MAGNETIC VECTOR POTENTIAL

FROM $\vec{\nabla} \times \vec{E} = \vec{0} \longrightarrow \vec{E} = -\vec{\nabla} V$ SCALAR POTENTIAL

FROM $\vec{\nabla} \cdot \vec{B} = 0 \longrightarrow \vec{B} = \vec{\nabla} \times \vec{A}$ VECTOR POTENTIAL

THIS AUTOMATICALLY $\vec{\nabla} \cdot \vec{B} = 0$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

AMPÈRE'S LAW

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \underbrace{\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}}_{\text{LOOKS COMPLICATED}} = \mu_0 \vec{J}$$

RECALL → CAN ADD ANY F, $\vec{\nabla} F = 0$ TO V

CAN ADD TO $\vec{A}$ ANY $\vec{F}$, $\vec{\nabla} \times \vec{F} = 0$ GRADIENT OF ANY SCALAR

→ MAKE $\vec{\nabla} \cdot \vec{A} = 0$

SUPPOSE $\vec{A}_0$ NOT DIVERGENCELESS

ADD GRADIENT of λ → $\vec{A} = \vec{A}_0 + \vec{\nabla}\lambda$

NEW DIVERGENCE

$$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{A}_0 + \nabla^2\lambda$$

CAN MAKE $\vec{\nabla}\cdot\vec{A} \to 0$ IF $\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}_0$

KNOW HOW TO SOLVE → THIS IS JUST POISSON

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{\mathscr{r}}\, d\tau' \longrightarrow \lambda = \frac{1}{4\pi}\int \frac{\vec{\nabla}\cdot\vec{A}_0}{\mathscr{r}}\, d\tau'$$

IF $\rho \to 0$ @ ∞

IF $\vec{\nabla}\cdot\vec{A}_0 \to 0$ @ ∞

ALWAYS POSSIBLE TO MAKE $\vec{\nabla}\cdot\vec{A} \to 0$

CAN ALWAYS MAKE $\bar{\nabla}\cdot\bar{A} \to 0$

THE DEFINITION $\bar{B} = \bar{\nabla}\times\bar{A}$ SAYS NOTHING ABOUT $\bar{\nabla}\cdot A$

CHOOSE 0 → CAN CHOOSE IT ANYWAY WE LIKE

$$\bar{\nabla}\times\bar{B} = \bar{\nabla}\times(\bar{\nabla}\times\bar{A}) = \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \nabla^2\bar{A} = \mu_0\bar{J}$$

BECOMES $\bar{\nabla}^2\bar{A} = -\mu_0\bar{J}$ ← AMPÈRE'S LAW

→ POISSON 3-d LAW

WE KNOW THE SOLUTION IS

$$\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi}\int\frac{\bar{J}(\bar{r}')}{\mathscr{r}}\,d\tau'$$

ASSUME $\bar{J}\to 0$ @ ∞

LINE CURRENT $\bar{A} = \frac{\mu_0}{4\pi} \int \frac{\bar{I}}{\mathscr{r}} \, dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{\mathscr{r}} \, d\bar{l}'$

SURFACE CURRENT $\bar{A} = \frac{\mu_0}{4\pi} \int \frac{\bar{K}}{\mathscr{r}} \, da'$

$\bar{A}$ NOT AS USEFUL AS V $\rightarrow$ IT'S A VECTOR

WOULD BE NICE IF $\vec{B} = -\vec{\nabla} u$ SCALAR

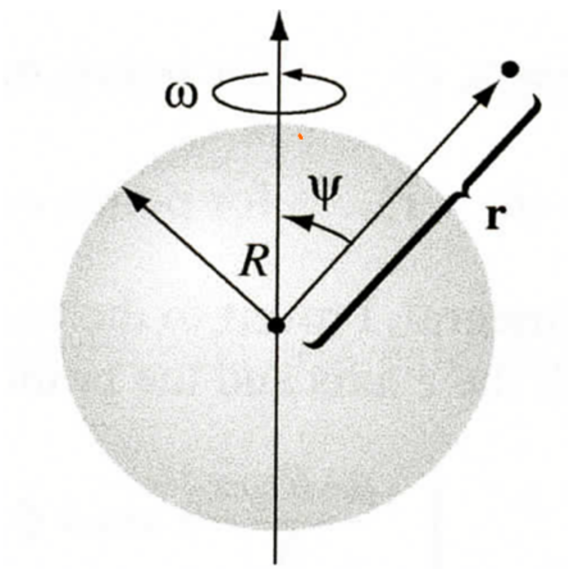
THIS CONTRADICTS $\nabla^2 \bar{A} = -\mu_0 \vec{J} \rightarrow \vec{\nabla} \times (\vec{\nabla} u) = 0$

