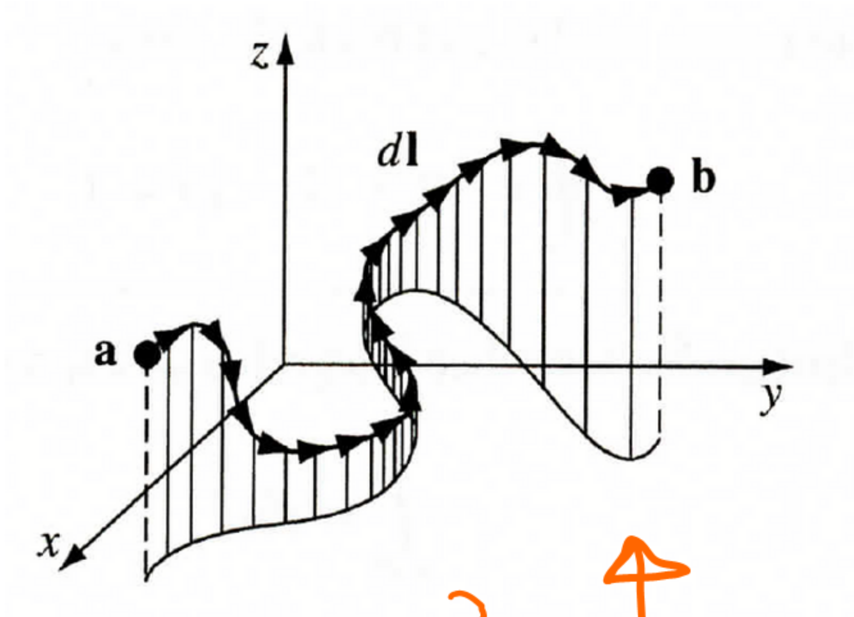
WE STILL HAVE SOME MATHS TO "REVIEW"

$\longrightarrow$ ON TO INTEGRALS OF VARIOUS KINDS

LINE INTEGRALS, SURFACE INTEGRALS & VOLUME INTEGRALS

LINE INTEGRALS:



YOU ARE FAMILIAR WITH

$$\int_a^b f\,dx$$

A LINE INTEGRAL INVOLVES VECTORS

$$\int_{\bar{a}}^{\bar{b}} \bar{v}\cdot d\bar{\ell}$$

INFINITESIMAL DISPLACEMENT

VECTOR FUNCTION

$\int$ IS CARRIED OUT ALONG PATH $\bar{a} \rightarrow \bar{b}$

FOR CLOSED PATH $\oint \bar{v}\cdot d\bar{\ell}$

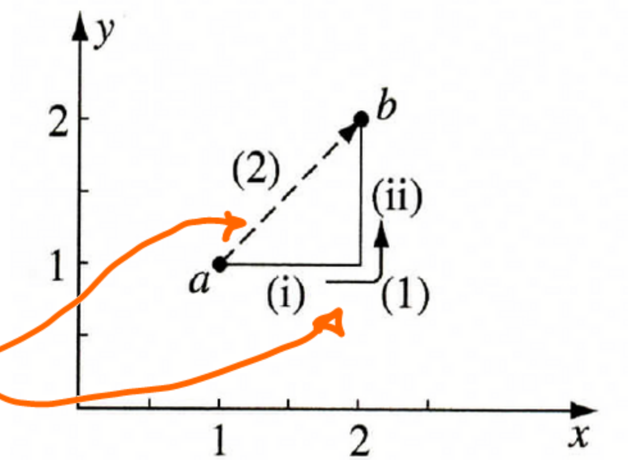
AT EACH POINT ON PATH TAKE DOT PRODUCT

$\bar{F}$: WORK $= \int \bar{F}\cdot d\bar{\ell}$

GENERALLY LINE INTEGRAL DEPENDS ON PATH

IF $\int$ IS INDEPENDENT OF PATH, DEPENDS ONLY ON $\bar{a}$ $\bar{b}$ $\rightarrow$ eg CONSERVATIVE FORCE.

LOOK AT SIMPLE EXAMPLE



CALC LINE INTEGRAL OF

$$\bar{V} = y^2 \hat{x} + 2y(y+1)\hat{y}$$

ALONG TWO DIFFERENT PATHS

$$d\bar{\ell} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} \rightarrow 0$$

ALONG HORIZONTAL PATH:

$$dy = dz = 0$$

$$\bar{v} \cdot d\bar{\ell} = \underset{\downarrow \atop y^2}{V_x}\, dx\,\hat{x} + \underbrace{v_y\, dy\,\hat{y} + v_z\, dz\,\hat{z}}_{0}$$

$$\bar{v}\cdot d\bar{l} = y^2\,dx = dx \quad = 1$$

$$\int \bar{v}\cdot d\bar{l} = \int_1^2 dx = 1$$

ALONG VERTICAL SEGMENT $dx = dz = 0$

$$d\bar{l} = dy\,\hat{y}\;;\quad x = 2\;;\quad \bar{v} = 2x\,(y+1)\,\hat{y}$$

$$\bar{v}\cdot d\bar{l} = 4(y+1)\,dy\;;\quad \int \bar{v}\cdot d\bar{l} = 4\int_1^2 (y+1)\,dy$$

$$= 4\left[\frac{y^2}{2} + y\right]_1^2 = 10$$

So ALONG PATH 1 $\int \bar{v}\cdot d\bar{l} = 1 + 10 = 11$:)

