MAXWELL'S EQUATIONS

SUMMARIZE WHAT WE KNOW:

(i) $\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0$ GAUSS

(ii) $\vec{\nabla} \cdot \vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\partial \vec{B} / \partial t$ FARADAY

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ AMPÈRE

$\Big\}$ EXPERIMENTAL FACTS

INHERENT INCONSISTENCY

FIRST TAKE DIVERGENCE OF $\vec{\nabla} \times E = -\partial \vec{B} / \partial t$

$$\vec{\nabla} \cdot (\vec{\nabla} \times E) = \vec{\nabla} \cdot \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{B} \right)$$

DIV. CURL $= 0$

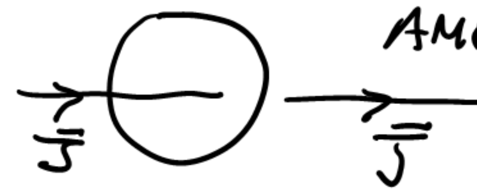
$\vec{\nabla} \cdot B = 0$

CONSISTENT

NOW TAKE DIVERGENCE OF $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$

$$\underbrace{\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right)}_{\text{DIV CURL}=0} = \mu_0 \left(\vec{\nabla}\cdot\vec{J}\right)$$

FOR STEADY CURRENT THIS IS ZERO

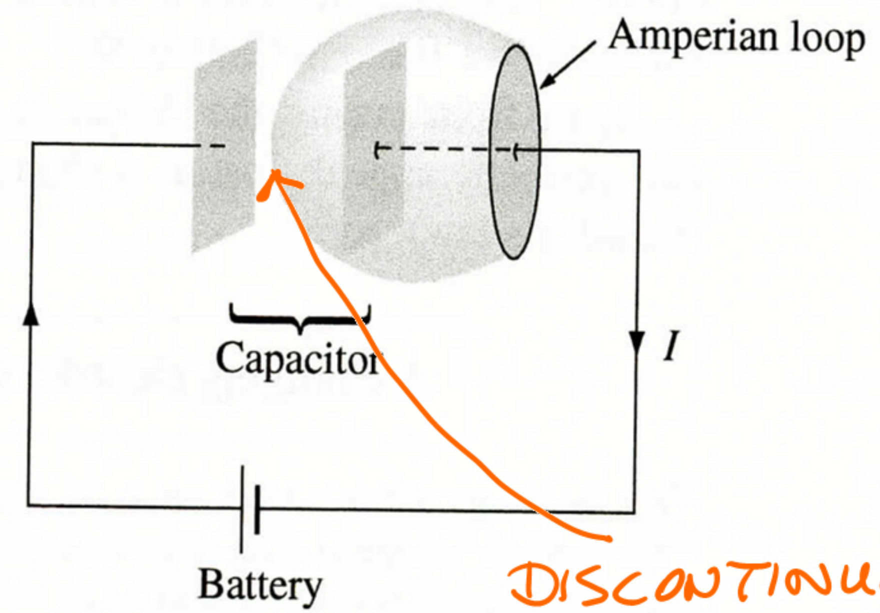


CHARGE DOES NOT PILE UP