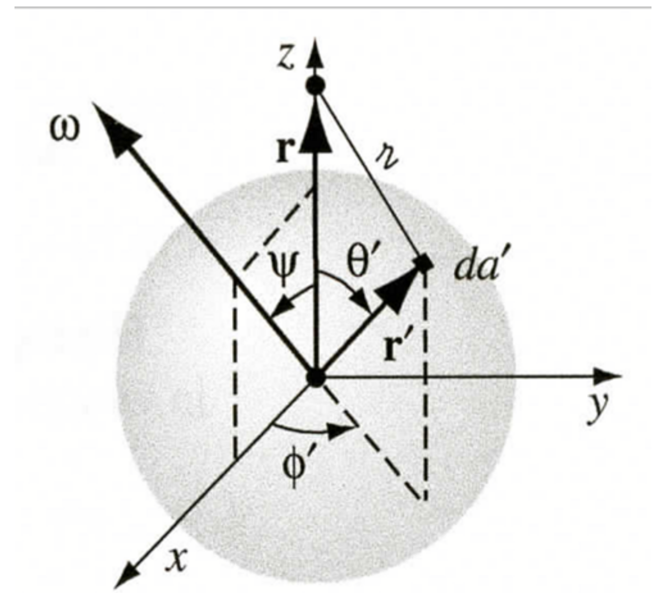
SINCE MAGNETIC FORCES DO NO WORK

$\bar{A}$ CAN NOT BE INTERPRETED SIMPLY

AS POTENTIAL ENERGY / UNIT CHARGE

EXAMPLE: SPHERICAL SHELL RADIUS R UNIFORM SURFACE CHARGE σ SPINNING AT ANGULAR VELOCITY ω. FIND VECTOR POTENTIAL AT A POINT r





PUT r ALONG z-AXIS, ω TILTED AT ANGLE ψ

ω LIES IN xz PLANE

SURFACE CURRENT

$$A(\bar{r}) = \frac{\mu_0}{4\pi} \int \frac{\bar{K}(\bar{r}')}{\mathscr{r}} \, da'$$

$$A(\bar{r}) = \frac{\mu_0}{4\pi}\int \frac{\bar{K}(\bar{r}')}{\eta}\, da'$$

$\bar{K} = \sigma \bar{V}$ — VELOCITY

$$\eta = \left(R^2 + r^2 - 2Rr\cos\theta'\right)^{\frac{1}{2}}$$

$$da' = R^2 \sin\theta'\, d\theta'\, d\phi'$$

LAW OF COSINES

$$\bar{\eta} = \bar{r} - R$$

$$\eta^2 = (\bar{r} - \bar{R})^2$$

$$\eta^2 = (r^2 + R^2 - 2Rr\cos\theta)$$

VELOCITY OF POINT $\bar{r}'$ ON ROTATING RIGID BODY

$$\bar{V} = \bar{\omega} \times \bar{r}'$$

$\bar{\omega}$ IS IN xy PLANE

$$\omega_z = \omega\cos\psi$$

$$\omega_x = \omega\sin\psi$$

$$\omega_y = 0$$

GENERALLY — SPHERICAL COOR DS

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

$$x = R'\sin\theta'\cos\phi',\ y = R\sin\theta'\sin\phi',\ z = R\cos\theta'$$

$$\bar{V} = \bar{\omega} \times \bar{r} \quad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega \sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega\left(-\cos\psi \cancel{\sin\theta' \sin\phi'}\right)\hat{x} + R\omega\left(\cos\psi \cancel{\sin\theta'\cos\phi'} - \sin\psi\cos\theta'\right)\hat{y}$$

$$- R\omega\left(\cos\psi \cancel{\sin\theta' \sin\phi'}\right)\hat{z}$$

$$\int_0^{2\pi} \sin\phi' \, d\phi' = 0 \quad ; \quad \int_0^{2\pi} \cos\phi' \, d\phi' = 0$$

