INTEGRATION BY PARTS

EXPLOITS PRODUCT RULE FOR DERIVATIVES

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} \quad (1)$$

$\int$ BOTH SIDES OF

$$fg\Big|_a^b = \int_a^b f\frac{dg}{dx}\cdot dx + \int_a^b g\frac{df}{dx}\cdot dx$$

SO
$$\int_a^b f\left(\frac{dg}{dx}\right) = fg\Big|_a^b - \int_a^b g\frac{df}{dx}\cdot dx$$

$\int$ (1) – TRANSFER DERIVATIVE FROM $g \to f$

CHANGE SIGN AND ADD BOUNDARY TERM

EXAMPLE: $\int_0^\infty x e^{-x}\, dx$; $e^{-x} = \frac{d}{dx}(-e^{-x})$

SO, IN THIS CASE $f(x) = x$, $g(x) = -e^{-x}$

$$\int_0^\infty x e^{-x} dx = -x e^{-x}\Big|_0^\infty - \int -e^{-x} dx$$

$$= e^{-x}\Big|_0^\infty = 1$$

CAN DO SAME THING USING VECTOR PRODUCT RULES

$$\bar{\nabla}\cdot(f\bar{A}) = f(\bar{\nabla}\cdot\bar{A}) + \bar{A}(\bar{\nabla}f)$$

OVER SOME VOLUME

$$\int \bar{\nabla}\cdot(f\bar{A})\,d\tau = \int f(\bar{\nabla}\cdot\bar{A})\,d\tau + \int \bar{A}(\bar{\nabla}f)\,d\tau$$

DIVERGENCE THEOREM SAYS

$$\int_V (\bar{\nabla}\cdot\bar{v})\,d\tau = \oint_S \bar{v}\cdot d\bar{a}$$

SO $$\int \bar{\nabla}\cdot(f\bar{A})\,d\tau = \oint_S f\bar{A}\cdot d\bar{a}$$

FROM ABOVE

$$\int_V f(\bar{\nabla}\cdot\bar{A})\,d\tau = -\int \bar{A}\cdot(\bar{\nabla} f)\,d\tau + \oint_S f\bar{A}\cdot d\bar{a}$$