IF GO BEYOND MAGNETOSTATICS NOT STEADY CURRENT

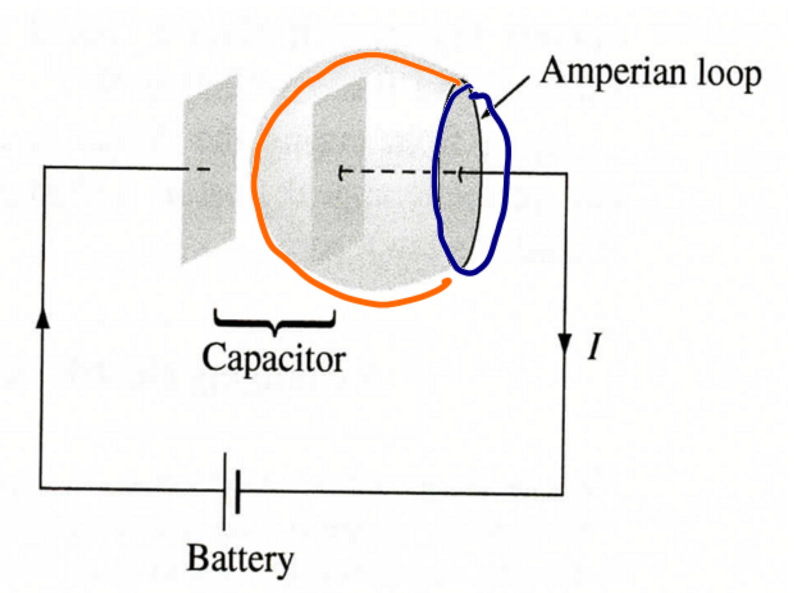
→ AMPERE CANNOT BE TRUE

UP UNTIL NOW HAVE CONSIDERED CONTINUOUS CIRCUITS, CHARGE CANNOT PILE UP



DISCONTINUOUS CIRCUIT

IN THIS CASE WE ARE CHARGING UP A CAPACITOR



IN GENERAL AMPÈRE SAYS

$$\oint \bar{B} \cdot d\bar{l} = \mu_0 I_{enc}$$

TOTAL CURRENT PASSING THROUGH LOOP

FOR SURFACE IN PLANE OF LOOP $I_{en} = I$ OK ✓

FOR ORANGE SURFACE

→ NO CURRENT PASSES THRU IT

$I_{enc} = 0$!

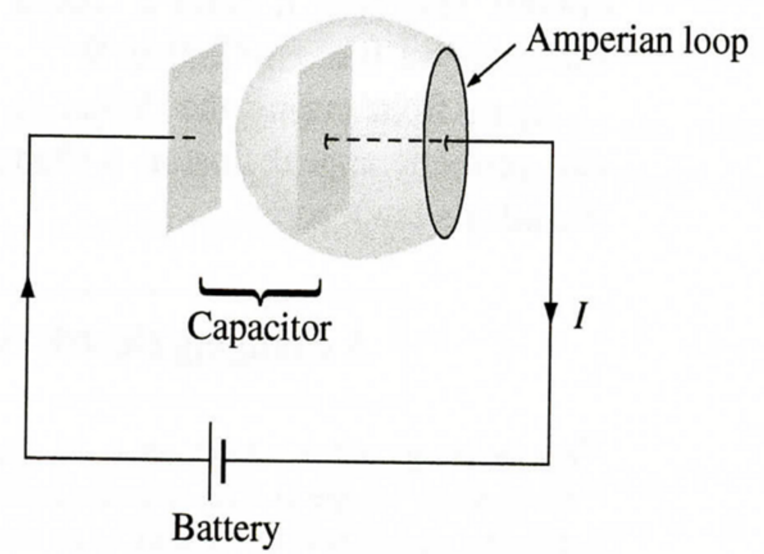
SITUATIONS LIKE THIS DO NOT ARISE IN MAGNETOSTATICS

→ ONLY WHEN CHARGE PILING UP.

FOR NON-STEADY CURRENTS

→ CURRENT ENCLOSED BY LOOP IS ILL-DEFINED

↳ DEPENDS ON SURFACE YOU CHOOSE



THIS IS A PURELY THEORETICL PROBLEM

↳ AT THE TIME OF MAXWELL THERE WAS NO EXPERIMENTAL REASON TO DOUBT AMPÈRE'S LAW

GO BACK TO

$$\underbrace{\bar{\nabla}\cdot(\bar{\nabla}\times\bar{B})}_{\text{ZERO}} = \mu_0\underbrace{(\bar{\nabla}\cdot\bar{J})}$$

SHOULD BE ZERO BUT ISN'T.