ALONG PATH 2 $x = y$, $dx = dy$, $dz = 0$

$$d\bar{l} = dx\,\hat{x} + dy\,\hat{y}, \quad \bar{v} = x^2\,\hat{x} + 2x(x+1)\,\hat{y}$$

$$\bar{v}\cdot d\bar{\ell} = x^2\,dx + 2x(x+1)\,dy$$
$$= (x^2 + 2x^2 + 2x)\,dx$$
$$= (3x^2 + 2x)\,dx$$

$dx = dy$

$$\int \bar{v}\cdot d\bar{\ell} = \int (3x^2 + 2x)\,dx = \left[\frac{3x^2}{3} + \frac{2x^2}{2}\right]_1^2 = 10$$

→ NOTICE WE EXPRESSED EVERYTHING IN ONE VBL $= x$

$$\oint \bar{v}\cdot d\bar{\ell} = \int_{\text{PATH 1}} - \underbrace{\int_{\text{PATH 2}}}_{\text{BACK ALONG PATH 2}}$$

$$\int \bar{v}\cdot d\bar{\ell} = 11 - 10 = 1.$$

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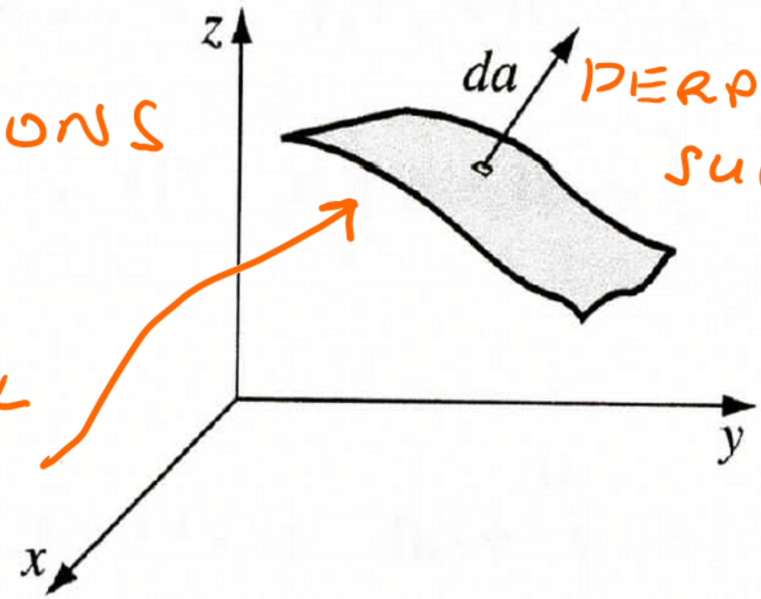
SURFACE INTEGRALS:

EXTENSION TO 2-DIMENSIONS

$$\int \bar{v} \cdot d\bar{a}$$

INFINITESIMAL PATCH OF AREA



PERP TO SURFACE

- 2 POSSIBLE PERPENDICULAR DIRECTIONS
  - AMBIGUITY
- FOR CLOSED SURFACE $\oint \bar{v} \cdot d\bar{a}$
- CONVENTIONALLY – OUTWARD +ve

