VECTOR OPERATORS — DIV, GRAD, CURL.

FAMILIAR WITH $df = \left(\frac{df}{dx}\right) \cdot dx$ HOW RAPIDLY A FN VARIES WRT x

$$dT = \left(\frac{\partial T}{\partial x}\right) \cdot dx + \left(\frac{\partial T}{\partial y}\right) \cdot dy + \left(\frac{\partial T}{\partial z}\right) \cdot dz$$

SEVERAL VARIABLES

IF WE THINK ABOUT SCALAR PRODUCTS — LOOKS LIKE

$$dT = \underbrace{\left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)}_{\text{VARIATION OF } T} \cdot \underbrace{\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)}_{\text{LINE INCREMENT}}$$

$$= \vec{\nabla} T \cdot d\vec{l} \rightarrow \text{SCALAR}$$

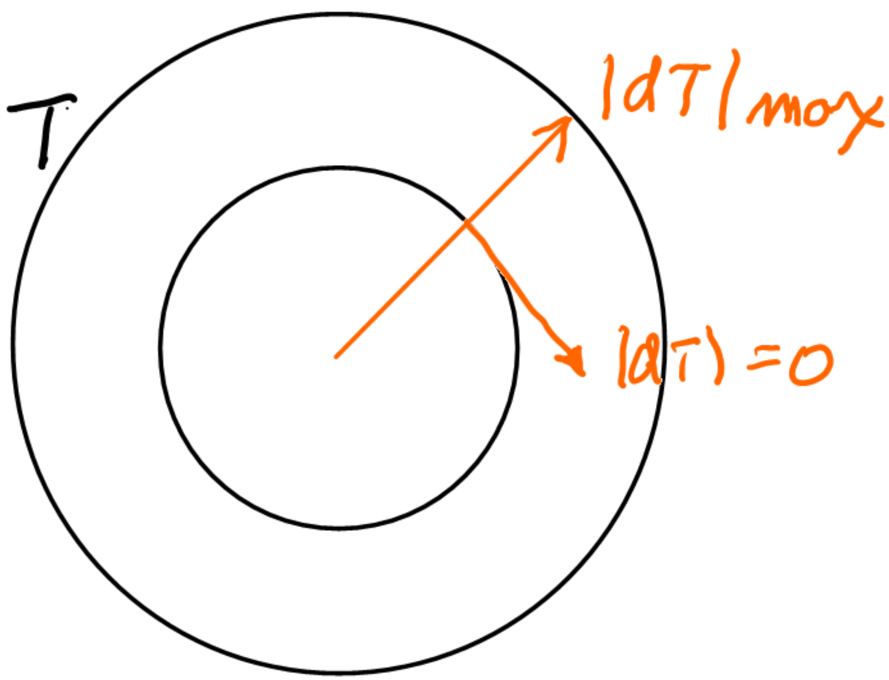
$\vec{\nabla} T$ IS GRADIENT OF T

$$\vec{\nabla} T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

$\bar{\nabla} T$ → GRADIENT — VECTOR (BECAUSE OF UNIT VECTORS)

$$dT = \bar{\nabla} T \cdot d\bar{l} = |\bar{\nabla} T| \, |d\bar{l}| \cos\theta$$

- MAXIMUM RATE OF CHANGE OF T → $\cos\theta = 1$
- GRADIENT $\bar{\nabla} T$ POINTS IN DIRECTION OF MAXIMUM INCREASE IN T
- MAGNITUDE $|\bar{\nabla} T|$ GIVES SLOPE (RATE OF INCREASE) ALONG MAXIMUM DIRECTION.



IF $\bar{\nabla} T = 0$ AT (x, y, z)

$dT = 0$ FOR SMALL DIPLACEMENTS

MAXIMUM, MINIMUM OR SADDLE POINT.