TERMS IN $\sin\phi'$, $\cos\phi' \rightarrow 0$

$$\bar{V} = -R\omega \sin\psi \cos\theta' \, \hat{y}$$

CONSTANT

$\sigma \bar{V}$

$$\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi}\int \frac{\bar{K}}{\mathscr{r}} \, d\tau = \hat{y}\,\frac{\mu_0}{4\pi}\int \frac{-R\omega \sin\psi\cos\theta' \, r^2 \sin\theta' \, dr \, d\theta' \, d\phi'}{\left(R^2 + r^2 - 2Rr\cos\theta'\right)}$$

NO INTEGRAL OVER r

$r \rightarrow R$

$$\bar{A}(\bar{r}) = -\frac{\mu_0}{4\pi} R\sigma\omega\sin\psi \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \int_0^{\pi} \frac{R^2\cos\theta'\sin\theta'\,d\theta'}{(\cdots\cdots)^{\frac{1}{2}}}$$

$$= -\frac{\mu_0}{2} R^3\sigma\omega\sin\psi \int_0^{\pi} \frac{\cos\theta'\sin\theta'\,d\theta'}{(\cdots\cdots)^{\frac{1}{2}}}$$

PUT $u = \cos\theta'$, $du = -\sin\theta'\,d\theta'$

HAVE $\displaystyle\int_{-1}^{+1} \frac{u\,du}{(R^2+r^2-2ru)}$

KNOW THAT $\displaystyle\int \frac{x\,dx}{(a+bx)^{\frac{1}{2}}} = \frac{-2(2a-bx)}{3b^2}\cdot\sqrt{a+bx}$

$a = R^2+r^2$, $b = -2Rr$

$$\int = \frac{-2(2R^2+2r^2+2Rru)}{3(-2Rr)^2}\sqrt{\cdots\cdots}$$

$$\int_{-1}^{+1} \longrightarrow \quad \frac{-(R^2+r^2+Rr\,u)}{3R^2r^2}\sqrt{\cdots\cdots}$$

FOR $u=+1$ $\quad = \dfrac{-1}{3R^2r^2}\left(R^2+r^2+Rr\right)(R-r)$

$u=-1$ $\quad = \dfrac{-1}{3R^2r^2}\left(R^2+r^2-Rr\right)\left(R^2+r^2+2Rr\right)^{\frac{1}{2}}$

$\quad = \dfrac{1}{3R^2r^2}\left(R^2+r^2-Rr\right)(R+r)$

HAVE TO TAKE +VE $\sqrt{\ }$ IN EACH CASE

$$\bar{A}(\bar{r}) \begin{cases} \mu_0 \dfrac{R\sigma}{3}(\bar{\omega}\times\bar{r}) & \text{INSIDE SPHERE} \\[2ex] \mu_0 \dfrac{R^4\sigma}{3r^2}(\bar{\omega}\times\bar{r}) & \text{OUT SIDE SPHERE} \end{cases}$$

$\bar{\omega} \times \bar{r} = \omega r \sin\theta$ in $\hat{\phi}$ DIRECTION

$$\bar{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R \omega \sigma}{3} \cdot r \sin\theta \, \hat{\phi} & r \leq R \\[2ex] \dfrac{\mu_0 R^4 \omega \sigma}{3} \dfrac{r}{r^3} \sin\theta \, \hat{\phi} & r \geq R \end{cases}$$

NOW USE $\bar{A}$ TO GET $\bar{B}$

$$\bar{B} = \bar{\nabla} \times \bar{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r} r A_\phi\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$\bar{A} \to A\hat{\phi}$ $\therefore$ ONLY A_ϕ TERMS CONTRIBUTE, $\underline{r \leq R}$

INSIDE

① $\frac{\mu_0 R\omega\sigma}{3}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin^2\theta\right)\right]\hat{r}$

$$= \frac{\mu_0 R\omega\sigma}{3}\cdot\frac{2\sin\theta\cos\theta\, r}{r\sin\theta} = \frac{\mu_0 R\omega\sigma}{3}\cdot 2\cos\theta\,\hat{r}$$

② $-\frac{\mu_0 R\sigma\omega}{3}\cdot\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot r\sin\theta\right)\hat{\phi} = -\frac{\mu_0 R\sigma\omega}{3}\frac{1}{r}2r\sin\theta\,\hat{\phi}$

so $\bar{\nabla}\times\bar{A} = \bar{B} = \frac{\mu_0 R\sigma\omega}{3}\left(\underbrace{2\cos\theta\,\hat{r} - 2\sin\theta\,\hat{\phi}}_{2\hat{z}}\right)$

$$\bar{B} = \frac{2\mu_0\sigma R\omega}{3}\,\hat{z}$$

$\bar{B} = \frac{2}{3}\mu_0\sigma R\bar{\omega}$ ← UNIFORM INSIDE SPHERE

EXAMPLE: FIND VECTOR POTENTIAL of ∞ SOLENOID

n TURNS PER UNIT LENGTH, RADIUS R

CURRENT I.