INTEGRAND → FUNCTION X DIVERGENCE

TRANSFER DERIVATIVE FROM $\bar{A} \to f$

ADDS BOUNDARY SURFACE $\oint_S$

CURVILINEAR COORDINATES

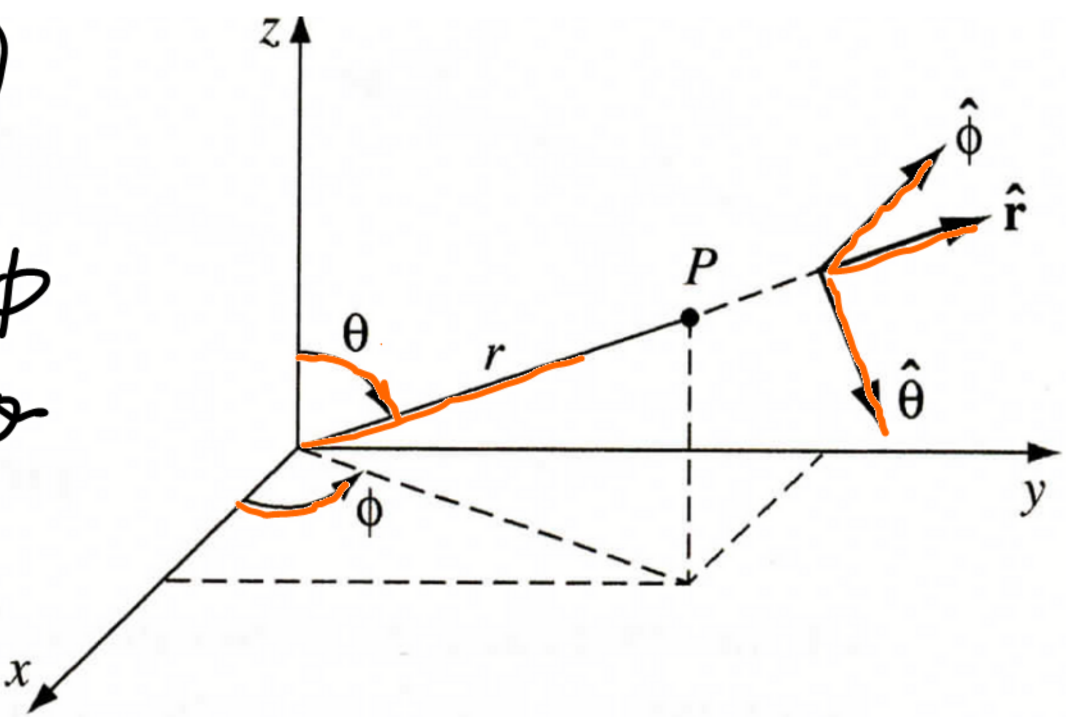
CARTESIAN COORDINATES MOST FAMILIAR

CAN DEFINE ANY SET OF (3) COORDINATES THAT WE WANT — MANY PHYSICAL SITUATIONS ARE SPHERICAL OR CYLINDRICAL.

$$(x, y, z) \longrightarrow (r, \theta, \phi)$$

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$



UNIT VECTORS

$$\underbrace{\hat{r}, \hat{\theta}, \hat{\phi}}_{\text{ORTHOGONAL}}$$

CAN WRITE A VECTOR AS

$$\bar{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_z \hat{z}$$

$$\hat{r} = \sin\theta \cos\phi \, \hat{x} + \sin\theta \sin\phi \, \hat{y} + \cos\theta \, \hat{z}$$

$$\hat{\theta} = \cos\theta \cos\phi \, \hat{x} + \cos\theta \sin\phi \, \hat{y} - \sin\theta \, \hat{z}$$

$$\hat{\phi} = -\sin\phi \, \hat{x} + \cos\phi \, \hat{y}$$

$\hat{r}, \hat{\theta}, \hat{\phi}$ CHANGE DIRECTION AS POINT MOVES AROUND

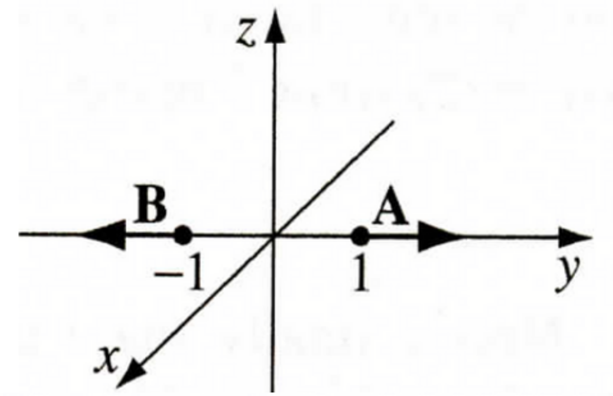
- CANNOT COMBINE SPHERICAL COORDS AT DIFF POINTS
- UNIT VECTORS ARE FUNCTIONS OF POSITION