IF $\bar{v}$ IS FLOW – MASS / UNIT AREA / TIME

$\int \bar{v} \cdot d\bar{a}$ TOTAL MASS / UNIT TIME THROUGH AREA

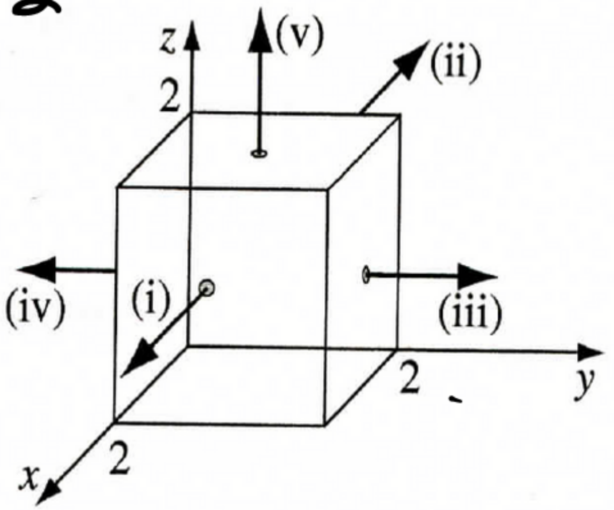
$\longrightarrow$ FLUX

IN GENERAL VALUE OF $\int \bar{V} \cdot d\bar{a}$ DEPENDS ON SURFACE

→ CLASS OF FUNCTIONS WHICH DO NOT DEPEND ON SURFACE, BUT ONLY BOUNDARY

→ LOOK AT SIMPLE EXAMPLE

SURFACE $\int$ OF $\bar{V} = 2xz\,\hat{x} + (x+2)\,\hat{y} + y(z^2-3)\,\hat{z}$



(i) $x=2$, $d\bar{a} = dy\,dz\,\hat{x}$

$$\bar{V} \cdot d\bar{a} = 2xz\,dy\,dz = 4z\,dy\,dz$$

$$\int \bar{V} \cdot d\bar{a} = 4\int_0^2 dy \int_0^2 z\,dz = 16$$

(ii) $x=0$, $d\bar{a} = -dz\,dy\,\hat{x}$, $\bar{V}\cdot d\bar{a} = -2\underbrace{x}_{=0}z\,dz\,dy = 0$

iii) $y=2$, $d\bar{a} = dx\,dz\,\hat{y}$, $\bar{V}\cdot d\bar{a} = (x+2)\,dx\,dz$

$$\int \bar{V}\cdot d\bar{a} = \int_0^2 (x+2)\,dx \int_0^2 dz = 12$$

iv) $y=0$, $d\bar{a} = -dz\,dx\,\hat{y}$

$$\bar{V}\cdot d\bar{a} = -(x+2)\,dz\,dx$$ ONLY $\hat{y} \neq 0$

$$\int \bar{V}\cdot d\bar{a}\,dz\,dx = -\int_0^2 (x+2)\,dx \int_0^2 dz = -12$$

(v) $z=2$, $d\bar{a} = dx\,dy\,\hat{z}$,

$$\bar{v} \cdot d\bar{a} = y\,(z^2 - 3)\,dx\,dy$$

$$= y\,(4-3)\,dx\,dy$$

$$\int \bar{v} \cdot d\bar{a} = \int_0^2 dx \int_0^2 y\,dy = [2]\left[\frac{y^2}{2}\right]_0^2 = 4$$

$$\text{TOTAL FLUX} = \int_{\text{SURFACE}} \bar{v} \cdot d\bar{a} = 16 + 0 + 12 - 12 + 4 = 20$$

VOLUME INTEGRALS

3-d AND THAT'S IT.

$$\int_{\mathcal{V}} T \, d\tau$$

SCALAR FUNCTION

INFINITESIMAL VOLUME ELEMENT

$d\tau = dx\,dy\,dz$

eg T = DENSITY $\rightarrow \int_{\mathcal{V}} T\,d\tau$ = TOTAL MASS

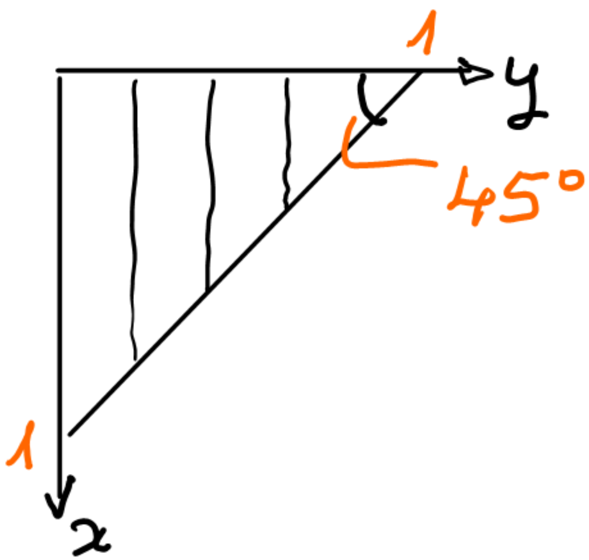
$T \rightarrow$ VECTOR FUNCTION $\bar{v}$

$$\int \bar{v} \, d\tau = \int \left(v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \right) d\tau$$

$$= \hat{x} \int v_x \, d\tau + \hat{y} \int v_y \, d\tau + \hat{z} \int v_z \, d\tau$$