CONTINUITY EQUATION

$$\bar{\nabla}\cdot\bar{J} = -\frac{\partial\rho}{\partial t}$$

GAUSS

$$-\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}\left(\varepsilon_0\bar{\nabla}\cdot\bar{E}\right)$$

$$= -\bar{\nabla}\cdot\left(\varepsilon_0\frac{\partial\bar{E}}{\partial t}\right)$$

IF $\quad \bar{J} \rightarrow \bar{J} + \varepsilon_0\frac{\partial\bar{E}}{\partial t}$

$$\bar{\nabla}\cdot\left(\bar{J} + \varepsilon_0\frac{\partial\bar{E}}{\partial t}\right) = 0$$

$$\bar{J} \to \bar{J} + \varepsilon_0 \frac{\partial \bar{E}}{\partial t}$$

REWRITE AMPÈRE AS:

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \partial \bar{E}/\partial t$$

THEN

$$\underbrace{\bar{\nabla} \cdot (\bar{\nabla} \times \bar{B})}_{0} = \underbrace{\bar{\nabla} \cdot (\mu_0 \bar{J}) + \bar{\nabla} \cdot \left(\varepsilon_0 \partial \bar{E}/\partial t\right)}_{0}$$

THIS CHANGES NOTHING IN MAGNETOSTATICS

$\bar{E}$ CONSTANT $\to$ $\partial \bar{E}/\partial t = 0$

EXTRA TERM HARD TO DETECT

$J \to$ LARGE

$\partial E/\partial t \to$ SMALL

NOW:

$$\frac{\partial \bar{B}}{\partial t} \to \bar{E}$$

$$\frac{\partial \bar{E}}{\partial t} \to \bar{B}$$

CHANGING $\bar{E}$ FIELD RESULTS IN $\bar{B}$ FIELD

MAXWELL CALLED EXTRA TERM DISPLACEMENT CURRENT

$$J_d \equiv \varepsilon_0 \, \partial \bar{E}/\partial t$$

GO BACK TO CHARGING CAPACITOR, PLATES CLOSE TOGETHER

ELECTRIC FIELD BETWEEN PLATES

GAUSS $\rightarrow$ $E = \dfrac{\sigma}{\varepsilon_0}$ ← SURFACE CHARGE DENSITY

$\oint \bar{E} \cdot d\bar{a} = \dfrac{Q_{en}}{\varepsilon_0}$

$EA = \dfrac{\sigma A}{\varepsilon_0}$

$= \dfrac{1}{\varepsilon_0} \dfrac{Q}{A}$ ← TOTAL CHARGE (Q), ← PLATE AREA (A)

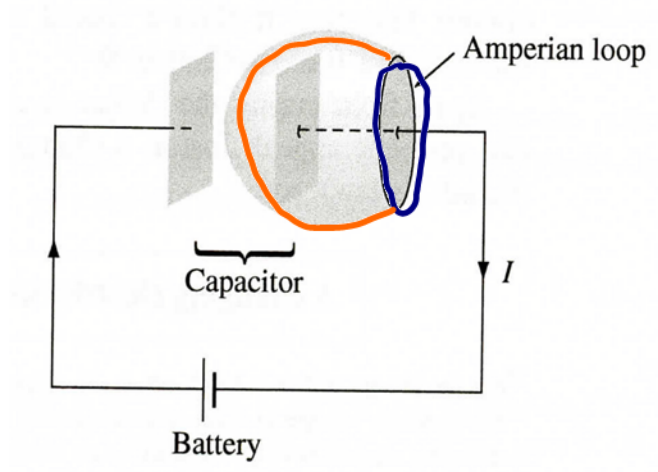
INTEGRAL FORM OF

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \, \partial \bar{E}/\partial t$$

BECOMES

$$\oint \bar{B} \cdot d\bar{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \int \frac{\partial \bar{E}}{\partial t} \cdot d\bar{a}$$

$$\oint \bar{B} \cdot d\bar{\ell} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \int \frac{\partial \bar{E}}{\partial t} \cdot d\bar{a}$$



FOR FLAT SURFACE

$E = 0$ AND $I_{enc} = I$

COMES FROM CONDUCTION CURRENT

FOR ORANGE SURFACE $I_{ENC} = 0$

BUT $\oint \bar{B} \cdot d\bar{\ell} = \mu_0 \varepsilon_0 \int \frac{\partial \bar{E}}{\partial t} \cdot d\bar{a}$

PUT $\frac{\partial E}{\partial t} = I / \varepsilon_0 A$

$$\oint \bar{B} \cdot d\bar{\ell} = \mu_0 \varepsilon_0 \int \frac{I}{\varepsilon_0 A} \cdot da$$