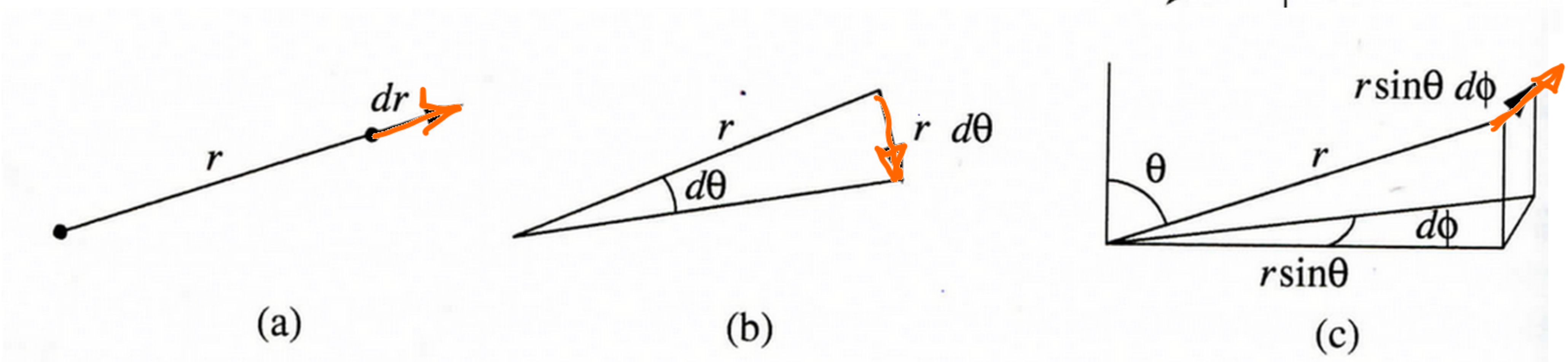
$$\frac{d\hat{r}}{d\theta} = \hat{\theta}$$

- CANNOT TAKE $\hat{r}, \hat{\theta}, \phi$ OUTSIDE $\int$ LIKE YOU CAN WITH $\hat{x}, \hat{y}, \hat{z}$

$\bar{A} + \bar{B} = \bar{0}$ NOT $2\hat{y}$

$\bar{A} \cdot \bar{B} = -1$ NOT $+1$





$d\ell_{\underline{r}} = d\underline{r}$

MORE COMPLEX FOR $d\ell_\theta = r\,d\theta$

$d\ell_\phi = r\sin\theta\, d\phi$

NOT $d\theta$

-ENERAL INFINITESIMAL LENGTH

$$d\bar{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

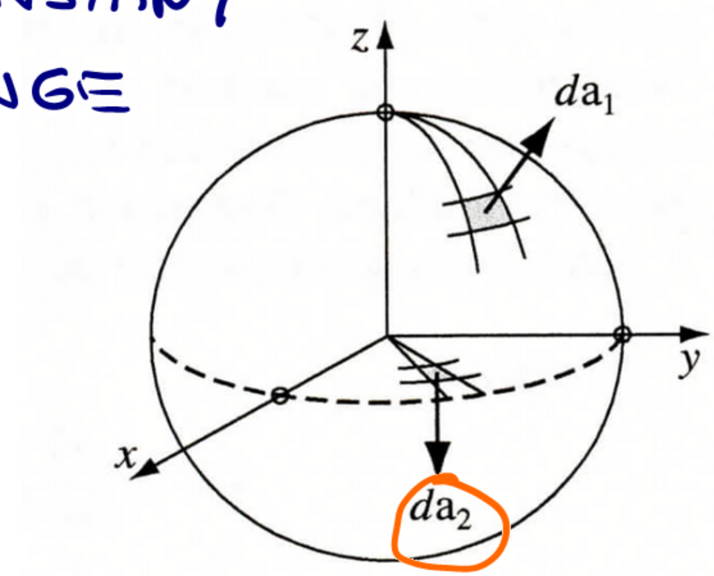
INFINITESIMAL VOLUME

$$d\tau = dl_r\,dl_\theta\,dl_\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