CONSTANT UNIT VECTORS

SIMPLE EXAMPLE:



$$\begin{cases} x = 1-y \\ \text{AT } y=0, \quad x=1 \\ \text{AT } y=1, \quad x=0 \\ \text{AT } y=0.5, \quad x=0.5 \end{cases}$$

$$\int T\,d\tau = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x\,dx \right] dy \right\} dz$$

$$\int_0^{1-y} x\,dx = \left[\frac{x^2}{2}\right]_0^{1-y} = \frac{(1-y)^2}{2}$$

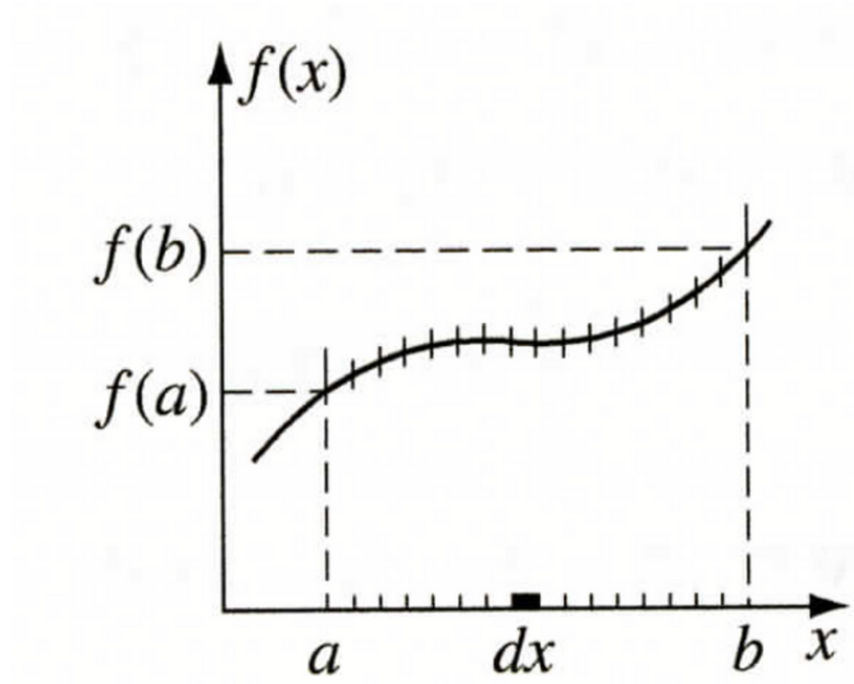
$$\frac{1}{2}\int_0^1 y(1-y)^2\,dy = \frac{1}{2}\int_0^1 \left(y - 2y^2 + y^3\right) dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right]_0^1 = \frac{1}{2}\cdot\frac{1}{12}$$

$$\int_0^3 z^2\,dz = \left[\frac{z^3}{3}\right]_0^3 = 9$$

$$\int T\,d\tau = \frac{1}{2}\cdot 9\cdot\frac{1}{12} = \frac{3}{8}$$

FUNDAMENTAL THEOREMS



WHAT IS AN INTEGRAL?

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$$

$$\int_a^b F(x)\, dx = f(b) - f(a)$$

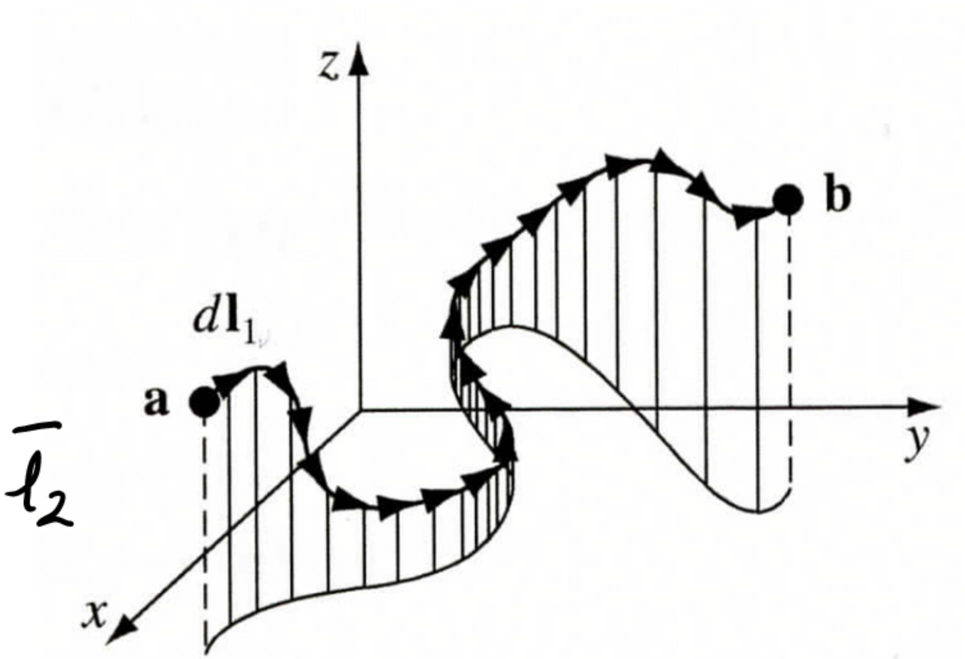
INTEGRAL OF $F(x)$ IS A FUNCTION WHOSE DERIVATIVE IS $F(x)$

- $df = (df/dx)\, dx$ → INFINITESIMAL CHANGE IN f, $x \to x + dx$
- SUM ALL INCREMENTS: TOTAL CHANGE IN $f = f(b) - f(a)$

$\int$ OF A DERIVATIVE OVER SOME REGION IS GIVEN BY VALUE OF FUNCTION AT END POINTS

LOOK AT THIS FOR GRAD, DIV, CURL

FUNDAMENTAL THEOREM FOR GRADIENTS.



