} \equiv \bar{r} - \bar{r}'$$

UNIT VECTOR $\bar{r}' \longrightarrow \bar{r}$

$$\hat{\mathscr{r}} = \frac{\bar{\mathscr{r}}}{\mathscr{r}} = \frac{\bar{r} - \bar{r}'}{|\bar{r} - \bar{r}'|}$$

SAVES WRITING. ∸?

IN CARTESIAN COORDINATES:

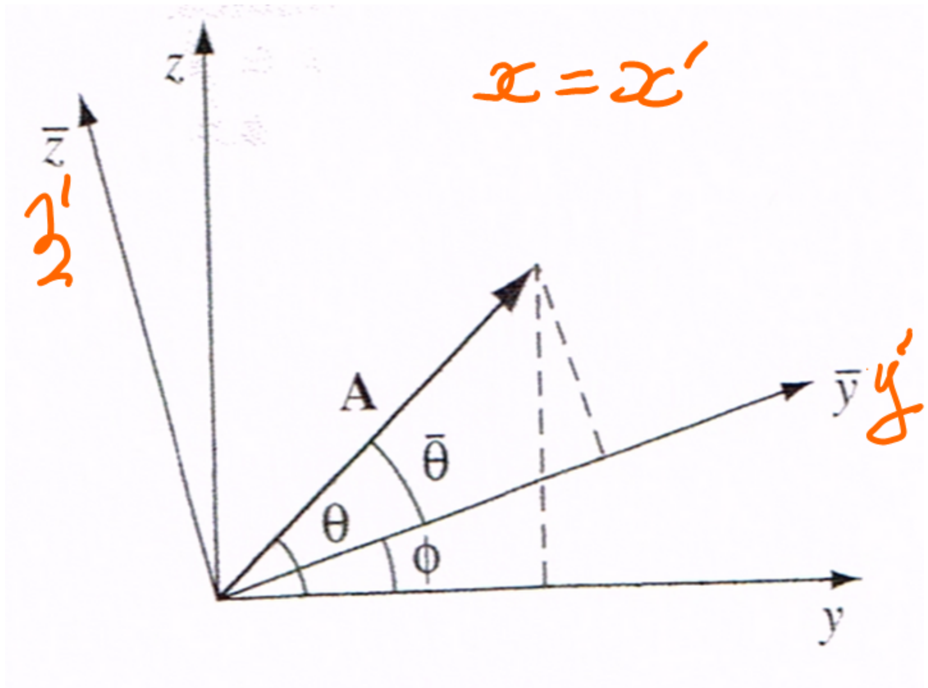
$$\bar{\mathscr{r}} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$

$$|\bar{\mathscr{r}}| = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

$$\hat{\mathscr{r}} = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}}$$

SO, I GEUSS HE MAKES HIS POINT ON "ECONOMY" OF $\mathscr{r}$

HOW VECTORS TRANSFORM ← TRANSFORMATION PROPERTIES ARE WHAT DEFINE VECTORS



– SPECIFIC COORDINATE SYSTEM IS ARBITRARY

(x', y', z') ROTATED BY ϕ W.R.T (x, y, z)

$$A_y = A\cos\theta, \quad A_z = A\sin\theta$$

$$\begin{aligned} A_{y'} = A\cos\theta' &= A\cos(\theta - \phi) \\ &= A(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ &= \cos\phi\, A_y + \sin\phi\, A_z \end{aligned}$$

CONTINUING .....

$$A'_z = A\sin\theta' = A\sin(\theta - \phi)$$

$$= A(\sin\theta\cos\phi - \cos\theta\sin\phi)$$

$$= -\sin\phi\, A_y + \cos\phi\, A_z$$

WRITE NEATLY AS...

$$\begin{pmatrix} A_y' \\ A_z' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

TRANSFORMATION MATRIX

ROTATION ABOUT ARBITRARY AXIS IN 3-DIMENSIONS

$$\begin{pmatrix} A_x' \\ A_y' \\ A_z' \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

COMPACTLY

$$A_i' = \sum_{j=1}^{3} R_{ij} A_j$$

VECTOR — 3 COMPONENTS TRANSFORM IN SAME WAY AS DISPLACEMENTS, AS COORD SYSTEM CHANGES

VECTOR IN ONE (VERY USEFUL!) CASE

— SECOND RANK TENSOR 9 - COMPONENTS

$$T'_{xx} = R_{xx}(R_{xx}T_{xx} + R_{xy}T_{xy} + R_{xz}T_{xz})$$
$$+ R_{xy}(R_{xx}T_{yx} + R_{xy}T_{yy} + R_{xz}T_{yz})$$
$$+ R_{xz}(R_{xx}T_{zx} + R_{xy}T_{zy} + R_{xz}T_{zz}) \ldots\ldots$$

$$T'_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} R_{ik}R_{jl}T_{kl}$$

n - RANK TENSOR, n INDICES & 3^n COMPONENTS

SCALAR $\longrightarrow$ TENSOR OF RANK 0

VECTOR $\longrightarrow$ TENSOR OF RANK 1