SURFACE DEPENDS ON ORIENTATION

SURFACE OF SPHERE $r \to$ CONSTANT

θ, ϕ CHANGE



$$d\bar{a}_1 = dl_\theta\,dl_\phi\,\hat{r} = r^2\sin\theta\,d\theta\,d\phi\,\hat{r}$$

$$d\bar{a}_2 = dl_r\,dl_\phi\,\hat{\theta} = r\,dr\,d\phi\,\hat{\theta}$$

θ CONST $\to \frac{\pi}{2}$

$0 < r < \infty$

$0 < \phi < 2\pi$

$0 < \theta < \pi \leftarrow$ NOT 2π

SIMPLE EXAMPLE $\rightarrow$ VOLUME OF SPHERE

RADIUS $\rightarrow$ R

$$V = \int d\tau = \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

VOLUME ELEMENT

$$= \left(\int_0^R r^2 dr \right) \left(\int_0^{\pi} \sin\theta \, d\theta \right) \left(\int_0^{2\pi} d\phi \right)$$

$$= \frac{R^3}{3} \cdot 2 \cdot 2\pi = \frac{4}{3}\pi R^3$$

GRAD, DIV, CURL $\nabla^2 T \rightarrow r, \theta, \phi$

eg GRAD $\quad \bar{\nabla} T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} + \frac{\partial T}{\partial \phi}\cdot\frac{\partial \phi}{\partial x} \rightarrow$$

SEE APPENDIX IF INTERESTED

<u>GRAD</u> $\quad \bar{\nabla} T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$

<u>DIV</u> $\quad \bar{\nabla}\cdot\bar{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, V_\theta\right)$

$$+ \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}$$

COMPONENT of V IN ϕ DIRECTION

CURL

$$\bar{\nabla} \times \bar{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

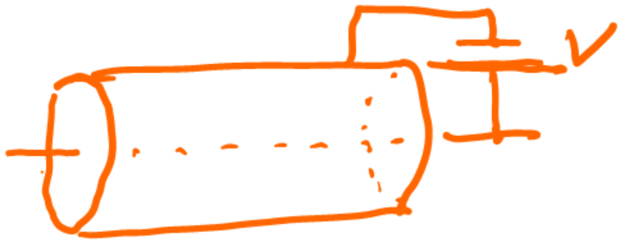
LAPLACIAN

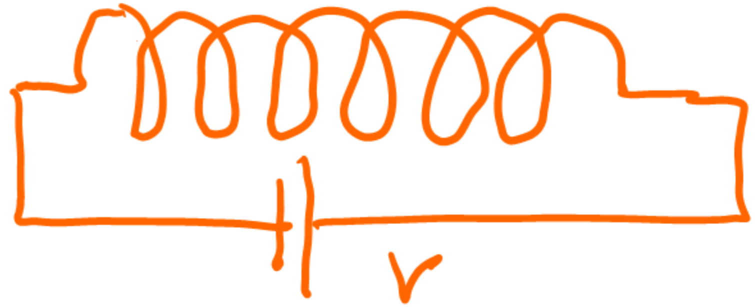
$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$$