- SCALAR FN $T(x, y, z)$
- $\bar{a} \rightarrow$ MOVE $d\bar{l}_1$
- $dT = (\bar{\nabla} T) \cdot d\bar{l}_1$
- MOVE $d\bar{l}_2 \rightarrow dT = (\bar{\nabla} \cdot T) \cdot d\bar{l}_2$

- EACH POINT COMPUTE $\bar{\nabla} \cdot T$

  TAKE DOT PRODUCT WITH $d\bar{l}$

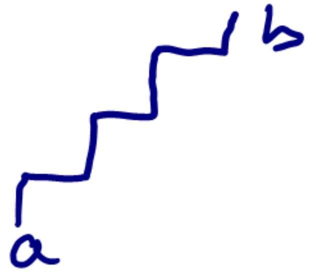
$$\int_a^b (\bar{\nabla} T) \cdot d\bar{l} = T(\bar{b}) - T(\bar{a})$$

FUNDAMENTAL THEOREM

$\int_a^b$ DERIVATE = VALUE OF FUNCTIONS ON BOUNDARIES

$\bar{\nabla} \cdot T$

$T(\bar{b}) - T(\bar{a})$

GEOMETRY $\int$ 

$\Rightarrow$ $T(\bar{b}) - T(\bar{a})$

$$\int_{\bar{a}}^{\bar{b}} (\bar{\nabla} T)\cdot d\bar{\ell} = T(\bar{b}) - T(\bar{a})$$

USUALLY LINE INTEGRALS DEPEND ON PATH

NOT TRUE FOR GRAD

1) $\int_{\bar{a}}^{\bar{b}} (\bar{\nabla} T)\cdot d\bar{\ell}$ INDEPENDANT OF PATH $\bar{a} \to \bar{b}$

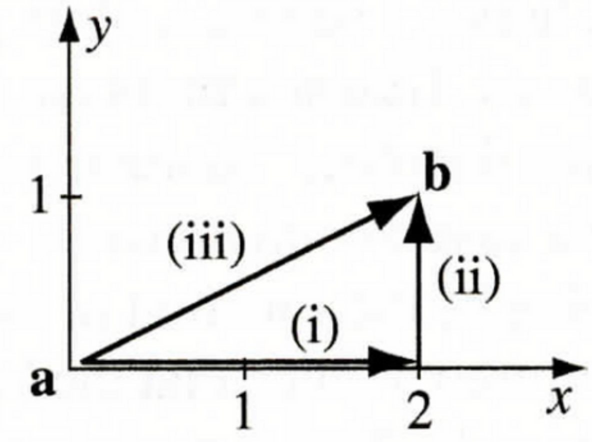
2) $\oint (\bar{\nabla} T)\cdot d\bar{\ell} = 0$ START & END SAME

$$T(\bar{b}) - T(\bar{a}) = 0$$

$$T(\bar{a}) - T(\bar{a})$$

EXAMPLE: $T = xy^2$

$(0,0,0) \rightarrow (2,1,0)$



HAVE TO CHOOSE A PATH FOR $\int$

(i) $\rightarrow$ (ii) $\quad d\bar{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

$$\bar{\nabla} T = \hat{x}\frac{\partial}{\partial x}(xy^2) + \hat{y}\frac{\partial}{\partial y}(xy^2)$$

$$= y^2\hat{x} + 2xy\,\hat{y}$$

i) $y = 0$; $\quad (\bar{\nabla} T)\cdot\hat{x}\,dx = 0$ $\quad , \int_i (\bar{\nabla} T)\cdot d\bar{l} = 0$

ii) $(\bar{\nabla} T)\cdot dy\,\hat{y} = 2xy\,dy\,\hat{y}$, $x = 2$, $4y\,dy\,\hat{y}$

$$\int_{ii} (\bar{\nabla} T)\cdot d\bar{l} = \int_0^1 4y\,dy = \frac{4y^2}{2}\Big|_0^1 = 2$$

TOTAL LINE $\int = 2$ & $T(\bar{b}) - T(\bar{a}) = 2$ $\underline{\text{OK}}$

IT'S SUPPOSED TO BE INDEPENDANT OF PATH

iii) $y = \frac{1}{2}x$, $dy = \frac{1}{2}\,dx$

$\hat{x}\cdot\hat{x} = 1$

$$(\bar{\nabla} T)\cdot d\bar{\ell} = (\bar{\nabla} T)_x\,dx + (\bar{\nabla} T)_y\,dy$$

$$= \frac{\partial}{\partial x}(xy^2)\,dx + \frac{\partial}{\partial y}(xy^2)\,dy = y^2\,dx + 2xy\,dy$$

$$= \frac{1}{4}x^2\,dx + 2x\cdot\left(\frac{x^2}{2}\right)dx = \frac{3}{4}x^2\,dx$$

$$\int_0^2 \frac{3}{4}x^2\,dx = \frac{3}{4}\cdot\frac{x^3}{3}\Big|_0^2 = \frac{3}{4}\cdot\frac{8}{3} = \underline{\underline{2}} \quad \text{Ok.}$$