STATIONARY POINT

DEL OPERATOR $\vec{\nabla}$

LOOKS LIKE AN ANCIENT ASSYRIAN HARP → "NABLA" :-)

GRADIENT LOOKS LIKE $\overline{\text{VECTOR}}$ × SCALAR

$$\vec{\nabla} T = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) T$$

$\vec{\nabla}$ IS NOT A VECTOR — IT IS A VECTOR <u>OPERATOR</u>

IT ACTS ON A FUNCTION

3 WAYS $\vec{\nabla}$ CAN ACT

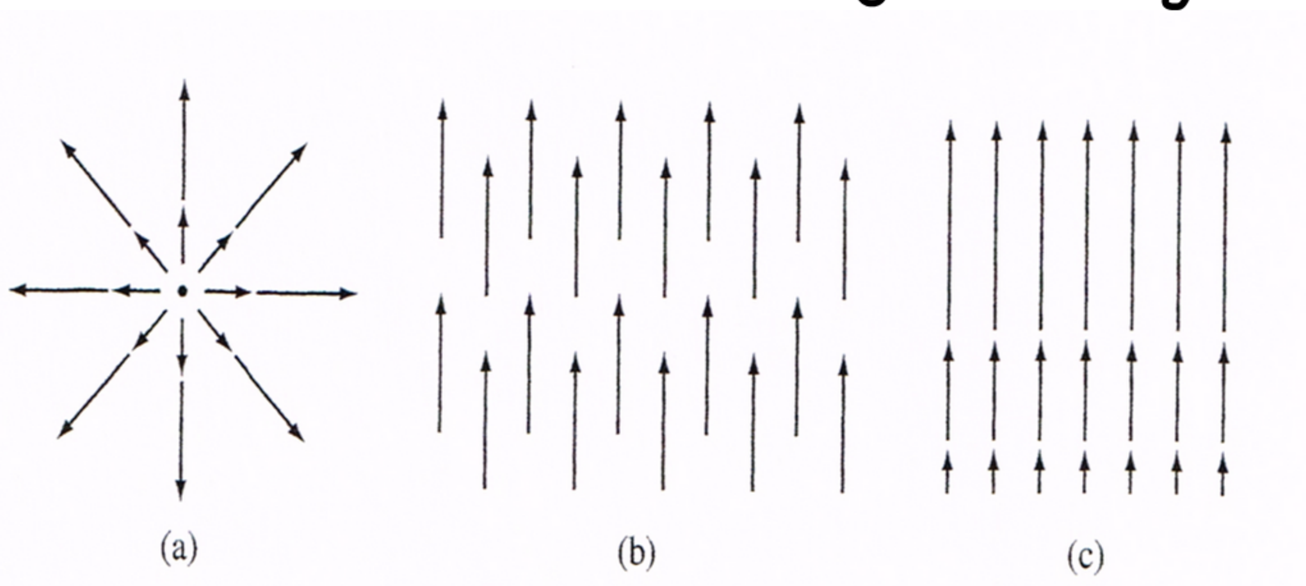
- ON A SCALAR FUNCTION T — $\vec{\nabla} T$ GRADIENT
- DOT PRODUCT ON VECTOR FN $\vec{v}$ — $\vec{\nabla} \cdot \vec{v}$ DIVERGENCE
- VECTOR PRODUCT ON VECTOR FN $\vec{v}$ — $\vec{\nabla} \times \vec{v}$ CURL

DIVERGENCE $\bar{\nabla} \cdot \bar{v}$

$$\bar{\nabla} \cdot \bar{v} = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left(v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \right)$$