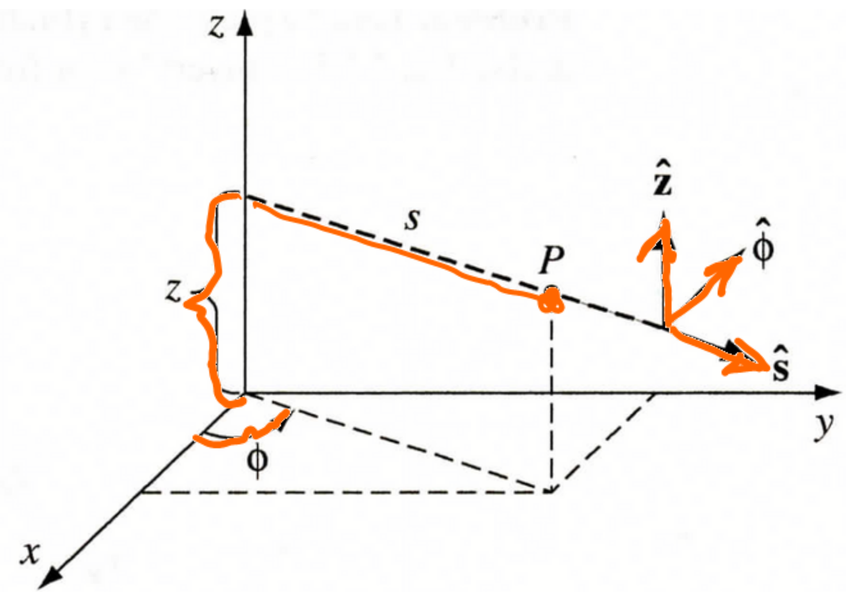
CYLINDRICAL COORDINATES

MANY SITUATIONS $\longrightarrow$ CYLINDRICAL SYMMETRY





$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$



UNIT VECTORS

$$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$
$$\hat{z} = \hat{z}$$

INFINITESIMAL DISPLACEMENTS ~

$$dl_s = ds, \quad dl_\phi = s\,d\phi, \quad dl_z = dz$$

LENGTH

$$d\bar{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

VOLUME

$$d\tau = s\,ds\,d\phi\,dz$$

$$0 < s < \infty$$

$$0 < \phi < 2\pi$$

$$-\infty < z < \infty$$

GRAD $$\bar{\nabla} T = \frac{\partial T}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$$

DIV $$\bar{\nabla}\cdot\bar{V} = \frac{1}{s}\frac{\partial}{\partial s}(sV_s) + \frac{1}{s}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$$

CURL $$\bar{\nabla}\times\bar{V} = \left(\frac{1}{s}\frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial V_s}{\partial z} - \frac{\partial V_z}{\partial s}\right)\hat{\phi}$$

$$+ \frac{1}{s}\left[\frac{\partial}{\partial s}(sV_\phi) - \frac{\partial V_s}{\partial \phi}\right]\hat{z}$$

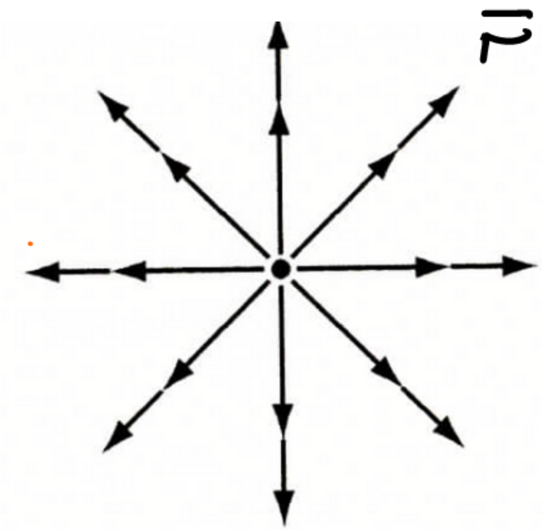
LAPLACIAN

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\left(\frac{\partial^2 T}{\partial \phi^2}\right) + \frac{\partial^2 T}{\partial z^2}$$

DIRAC DELTA FUNCTION

MATHEMATICS DENY THAT THIS IS A FUNCTION INVENTED IN QUANTUM MECHANICS

DIVERGENCE OF $\dfrac{\hat{r}}{r^2}$



WOULD EXPECT LARGE DIV

$$\bar{\nabla}\cdot\bar{v} = \underbrace{\frac{1}{r^2}\,\frac{\partial}{\partial r}}_{\text{SPHERICAL COORD}}\left(r^2\cdot\frac{1}{r^2}\right)$$

VECTOR

$$= 0$$

APPLY DIVERGENCE THEORM $\int$ OVER SPHERE RADIUS R

$$\int_V \bar{\nabla}\cdot\bar{v}\,d\tau = \oint_S \bar{v}\cdot d\bar{a}$$

$$\oint_S \bar{v}\cdot d\bar{a} = \int \left(\frac{1}{R^2}\hat{r}\right)\left(R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}\right)$$

$$= \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2} \; \underbrace{\int_0^{2\pi} d\phi}_{2\pi} = 4\pi$$

cf ?