FUNDAMENTAL THEOREM FOR DIVERGENCE

$$\int_{\text{VOLUME}} (\bar{\nabla} \cdot \bar{v})\, d\tau = \oint_{\text{SURFACE}} \bar{v} \cdot d\bar{a}$$

GAUSS'S THEOREM

$\int_{\text{VOL}}$ DERIVATIVE = VALUE OF FUNCTION AT BOUNDARY

DIVERGENCE

$\int$ OVER SURFACE

GEOMETRY

SAY $\bar{v}$ FLOW OF FLUID, THEN

$\oint_{\text{SURFACE}} \longrightarrow$ FLUX OF FLUID LEAVING VOLUME

- DIVERGENCE MEASURES SPREADING OUT OF FLUID FROM POINTS (TAPS)

- TAPS PRODUCING FLUID → EQUAL AMOUNT FLOWS OUT OF SURFACE

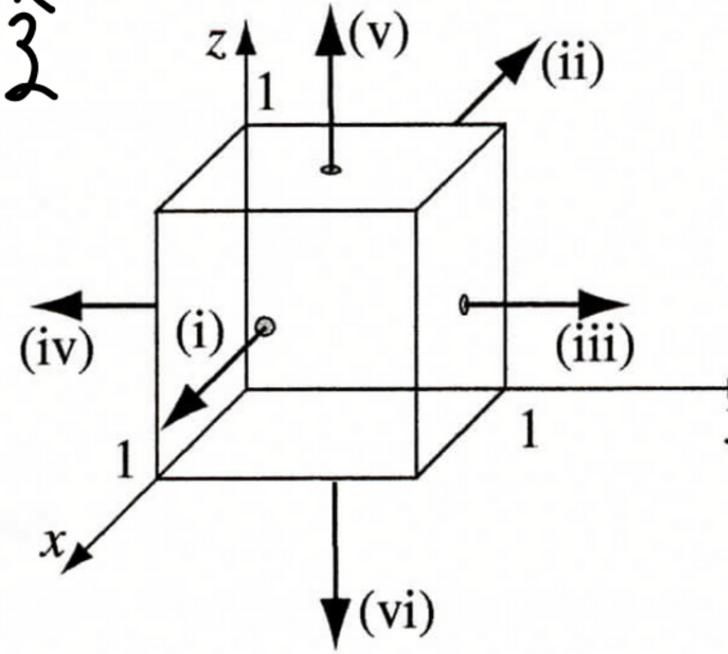
- TWO WAYS TO MEASURE FLUID FLOW
  - COUNT UP ALL SOURCES + HOW MUCH EACH PRODUCES
  - SUM OVER BOUNDARY FLOW

    ↳ EQUAL

$$\int \text{SOURCES WITHIN VOLUME} = \oint \text{FLOW OUT THROUGH SURFACE}$$

EXAMPLE CHECK GAUSS'S THEOREM FOR:

$$\bar{v} = y^2\,\hat{x} + (2xy + z^2)\,\hat{y} + (2yz)\,\hat{z}$$



SURFACE IS CUBE

$$\bar{\nabla}\cdot\bar{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot$$

$$= 0 + 2x + 2y = 2(x+y)$$

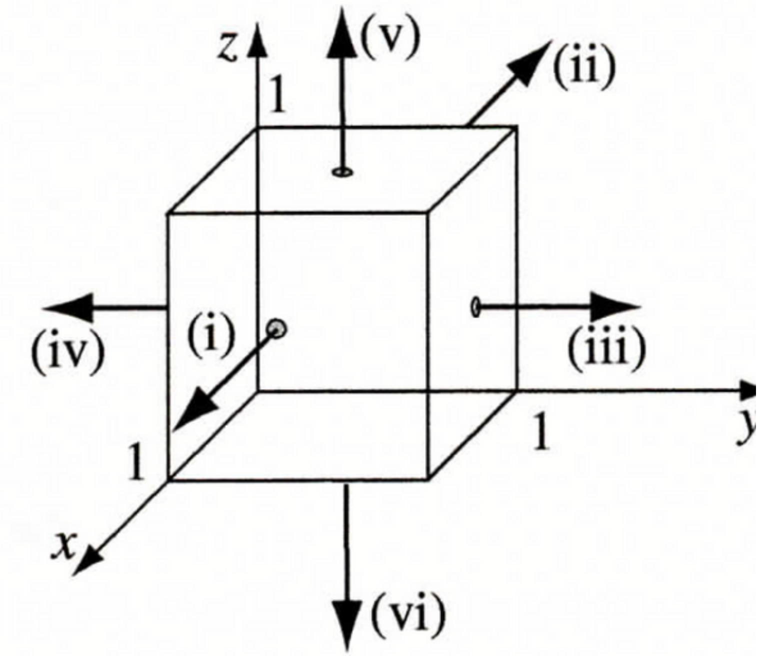
$$\int_V 2(x+y) = 2\int_0^1\int_0^1\int_0^1 (x+y)\,dx\,dy\,dz$$