CANNOT USE $\bar{A} = \frac{\mu_0}{4\pi}\int \frac{I}{\eta}\, dl'$ CURRENT EXTENDS TO ∞

$$① \underbrace{\oint \bar{A}\cdot d\bar{l} = \int (\bar{\nabla}\times\bar{A})\cdot d\bar{a}'}_{\text{STOKES}} = \int \underbrace{\bar{B}\cdot d\bar{a}}_{\bar{B} = \bar{\nabla}\times\bar{A}} = \underline{\Phi}$$

$\underline{\Phi}$ IS FLUX OF $\bar{B}$ THRU ONE TURN LOOP

AMPÈRE'S LAW $\oint \bar{B}\cdot d\bar{l} = \mu_0 I_{enc}$ CURRENT ENCLOSED BY AMPER LOOP

THIS IS SAME AS ① ABOVE WITH

$$\bar{B} \to \bar{A}$$
$$\mu_0 I_{en} \to \underline{\Phi}$$

IN SYMMETRICAL SITUATION CAN DETERMINE

$\bar{A}$ from $\bar{\Phi}$

JUST AS $\bar{B}$ from I_{encl}

WE HAVE UNIFORM LONGITUDINAL MAGNETIC FIELD INSIDE SOLENOID, NO FIELD OUTSIDE

ANALAGOUS TO AMPERE LAW TO SOLVE THICK WIRE CARRYING UNIFORM CURRENT

– VECTOR POTENTIAL

USE CIRCULAR AMPERIAN LOOP RADIUS s INSIDE SOLENOID

$$\oint \bar{A}\cdot d\bar{l} = A\cdot 2\pi s = \int \bar{B}\cdot d\bar{a}$$

$\int$ AROUND LOOP

πs^2

$\bar{B} = \mu_0 n I$

$$\int \bar{B} \cdot d\bar{a} = \mu_0 n I \pi s^2$$

$$\bar{A} = \frac{\mu_0 n I s}{2} \hat{\phi} \quad \text{for } s \leq R$$

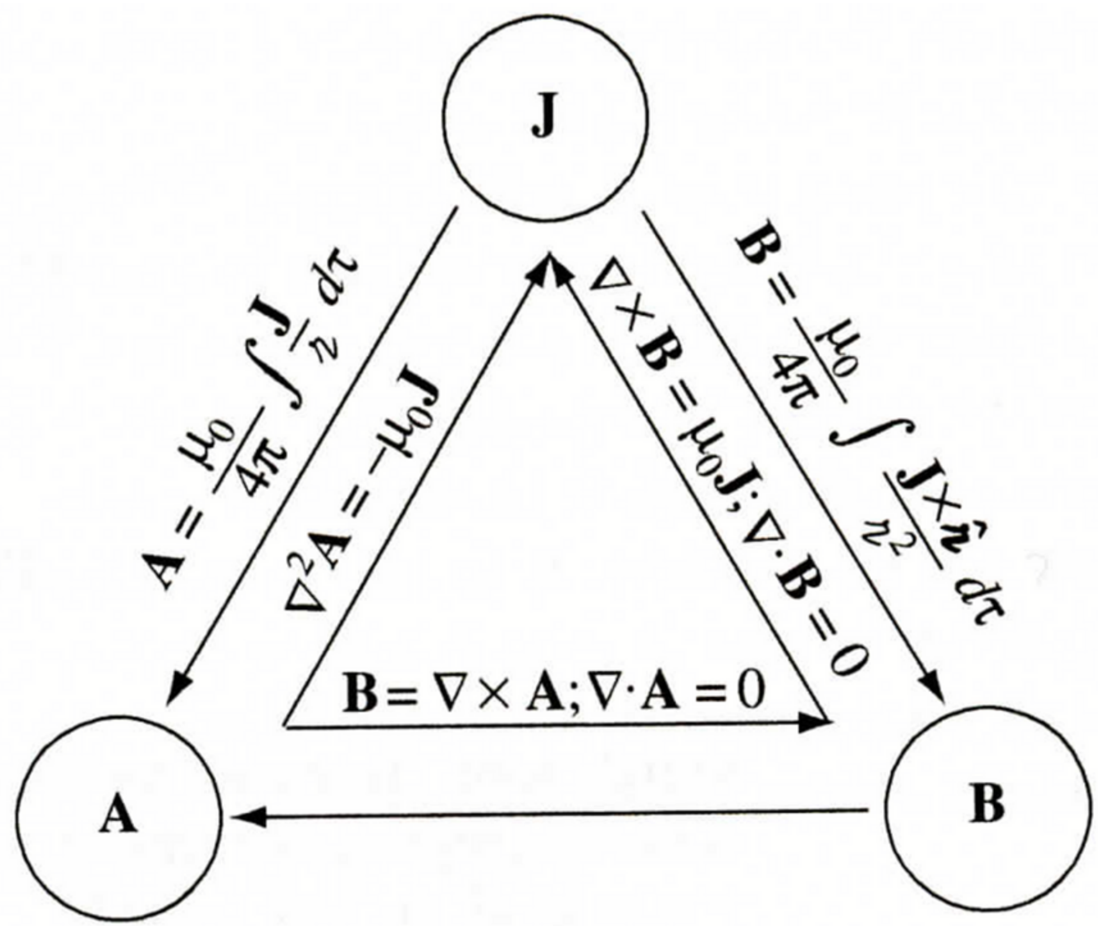
$\bar{B}$ is ZERO OUTSIDE SOLENOID

SO MAXIMUM AMPERIAN LOOP $\rightarrow$ RADIUS R

$$\int \bar{B} \cdot d\bar{a} = \underbrace{\mu_0 n I}\cdot \underbrace{\pi R^2}_{\text{AREA OF AMERIAN LOOP}}$$

$$\bar{A} = \frac{\mu_0 n I}{2} \cdot \frac{R^2}{s} \hat{\phi} \quad \text{for } s \geq R$$