$$= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

← SCALAR $\bar{\nabla} \cdot \bar{v}$



$\bar{\nabla} \cdot \bar{v}$ +ve

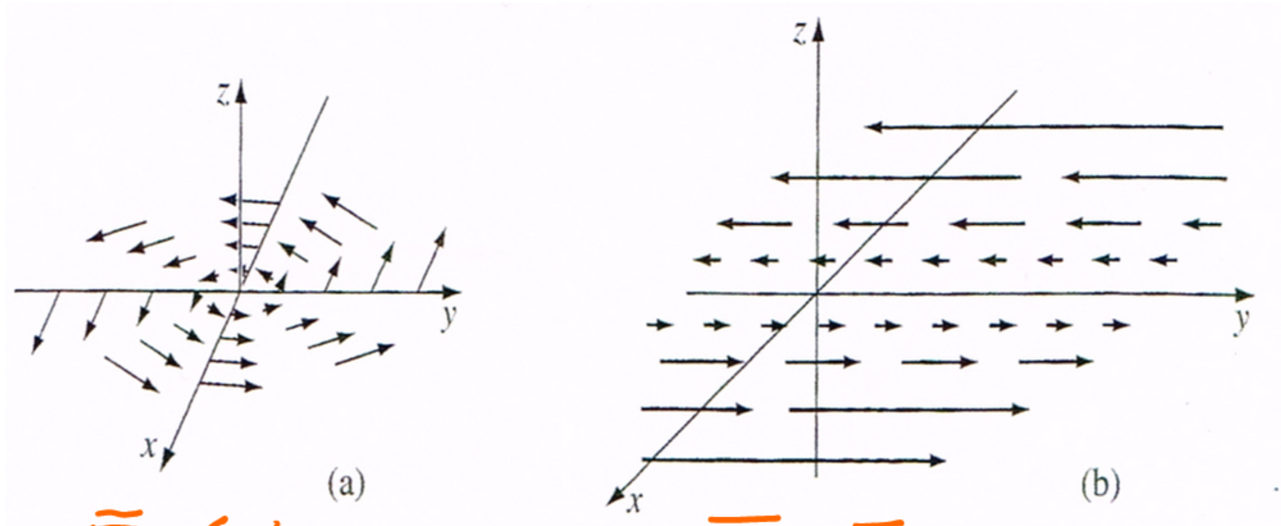
$\bar{\nabla} \cdot \bar{v} = 0$

$\bar{\nabla} \cdot \bar{v} = 0$

A MEASURE OF HOW MUCH A VECTOR FUNCTION DIVERGES FROM A POINT

CURL $\bar{\nabla} \times \bar{v}$

$$\bar{\nabla} \times \bar{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$



$\bar{\nabla} \times V \neq 0$ $\qquad$ $\bar{\nabla} \times \bar{V} = 0$

$$= \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{y}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$= \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

VECTOR FUNCTION – A MEASURE OF HOW MUCH A VECTOR FIELD ROTATES AROUND A POINT

$\nabla \rightarrow$ DERIVATIVES $\rightarrow$ BEHAVES LIKE "NORMAL" CALCULUS

PRODUCT RULES

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{\partial g}{\partial x}$$

$$\frac{d}{dx}(kf) = k\frac{df}{dx}$$

$$\frac{d}{dx}(f \cdot g) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

SIMILARLY $\longrightarrow$

$$\bar{\nabla}(f+g) = \bar{\nabla} f + \bar{\nabla} g \; ; \; \bar{\nabla}\cdot(\bar{A}+\bar{B}) = (\bar{\nabla}\cdot\bar{A}) + (\bar{\nabla}\cdot\bar{B})$$

$$\bar{\nabla}\times(\bar{A}+\bar{B}) = (\bar{\nabla}\times\bar{A}) + (\bar{\nabla}\times\bar{B})$$

$$\bar{\nabla}(kf) = k\bar{\nabla} f \, , \; \bar{\nabla}\cdot(k\bar{A}) = k\bar{\nabla}\cdot\bar{A} \, , \; \bar{\nabla}\times(k\bar{A}) = k(\bar{\nabla}\times\bar{A})$$

PRODUCT RULES A BIT MORE COMPLEX

- 2 WAYS TO CONSTRUCT A SCALAR

$$\left.\begin{array}{l} fg \\ \bar{A}\cdot\bar{B} \\ \text{- 2 WAYS TO MAKE A VECTOR} \\ f\bar{A} \\ \bar{A}\times\bar{B} \end{array}\right\} \text{6 PRODUCT RULES}$$

PRODUCT RULES:

TWO FOR GRADIENTS:

$$\nabla(fg) = f\bar{\nabla}g + g\bar{\nabla}f$$

$$\bar{\nabla}(\bar{A}\cdot\bar{B}) = \bar{A}\times(\bar{\nabla}\times\bar{B}) + \bar{B}\times(\bar{\nabla}\times\bar{A}) + (\bar{A}\cdot\bar{\nabla})\bar{B} + (\bar{B}\cdot\bar{\nabla})\bar{A}$$

TWO FOR DIVERGENCES:

$$\bar{\nabla}\cdot(f\bar{A}) = f(\bar{\nabla}\cdot\bar{A}) + \bar{A}\cdot(\bar{\nabla}f)$$

$$\bar{\nabla}\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\bar{\nabla}\times\bar{A}) - \bar{A}\cdot(\bar{\nabla}\times\bar{B})$$