$$= 2\int_0^1\int_0^1 \left[\frac{x^2}{2} + xy\right]_0^1 dy\,dz = 2\int_0^1\int_0^1 \left(\frac{1}{2} + y\right) dy\,dz$$

$$= 2\int_0^1 \left[\frac{y}{2} + \frac{y^2}{2}\right]_0^1 dz = 2\int_0^1 dz = 2$$

$$\int \bar{\nabla}\cdot\bar{v}\,d\tau = 2 \leftarrow \text{LHS.}$$

HAVE TO EVALUATE $\int \bar{v} \cdot d\bar{a}$ OVER 6 FACES OF CUBE



i) $d\bar{a} = \hat{x}\, dy\, dz$

$$\int \bar{v} \cdot d\bar{a} = \int_0^1 \int_0^1 y^2 dy\, dz$$

$$= \int_0^1 \frac{1}{3}\, dz = 1/3$$

ii) $d\bar{a} = -\hat{x}\, dy\, dz \qquad \int \rightarrow -1/3$

iii) $d\bar{a} = \hat{y}\, dx\, dz, \quad \bar{v} \cdot d\bar{a} = (2xy + z^2)\, dx\, dz$

$$\int_0^1 \int_0^1 (2xy + z^2)\, dx\, dz = \int_0^1 \left[x^2 y + x z^2\right]_0^1 dz$$

$$= \int_0^1 [1 + z^2]\, dz = \left[z + z^3/3 \right]_0^1 = 4/3$$

iv) cf ii) $= -1/3$

v) $d\bar{a} = \hat{z}\, dy\, dz$, $z = 1$

$\bar{v} \cdot d\bar{a} = 2yz$ at $z = 1$

$$\int \bar{v} \cdot d\bar{a} = \int_0^1 \int_0^1 2y\, dx\, dy = \int_0^1 [2xy]_0^1\, dy = \int_0^1 2y\, dy = \underline{1}$$

vi) $d\bar{a} = -\hat{z}\, dx\, dy$ at $z = 0$; $\int \bar{v} \cdot d\bar{a} = 0$

TOTAL FLUX THRU SURFACE $\oint \bar{v} \cdot d\bar{a} = -\frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2$

$= LHS$

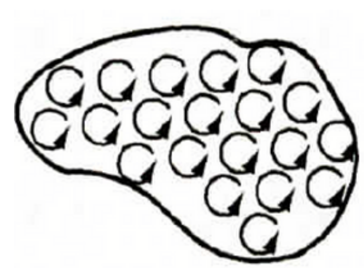
FUNDAMENTAL THEOREM FOR CURL

→ ALSO KNOWN AS STOKES' THEOREM

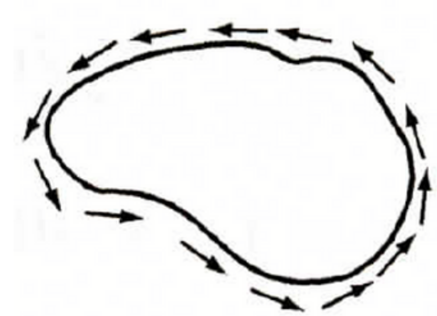
$$\int_{\text{surface}} (\bar{\nabla} \times \bar{v}) \cdot d\bar{a} = \oint_{\text{patch}} \bar{v} \cdot d\bar{\ell}$$