TRIANGLE SUMMARY of MAGNETOSTATICS



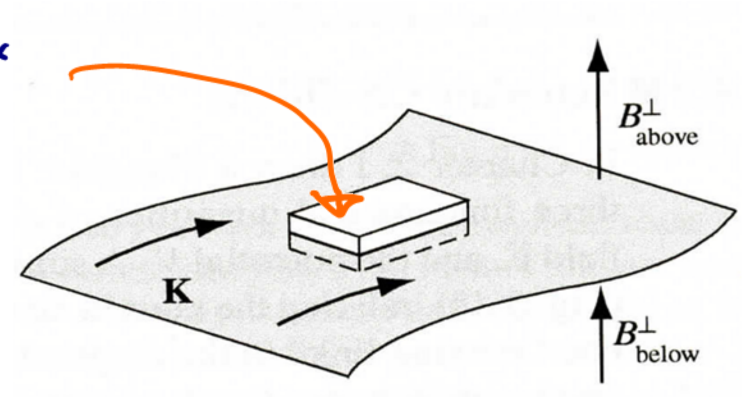
BOUNDARY CONDITIONS

ELECTROSTATICS → $\bar{E}$ DISCONTINUOUS @ SURFACE CHARGE

MAGNETOSTATICS → $\bar{B}$ DISCONTINUOUS @ SURFACE CURRENT

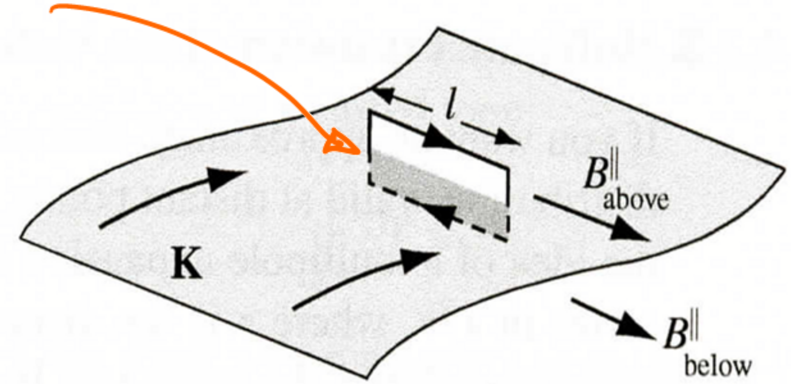
↳ TANGENTIAL COMPONENT CHANGES

APPLY $\oint \bar{B} \cdot d\bar{a} = 0$ TO PILL BOX



$$B^{\perp}_{ABOVE} = B^{\perp}_{BELOW}$$

AMPERIAN LOOP PERP TO CURRENT



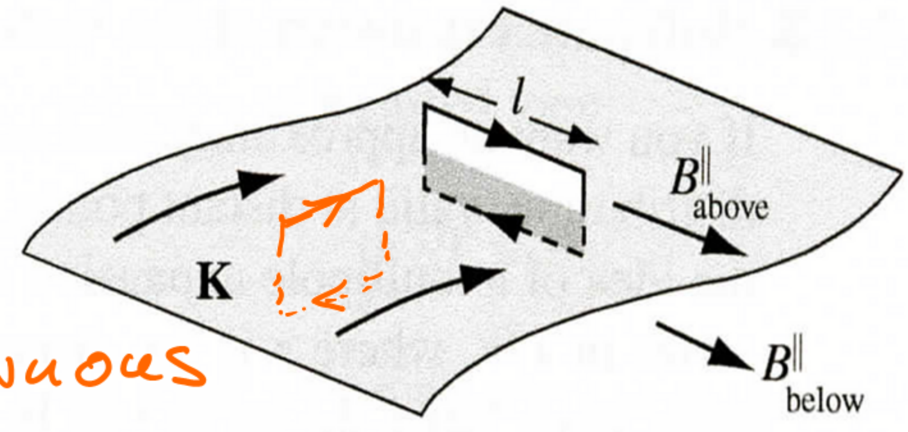
$$\oint \bar{B} \cdot d\bar{l} = \left(B^{\parallel}_{ABOVE} - B^{\parallel}_{BELOW} \right) l$$

$$= \mu_0 I_{enc} = \mu_0 K l$$

$$B^{\parallel}_{ABOVE} - B^{\parallel}_{BELOW} = \mu_0 K$$

AMPERIAN LOOP $\parallel^{el}$ TO CURRENT

$\bar{B}$ PARALLEL TO LOOP IS SAME ON EACH SIDE OF LOOP



SO $\bar{B}$ PARALLEL TO CURRENT CONTINUOUS