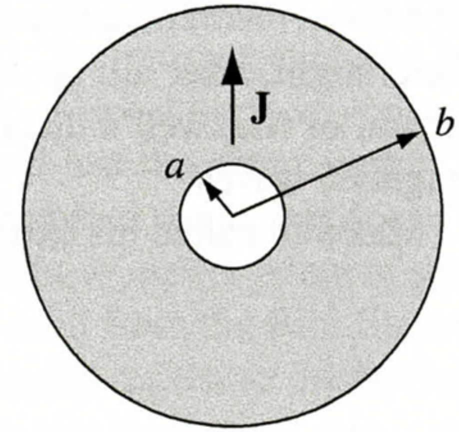
$$= \frac{\mu_0 \varepsilon_0}{\varepsilon_0} \cdot \frac{I}{A} \overbrace{\int da}^{A}$$

$$= \mu_0 I$$

COMES FROM DISPLACEMENT CURRENT

EXAMPLE: TWO CONCENTRIC SPHERES

INNER, RADIUS a CARRIES A VARYING CHARGE $+Q(t)$. OUTER, RADIUS b, CARRIES A CHARGE $-Q(t)$. THE SPACE BETWEEN IS AN OHMIC MATERIAL, VOLUME CONDUCTIVITY σ



THERE IS A RADIAL (OHM'S LAW) CURRENT

$$\bar{J} = \sigma \bar{E} = \sigma \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r} \qquad (1)$$

$$\bar{I} = -\frac{dQ}{dt} = \int \bar{J} \cdot d\bar{a}$$

$$= \frac{\sigma}{4\pi\varepsilon_0} \cdot Q \int \frac{\hat{r}}{r^2}$$

$$= \frac{\sigma}{4\pi\varepsilon_0} \frac{Q}{R^2} \int r^2 \sin\theta \, d\theta \, d\phi$$

$$\bar{I} = \frac{\sigma Q}{4\pi\varepsilon_0}\frac{1}{R^2}\cdot R^2\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi}d\phi = \frac{\sigma}{4\pi\varepsilon_0}\cdot 4\pi$$

$$\bar{I} = \frac{\sigma Q}{\varepsilon_0}$$

THIS GEOMETRY IS RADIALLY SYMMETRIC $\rightarrow \bar{B} = 0$

$\bar{B}$ WOULD BE RADIAL, $\bar{\nabla}\cdot\bar{B} = 0 \rightarrow \oint \bar{B}\cdot da = \underbrace{B\, 4\pi r^2}_{=0}$

$Q, \bar{E}, \bar{J}$ ARE ALL FUNCTIONS OF TIME $\therefore\ \bar{B} = 0$

AMPÈRE & BIOT-SAVART DO NOT APPLY

$\rightarrow$ USE DISPLACEMENT CURRET

DISPLACEMENT CURRENT

$$J_d = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{1}{4\pi r^2} \frac{dQ}{dt} \cdot \hat{r}$$

$$= \frac{-\sigma Q}{\varepsilon_0}$$

$$J_d = -\sigma \frac{Q}{4\pi\varepsilon_0 r^2} \cdot \hat{r} \qquad (2)$$

$$(1) - (2) = 0 = J_{OHM} - J_{DISP} \rightarrow \text{NO CURRENT}$$

NO $\vec{B}$

ALSO

$$\vec{\nabla} \times \vec{B} = \mu_0 J - \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} = 0$$

WE NOW HAVE

$$\text{(i)} \quad \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

$$\text{(ii)} \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\text{(iii)} \quad \vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$$

$$\text{(iv)} \quad \vec{\nabla}\times\vec{B} = \mu_0 J + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$+ \quad \vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

THIS IS ALL OF CLASSICAL ELECTROMAGNETISM

EVEN CONTINUITY EQUATION IS IN THERE

$$\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{0} = \mu_0\,\vec{\nabla}\cdot\vec{J} + \mu_0\varepsilon_0\,\vec{\nabla}\partial\vec{E}/\partial t$$

$$\mu_0\vec{\nabla}\cdot J = -\mu_0\varepsilon_0\cdot\frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial t} \Rightarrow \vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$$

USUAL WAY OF WRITING MAWELL SEEMS TO IMPLY

$\bar{E}$ PRODUCED BY ρ OR $\partial \bar{B}/\partial t$

$\bar{B}$ PRODUCED BY $\bar{J}$ OR $\partial \bar{E}/\partial t$.