$\int$ DERIVATIVE OVER REGION $=$ $\int$ VALUE OF FUNCTION AT BOUNDARY

- PATCH OF SURFACE — LINE INTEGRAL OVER BOUNDARY
- ADD UP ALL SWIRLS — TOTAL SWIRL



$=$

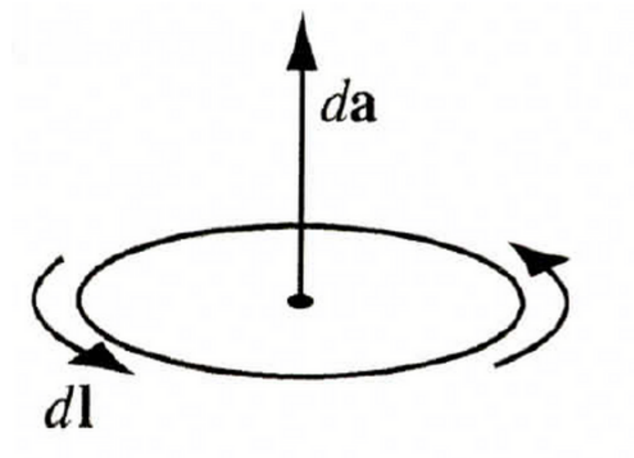


$\int \bar{v} \cdot d\bar{\ell} \Rightarrow$ CIRCULATION OF $\bar{v}$

NORMALLY FLUX $\int \sim \int_{SURFACE}$

NOT TRUE FOR CURLS

$d\bar{a} \leftrightarrow d\bar{\ell}$ RIGHT HAND RULE

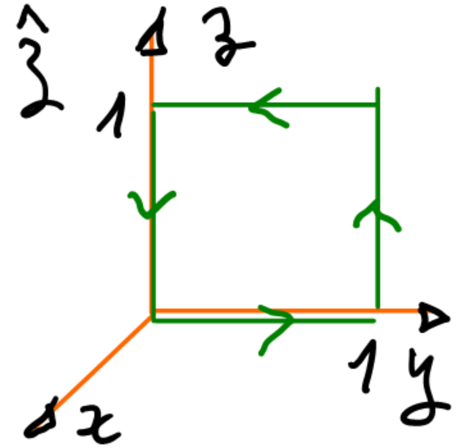


- $\int (\bar{\nabla} \times \bar{v}) \cdot d\bar{a}$ DEPENDS ONLY ON BOUNDARY LINE
  $\rightarrow$ NOT SURFACE CHOSEN

- $\oint (\bar{\nabla} \times \bar{v}) \cdot d\bar{a} = 0$ FOR CLOSED SURFACE

BOUNDARY LINE SHRINKS DOWN TO A POINT.

EXAMPLE: $\vec{v} = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}$



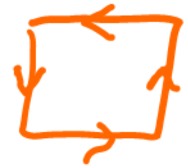
$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ 0 & (2xz + 3y^2) & 4yz^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(4yz^2) - \frac{\partial}{\partial z}(2xz + 3y^2) \right)\hat{x}$$

$$+ \left(\frac{\partial}{\partial x}(4yz^2) - \frac{\partial}{\partial z} 0 \right)\hat{y}$$

$$+ \left(\frac{\partial}{\partial x}(2xz + 3y^2) \right)\hat{z}$$

$$\bar{\nabla} \times \bar{v} = (4z^2 - 2x)\hat{x} + 2z\hat{z}$$

$$d\bar{a} = dy \cdot dz \cdot \hat{x}$$ → x +ve  RHS

$$\int_S (\bar{\nabla} \times \bar{v}) \cdot d\bar{a} = \iint \left[(4z^2 - 2x)\hat{x} + 2z\hat{z}\right] \cdot (dy\, dz)\,\hat{x}$$

$\hat{x} \cdot \hat{x} = 1$, $\hat{x} \cdot \hat{z} = 0$ & $x = 0$

So, $$= \int_0^1 \int_0^1 4z^2\, dy\, dz = \int_0^1 \left[\tfrac{4}{3} z^3\right]_0^1 dy$$

$$= \int_0^1 \tfrac{4}{3} \cdot dy = \left[\frac{4 \cdot y}{3}\right]_0^1 = \frac{4}{3}$$ ← THAT IS THE SURFACE $\int$

WE ARE AFTER $$\int_S (\bar{\nabla} \times \bar{v}) \cdot d\bar{a} = \oint \bar{v} \cdot d\bar{l}$$

DONE THIS ↑

↑ HAVE TO DO THIS ALONG EACH LINE.

LINE (i) $\bar{v} = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}$

ALONG (i) $z=0, \ x=0$

$\bar{v} = (3y^2)\,\hat{y}, \quad d\bar{\ell} = dy\,\hat{y}$

$$\int \bar{v}\cdot d\bar{\ell} = \int_0^1 3y^2\,dy = \left[\frac{3y^3}{3}\right]_0^1 = 1$$

:)

ii) $x=0, \ y=1, \ \bar{v} = 3\hat{y} + 4z^2\hat{z}, \ d\bar{\ell} = dz\,\hat{z}$

$$\int \bar{v}\cdot d\bar{\ell} = \int_0^1 4z^2\,dz = \left[\frac{4z^3}{3}\right]_0^1 = \frac{4}{3}$$

iii) $z=1, \ x=0, \ \bar{v} = (3y^2)\hat{y} + 4y\hat{z} \ ; \ d\bar{\ell} = dy\,\hat{y}$

$$\int \bar{v}\cdot d\bar{\ell} = \int_0^1 3y^2\,dy = -1$$

iv) $y=0,\ x=0 \quad \bar{v}=0 \quad \int_1^0 0\,dz = 0$

WANTED TO TEST!

$$\underbrace{\int_S (\bar{\nabla}\times\bar{v})\cdot d\bar{a}}_{4/3} = \underbrace{\oint \bar{v}\cdot d\bar{l}}_{1+\frac{4}{3}-1+0}$$

SO STOKES' THEOREM WORKS