TWO FOR CURLS:

$$\bar{\nabla}\times(f\bar{A}) = f(\bar{\nabla}\times\bar{A}) - \bar{A}\times(\nabla f)$$

$$\bar{\nabla}\times(\bar{A}\times\bar{B}) = (\bar{B}\cdot\bar{\nabla})\bar{A} - (\bar{A}\cdot\bar{\nabla})\bar{B} + \bar{A}(\bar{\nabla}\cdot\bar{B}) - \bar{B}(\bar{\nabla}\cdot\bar{A})$$

PROOFS OF PREVIOUS (SELF EVIDENT)

JUST COME FROM RULES FOR ORDINARY DERIVATIVES

eg $\bar{\nabla}\cdot(f\bar{A}) = \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$

$= \left(\frac{\partial f}{\partial x}A_x + f\frac{\partial A_x}{\partial x}\right) + \left(\frac{\partial f}{\partial y}A_y + f\frac{\partial A_y}{\partial y}\right) + \left(\frac{\partial f}{\partial z}A_z + f\frac{\partial A_z}{\partial z}\right)$

$= (\bar{\nabla}f)\cdot\bar{A} + f(\bar{\nabla}\cdot\bar{A})$ ←

ALSO HAVE 3 QUOTIENT RULES FOR

$\bar{\nabla}(f/g)$, $\bar{\nabla}\left(\frac{\bar{A}}{g}\right)$, $\bar{\nabla}\times\left(\frac{\bar{A}}{g}\right)$

→ SEE TEXT

JUST LIKE NORMAL CALCULUS CAN TAKE DERIVATIVES OF DERIVATIVES $\rightarrow$ 2nd DERIVATIVES

1) DIV GRAD $\quad \bar{\nabla} \cdot (\bar{\nabla} T) \quad \bar{\nabla} \cdot (\text{VECTOR})$

2) CURL GRAD $\quad \bar{\nabla} \times (\bar{\nabla} T) \quad \bar{\nabla} \times (\text{VECTOR})$

3) GRAD DIV $\quad \bar{\nabla} (\bar{\nabla} \cdot \bar{v}) \quad \bar{\nabla} (\text{SCALAR})$

CURL $\bar{\nabla} \times \bar{\nabla}$ IS A VECTOR $\rightarrow$ 2 POSSIBILITIES

4) DIV CURL $\quad \bar{\nabla} \cdot (\bar{\nabla} \times \bar{v})$

5) CURL CURL $\quad \bar{\nabla} \times (\bar{\nabla} \times \bar{v})$

THESE ARE ALL POSSIBILITIES

$\hookrightarrow$ NOT ALL OF THEM ARE USEFUL

LOOK AT THEM IN DETAIL $\longrightarrow$

1) $\bar{\nabla}\cdot(\bar{\nabla}T) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}\right)$

$\hat{x}\cdot\hat{x} = 1, \quad \hat{x}\cdot\hat{y} = \hat{x}\cdot\hat{z} = 0$

$$\bar{\nabla}\cdot(\bar{\nabla}T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T \leftarrow \text{LAPLACIAN}$$

SCALAR

CAN EXTEND THE MEANING OF ∇^2 TO A VECTOR

$\nabla^2\bar{v}$ – VECTOR QUANTITY WHOSE x-COMP IS LAPLACIAN OF v_x etc.

$$\nabla^2\bar{v} \equiv (\nabla^2 v_x)\hat{x} + (\nabla^2 v_y)\hat{y} + (\nabla^2 v_z)\hat{z}$$

2) CURL GRAD $= 0$ ALWAYS

$\bar{\nabla} \times (\bar{\nabla} T) = 0$ ALWAYS

$$\bar{\nabla} T = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}$$

$$\bar{\nabla} \times (\bar{\nabla} T) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial T/\partial x & \partial T/\partial y & \partial T/\partial z \end{vmatrix}$$

$$= \hat{x}\left(\cancel{\frac{\partial^2 T}{\partial y \partial z}} - \cancel{\frac{\partial^2 T}{\partial z \partial y}}\right) - \hat{y}\left(\cancel{\frac{\partial^2 T}{\partial x \partial z}} - \cancel{\frac{\partial^2 T}{\partial z \partial x}}\right)$$

$$+ \hat{z}\left(\cancel{\frac{\partial^2 T}{\partial x \partial y}} - \cancel{\frac{\partial^2 T}{\partial y \partial x}}\right) = 0$$