BUT $\partial \bar{B}/\partial t$, $\partial \bar{E}/\partial t$ THEMSELVES PRODUCED BY ρ, $\bar{J}$

MORE LOGICAL (?)

$$\bar{\nabla}\cdot\bar{E} = \rho/\varepsilon_0 \quad , \quad \bar{\nabla}\times\bar{E} + \frac{\partial \bar{B}}{\partial t} = 0$$

$$\bar{\nabla}\cdot\bar{B} = 0 \qquad \bar{\nabla}\times\bar{B} - \mu_0\varepsilon_0\frac{\partial \bar{E}}{\partial t} = \mu_0\bar{J}$$

FIELDS PRODUCED BY SOURCES

ALL OF E & M COMES FROM CURRENTS & CHARGES

MAXWELL $\longrightarrow$ SOURCES PRODUCE FIELDS

LORENTZ $\longrightarrow$ HOW FIELDS EFFECT CHARGES

MAGNETIC CHARGES?

STRIKING SYMMETRY IN MAXWELL

$\rho \to 0$
$\bar{J} \to 0$

① $\bar{\nabla}\cdot\bar{E} = 0$ $\qquad \bar{\nabla}\times\bar{E} = -\partial\bar{B}/\partial t$

② $\bar{\nabla}\cdot\bar{B} = 0$ $\qquad \bar{\nabla}\times\bar{B} = \mu_0\varepsilon_0\,\partial\bar{E}/\partial t$

IF $\bar{E} \to \bar{B}$, $\bar{B} \to -\mu_0\varepsilon_0\bar{E}$ $\qquad$ ① → ②

SYMMETRY SPOILED BY $\bar{\nabla}\cdot\bar{E} = \rho/\varepsilon_0$, $\bar{\nabla}\times\bar{B} = \mu_0\bar{J}$

WHAT IF HAD

$$\bar{\nabla}\cdot\bar{E} = \rho_e/\varepsilon_0 \qquad \bar{\nabla}\times\bar{E} = -\mu_0 J_m - \partial B/\partial t$$

$$\bar{\nabla}\cdot\bar{B} = \mu_0\rho_m \qquad \bar{\nabla}\times\bar{B} = \mu_0\bar{J}_e + \mu_0\varepsilon_0\,\partial\bar{E}/\partial t$$

$\rho_e \leftrightarrow \rho_m \longrightarrow$ BOTH CHARGES CONSERVED

$$\bar{\nabla}\cdot\bar{J}_m = -\frac{\partial\rho_m}{\partial t} \; ; \qquad \bar{\nabla}\cdot\bar{J}_e = \frac{-\partial\rho_e}{\partial t}$$