SUMMARIZE AS: $\bar{B}_{\text{ABOVE}} - \bar{B}_{\text{BELOW}} = \mu_0 (\bar{K} \times \hat{n})$

LIKE SCALAR $\bar{E}$ POTENTIAL, $\bar{A}$ CONTINUOUS OVER BOUNDARY

$$\bar{A}_{\text{ABOVE}} = \bar{A}_{\text{BELOW}}$$

$\bar{\nabla} \cdot \bar{A} = 0 \Rightarrow$ IMPLIES NORMAL COMPONENT CONTINUOUS

$\uparrow \bar{A}$

$\uparrow \bar{A}$

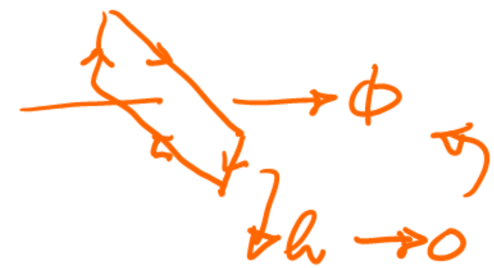
$\bar{\nabla} \cdot \bar{A} = 0$

USED STOKES TO SHOW THAT

$$\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}\cdot d\vec{a} = \Phi$$

← FLUX THRU LOOP

LOOP OF ZERO THICKNESS $\Phi \to 0$



$$\vec{B}_{ABOVE} - \vec{B}_{BELOW} = \mu_0 \vec{K} \times \hat{n}$$

$$\vec{\nabla}\times\vec{A}_{ABOVE} - \vec{\nabla}\times\vec{A}_{BELOW} = \mu_0 \vec{K}\times\hat{n}$$

← DISCONTINUOUS

$$\frac{\partial \vec{A}_{ABOVE}}{\partial n} - \frac{\partial \vec{A}_{BELOW}}{\partial n} = \mu_0 \vec{K}$$