LOOK AT $(\vec{\nabla} T)\times(\vec{\nabla} S)$

$$\vec{\nabla} T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

$$\vec{\nabla} S = \frac{\partial S}{\partial x}\hat{x} + \frac{\partial S}{\partial y}\hat{y} + \frac{\partial S}{\partial z}\hat{z}$$

$$(\vec{\nabla} T)\times(\vec{\nabla} S) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial T/\partial x & \partial T/\partial y & \partial T/\partial z \\ \partial S/\partial x & \partial S/\partial y & dS/dz \end{vmatrix}$$

$$= \hat{x}\underbrace{\left(\frac{\partial T}{\partial y}\frac{\partial S}{\partial z} - \frac{\partial S}{\partial y}\frac{\partial T}{\partial z}\right)}_{\neq 0} + \text{-----}$$

3) $\bar{\nabla}(\bar{\nabla}\cdot\bar{v})$ RARELY OCCURS

$\bar{\nabla}(\bar{\nabla}\cdot\bar{v})$ NOT SAME AS LAPLACIAN

$$\nabla^2\bar{v} = (\bar{\nabla}\cdot\bar{\nabla})\,\bar{v} \neq \bar{\nabla}(\bar{\nabla}\cdot\bar{v})$$

BRACKETS !

4) DIV CURL $\bar{\nabla}\cdot(\bar{\nabla}\times\bar{v}) = 0$ ALWAYS

$$\bar{\nabla}\times\bar{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{y}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)$$

$$+ \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$\bar{\nabla}\cdot(\bar{\nabla}\times\bar{v}) = \frac{\partial^2}{\partial x\partial y}v_z - \frac{\partial^2 v_y}{\partial x\partial z} - \frac{\partial^2 v_z}{\partial y\partial x} + \cancel{\frac{\partial^2 v_x}{\partial y\partial z}}$$

$$+\frac{\partial^2 v_y}{\partial z\partial x} - \cancel{\frac{\partial^2 v_x}{\partial z\partial y}} \quad \text{etc} = 0$$

5) $\bar{\nabla}\times(\bar{\nabla}\times\bar{V})$

$$\bar{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \quad ;$$

$$\bar{\nabla}\times\bar{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{y}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)$$

$$+\hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} & \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{vmatrix}$$

LET'S JUST DO x-COMPONENT ← LIFE IS SHORT!

$$= \hat{x}\left(\frac{\partial^2 v_y}{\partial y \partial x} - \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_z}{\partial z \partial x} \right) + \cdots \qquad (1)$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{v}) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

$$= \hat{x}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial x \partial y} + \frac{\partial^2 v_z}{\partial x \partial z} \right) + \hat{y} \cdots \qquad (2)$$

$$(-\nabla^2 \bar{v}) = -\hat{x}(\nabla^2 v_x) - \hat{y} \cdots\cdots$$

$$= -\hat{x}\frac{\partial^2 v_x}{\partial x^2} - \hat{y} \cdots\cdots \quad (3)$$

$$(2) - (3) = \hat{x}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial x \partial y} + \frac{\partial^2 v_z}{\partial x \partial z} - \frac{\partial^2 v_x}{\partial x^2}\right)$$

CAN WRITE (1) AS :

$$\hat{x}\left(\underbrace{\frac{\partial^2 v_y}{\partial y \partial x} + \frac{\partial^2 v_z}{\partial z \partial x} + \frac{\partial^2 v_x}{\partial x^2}}_{\bar{\nabla}(\bar{\nabla}\cdot\bar{v})_x} \underbrace{- \frac{\partial^2 v_x}{\partial x^2} - \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial z^2}}_{(-\nabla^2 \bar{v})_x}\right)$$

VECTOR QUANTITY WHOSE
x COMPONENT IS LAPLACIAN
OF v_x