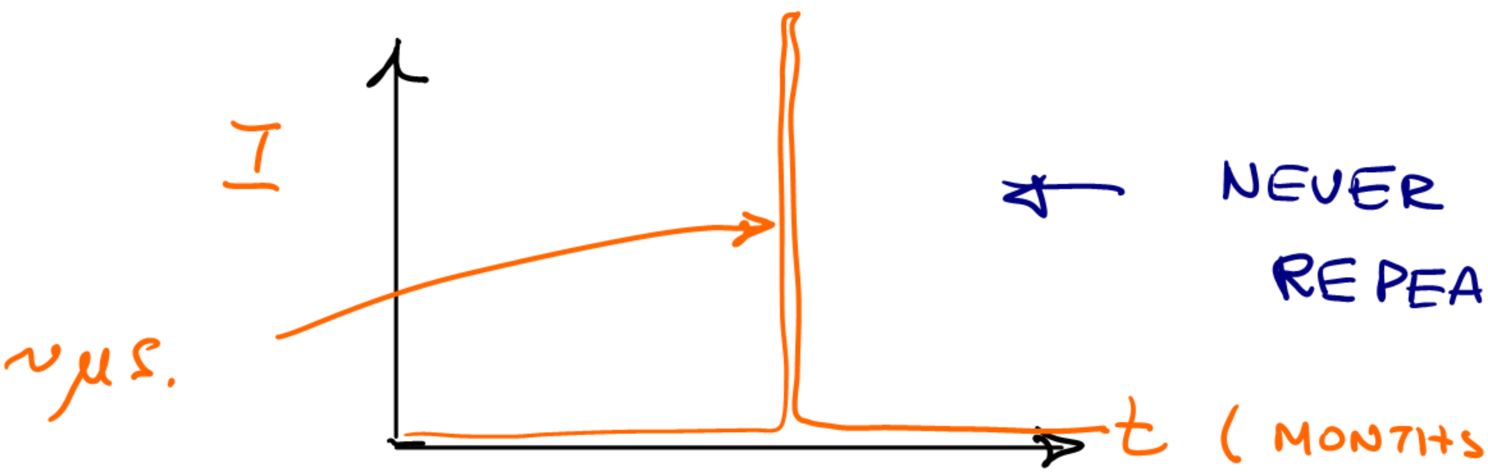
THERE APPEAR TO BE NO MAGNETIC MONOPOLES (CHARGES) IN OUR UNIVERSE

→ DIRAC SHOWED MAGNTIC MONOPOLES → ELECTRIC CHARGE QUANTIZATION

→ EXPECT MONOPOLES PRODUCED IN BIG BANG

↳ INFLATION SPREAD THEM OUT ~ 1 IN OUR UNIVERSE

ONE EXPERIMENT CLAIMED TO HAVE OBSERVED

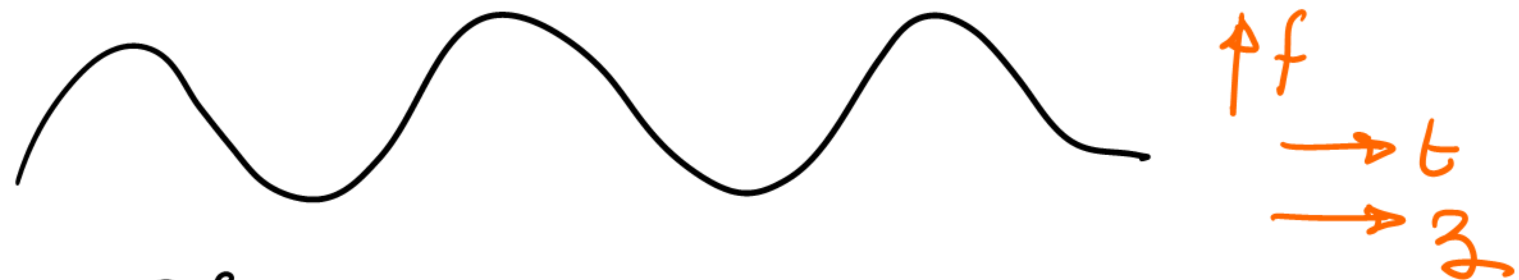


← NEVER REPEATED

ELECTROMAGNETIC WAVES

1-D WAVES

STRING



$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad f = f(z,t)$$

VELOCITY

$$f(z,t) = A\cos\left[\underline{k(z - vt)} + \delta\right]$$

PHASE

AMPLITUDE

$$v = \sqrt{\frac{T}{m}}$$

TENSION

MASS PER UNIT LENGTH

$$f(z,t) = \mathrm{Re}\left[A e^{i(kz - \omega t + \delta)}\right]$$

VELOCITY

FREQUENCY

MAXWELL IN FREE SPACE NO CHARGE OR CURRENT

(i) $\bar{\nabla} \cdot \underline{E} = 0$

(ii) $\bar{\nabla} \cdot \bar{B} = 0$

iii) $\bar{\nabla} \times \bar{E} = -\partial \bar{B} / \partial t$

iv) $\bar{\nabla} \times \bar{B} = \mu_0 \varepsilon_0 \partial \bar{E} / \partial t$