SINCE WE SHOWED $\bar{\nabla}\cdot\bar{v} = 0$ $\quad \int_{VOL} \bar{\nabla}\cdot\bar{v}\, d\tau = 0$

PROBLEM is $r = 0$ $\quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}\right)$

$= 1$??

AS $r \to 0$ UNDEFINED

$\bar{\nabla}\cdot\bar{V} = 0$ EVERYWHERE EXCEPT ORIGIN

$\int \left(\frac{1}{R^2}\hat{r}\right) R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$ INDEP OF R

SO SHOULD GET $\int (\bar{\nabla}\cdot\bar{v})\, d\tau = 4\pi$ FOR ANY RADIUS

NO MATTER HOW SMALL

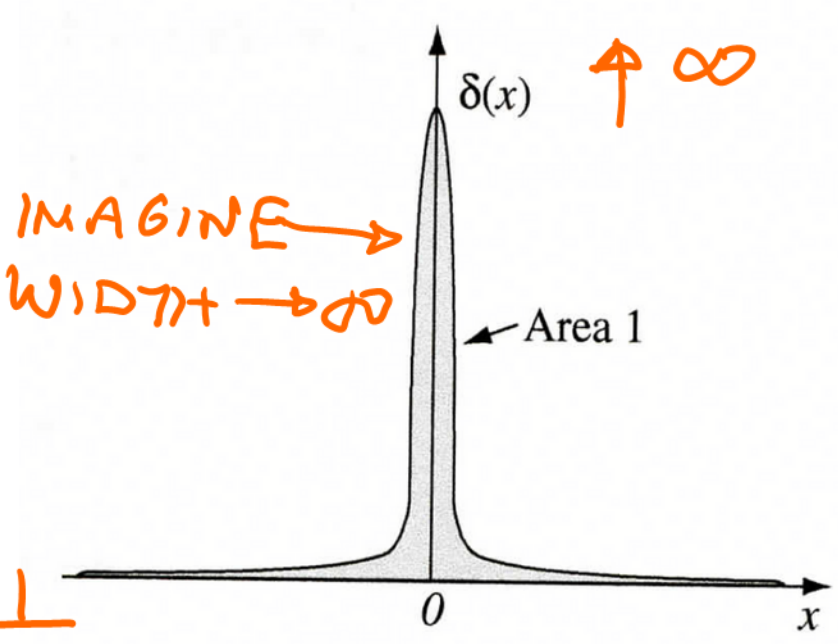
$\bar{\nabla}\cdot\left(\frac{1}{r^2}\cdot\hat{r}\right)$ HAS STRANGE PROPERTY THAT IT IS ZERO EVERYWHERE EXCEPT $r=0$

ALL OF $\int_V \bar{\nabla}\cdot\left(\frac{1}{r^2}\hat{r}\right) d\tau = 4\pi$

COMES FROM ORIGIN

DELTA FUNCTION IN 1-d $\delta(x)$

$$\delta(x) = \begin{cases} 0 & \rightarrow x \neq 0 \\ \infty & \rightarrow x = 0 \end{cases}$$



$$\int_{-\infty}^{+\infty} \delta(x)\, dx = \underline{1}$$

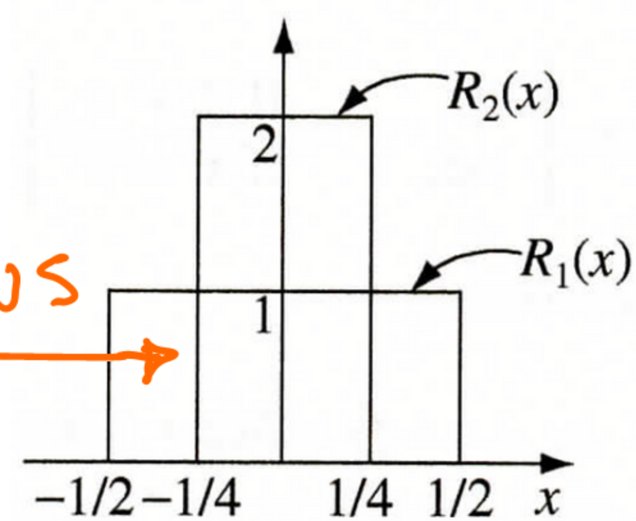
DIMENSIONS $\dfrac{1}{x}$

NOT A FUNCTION

SINCE NOT FINITE AT $x = 0$

– LIMIT OF SEQUENCE OF FUNCTIONS

"DISTRIBUTION"



$f(x) \rightarrow$ ORDINARY CONTINUOUS FUNCTION

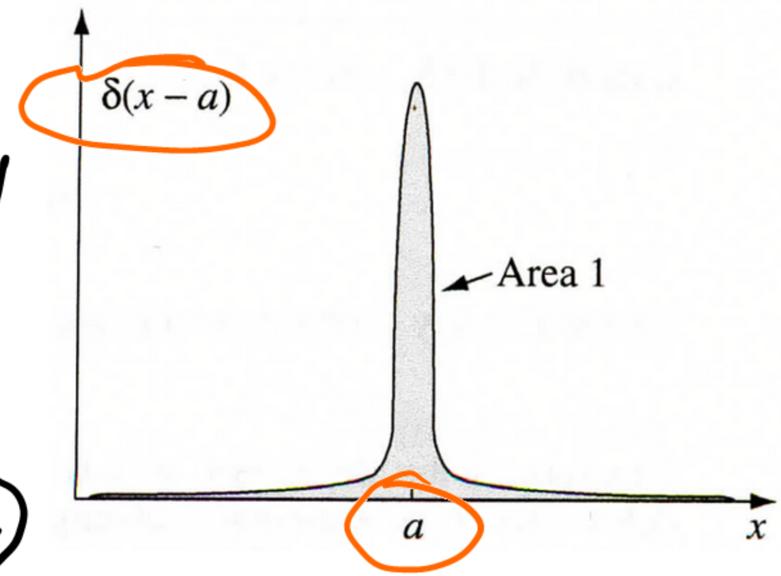
$f(x)\delta(x) = 0$ EVERY WHERE EXCEPT $x = 0$

So $f(x)\delta(x) = f(0)\delta x$ ← !!!

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\,dx = f(0)\int_{-\infty}^{+\infty}\delta(x)\,dx = f(0)$$

δ FN PICKS OUT VALUE OF $f(x)$ AT 0

$$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases} \qquad \int_{-\infty}^{+\infty} \delta(x-a)\,dx = 1$$



$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

eg $\int_0^3 x^3\,\delta(x-2)\,dx$

PICKS OUT VALUE AT $x=2$

$\int = 2^3 = 8$

$\int_0^1 = 0 \leftarrow$ δ FN SPIKE OUTSIDE ∫ LIMITS.

δ FN ONLY "WORKS" INSIDE INTEGRAL

$$\int_{-\infty}^{+\infty} f(x)\, D_1(x)\, dx = \int_{-\infty}^{+\infty} f(x)\, D_2(x)\, dx$$

MEANS $D_1 = D_2$

<u>EXAMPLE</u> SHOW $\delta(kx) = \dfrac{1}{|k|}\delta(x)$

$$\int_{-\infty}^{+\infty} f(x)\, \delta(kx)\, dx$$

PUT $y \equiv kx$, $x = \dfrac{y}{k}$

$dy = k\,dx$, $dx = \dfrac{1}{k} dy$

k +ve $\int -\infty \to +\infty$

k -ve $x = \infty \to y = -\infty$ $\Rightarrow \int_{+\infty}^{-\infty} = -\int_{-\infty}^{+\infty}$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(kx)\,dx = \pm\int_{-\infty}^{+\infty} f\left(\frac{y}{k}\right)\delta(y)\,\frac{dy}{k}$$