COUPLED 1ST ORDER DIFFERENTIAL EQUATIONS

DECOUPLE BY TAKING CURL of (iii) (iv)

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) = \bar{\nabla}(\bar{\nabla} \cdot \bar{E}) - \bar{\nabla}^2 \bar{E} = \bar{\nabla} \times \left(-\frac{\partial \bar{B}}{\partial t} \right)$$

$$= -\frac{\partial}{\partial t}(\bar{\nabla} \times \bar{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

SINCE $\bar{\nabla} \cdot \bar{E} = 0$

$$\nabla^2 \bar{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{B}) = \bar{\nabla}(\bar{\nabla}\cdot\bar{B}) - \nabla^2\bar{B} = \bar{\nabla}\times\left(\mu_0\varepsilon_0\frac{\partial\bar{E}}{\partial t}\right)$$

$$= \mu_0\varepsilon_0\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{E}) = -\mu_0\varepsilon_0\,\partial^2\bar{B}/\partial t^2$$

$$\nabla^2\bar{B} = \mu_0\varepsilon_0\,\partial^2\bar{B}/\partial t^2$$

AND HAD

$$\nabla^2\bar{E} = \mu_0\varepsilon_0\,\partial^2\bar{E}/\partial t^2$$

3-D WAVE EQUATION $\frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \Rightarrow \nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

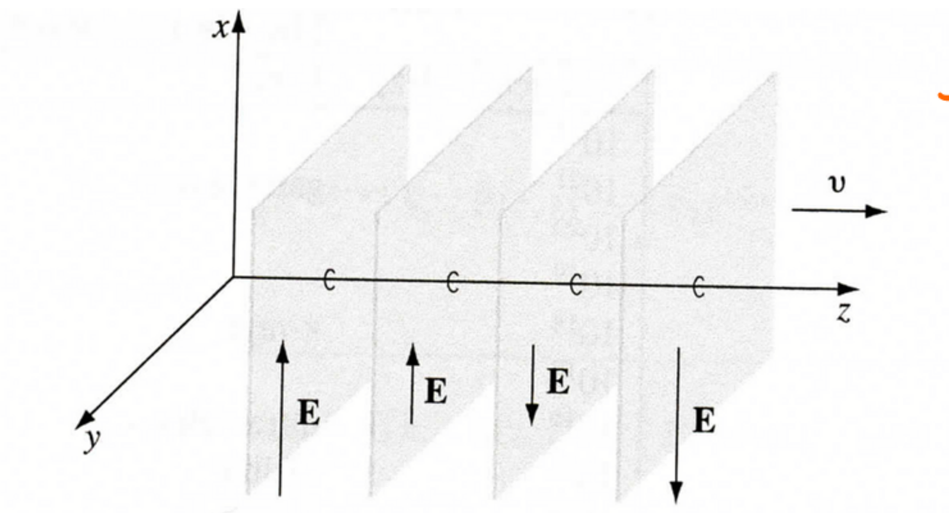
WAVE EQUATIONS FOR $\bar{E}$ & $\bar{B}$

$$v = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 3\times10^8\ m/sec$$

WITHOUT $\varepsilon_0\mu_0\frac{\partial\bar{E}}{\partial t}$ $\longrightarrow$ NO WAVES

$\varepsilon_0\,\partial\bar{E}/\partial t$ $\longrightarrow$ DISPLACEMENT CURRENT

MONOCHROMATIC PLANE WAVES



$\rightarrow v_z$

NO x, y DEPENDENCE

FIXED FREQUENCY ω

GENERALLY $\quad f(z,t) = \mathrm{Re}\left[A e^{i(kz - \omega t + \delta)}\right]$

IN OUR CASE

COMPLEX $\rightarrow$

$$\tilde{E} = \tilde{E}_0 e^{i(kz - \omega t)}$$

$$\tilde{B} = \tilde{B}_0 e^{i(kz - \omega t)}$$

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$$

$$\tilde{\mathbf{B}} = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)}$$

EVERY SOLUTION of MAXWELL IS A WAVE

EVERY WAVE IS NOT A SOLUTION OF MAXWELL

MAXWELL → $\vec{\nabla}\cdot\vec{E} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0$

$$(\tilde{\mathbf{E}}_0)_z = (\tilde{\mathbf{B}}_0)_z = 0$$

ELECTROMAGNETIC WAVES ARE TRANSVERSE

NO COMPONENT OF FIELDS IN DIRECTION OF MOTION

ALSO $\quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\quad$ E, $\vec{B}$ AMPLITUDES ARE RELATED