$$= \pm\,\frac{1}{k} f(0) = \frac{1}{|k|} f(0)$$

$\delta(kx)$ SERVES SAME PURPOSE AS $\frac{1}{k}\delta(x)$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(kx)\,dx = \int_{-\infty}^{+\infty} f(x)\left[\frac{1}{|k|}\delta(x)\right]dx$$

$$\therefore\ \delta(kx) = 1/|k|\ \delta(x)$$

LAST PAGE

$D_1 = D_2$

$$\delta(-x) = \delta(x)$$

3-d δ-FUNCTION

$$\delta^3(\bar{r}) = \delta(x)\,\delta(y)\,\delta(z) \qquad \bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$\delta^3(\bar{r}) = 0$ EVERYWHERE EXCEPT $(0,0,0)$

$$\int_{\substack{\text{ALL}\\ \text{SPACE}}} \delta^3(\bar{r})\,dr = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x)\delta(y)\delta(z) = 1$$

$$\int f(\bar{r})\,\delta^3(\bar{r}-\bar{a})\,d\tau = f(\bar{a})$$

EVIDENTLY $\quad \bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\bar{r})$

"SOLID ANGLE"

GENERALLY $\bar{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\mathscr{r})$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \implies \nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\mathscr{r})$$

EXAMPLE: EVALUATE $J = \int (r^2+2)\,\bar{\nabla}\cdot\left(\frac{\bar{r}}{r^2}\right)d\tau$

$$\bar{\nabla}\cdot\left(\frac{\bar{r}}{r^2}\right) = 4\pi\delta^3(\bar{r})$$

$$\bar{J} = \int_V (r^2+2)\,4\pi\delta^3(\bar{r})\,d\tau$$

V ← SPHERE OF RADIUS R

δ FN PICKS OUT $r=0$

$$\int = 4\pi(0+2) = 8\pi$$

DOING THIS USING $\int$ BY PARTS SHOWS THE POWER OF THE δ-fn

DIV THEOREM

$$\int_V f(\bar{\nabla}\cdot\bar{A})\,d\tau = -\int_V \bar{A}\cdot(\bar{\nabla}f)\,d\tau + \oint f\,\bar{A}\cdot d\bar{a}$$

HERE $\bar{\nabla}\cdot\bar{A} \longrightarrow \bar{\nabla}\cdot\frac{\hat{r}}{r^2}$

$$\int (r^2+2)\,\bar{\nabla}\cdot\frac{\hat{r}}{r^2} = -\int \frac{\hat{r}}{r^2}\cdot\bar{\nabla}(r^2+2)\,d\tau$$

$$+\oint_S (r^2+2)\frac{\hat{r}}{r^2}\,d\bar{a}$$

$$\bar{\nabla}(r^2+2) = \frac{\partial}{\partial r}(r^2+2)\,\hat{r} = 2r\,\hat{r}$$

VOLUME $\int = \int_V \frac{\hat{r}}{r^2}\cdot 2r\,\hat{r}\,d\tau$ $\qquad \hat{r}\cdot\hat{r}=1$

$$= \int_{\text{VOL}} \frac{2}{r}\, d\tau = \int \frac{2}{r} r^2 \sin\theta\, d\theta\, d\phi$$

$$= 2 \times 4\pi \int_0^R r\, dr = 8\pi \left[\frac{R^2}{2} \right]_0^R = 4\pi R^2$$

ON SURFACE OF SPHERE $d\vec{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

$$\text{SURFACE} \int = \int (R^2+2)\, \hat{r}\, R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$

$\hat{r} \cdot \hat{r} = 1$

$$= 4\pi (R^2 + 2)$$

$J = -$ VOLUME $+$ SURFACE

$$= -4\pi R^2 + 4\pi (R^2+2) = 8\pi \quad \checkmark$$