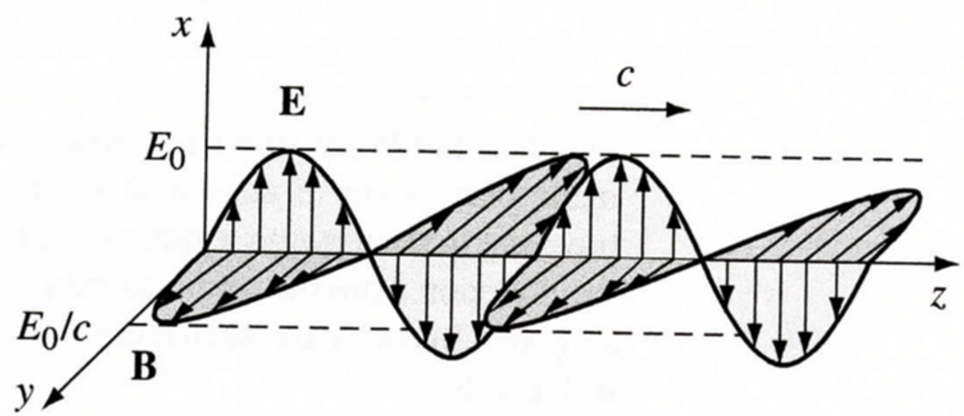
$$-k(\tilde{\vec{E}}_0)_y = -\omega(\tilde{\vec{B}}_0)_x, \quad k(\tilde{\vec{E}}_0)_x = \omega(\tilde{\vec{B}}_0)_y$$

$$\tilde{\vec{B}}_0 = \frac{k}{\omega}(\hat{z} \times \tilde{\vec{E}}_0)$$

$\vec{E}$ & $\vec{B}$ IN PHASE & PERPENDICULAR

AMPLITUDES RELATED $\quad B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0$

$$\mathbf{E}(z,t) = E_0 \cos(kz - \omega t + \delta)\,\hat{\mathbf{x}}, \quad \mathbf{B}(z,t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\,\hat{\mathbf{y}}. \tag{9.48}$$



The Electromagnetic Spectrum		
Frequency (Hz)	Type	Wavelength (m)
10^{22}		10^{-13}
10^{21}	gamma rays	10^{-12}
10^{20}		10^{-11}
10^{19}		10^{-10}
10^{18}	x-rays	10^{-9}
10^{17}		10^{-8}
10^{16}	ultraviolet	10^{-7}
10^{15}	visible	10^{-6}
10^{14}	infrared	10^{-5}
10^{13}		10^{-4}
10^{12}		10^{-3}
10^{11}		10^{-2}
10^{10}	microwave	10^{-1}
10^{9}		1
10^{8}	TV, FM	10
10^{7}		10^{2}
10^{6}	AM	10^{3}
10^{5}		10^{4}
10^{4}	RF	10^{5}
10^{3}		10^{6}

The Visible Range		
Frequency (Hz)	Color	Wavelength (m)
1.0×10^{15}	near ultraviolet	3.0×10^{-7}
7.5×10^{14}	shortest visible blue	4.0×10^{-7}
6.5×10^{14}	blue	4.6×10^{-7}
5.6×10^{14}	green	5.4×10^{-7}
5.1×10^{14}	yellow	5.9×10^{-7}
4.9×10^{14}	orange	6.1×10^{-7}
3.9×10^{14}	longest visible red	7.6×10^{-7}
3.0×10^{14}	near infrared	1.0×10^{-6}

NOW YOU HAVE TOOLS TO STUDY MANY FIELDS OF PHYSICS

→ OPTICS

→ PLASMAS

→ ELECTRICITY GENERATION

→ EM WAVES IN TECHNOLOGY

→ EM WAVES IN ASTRO PHYSICS

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