DIVERGENCE OF ELECTROSTATIC FIELD

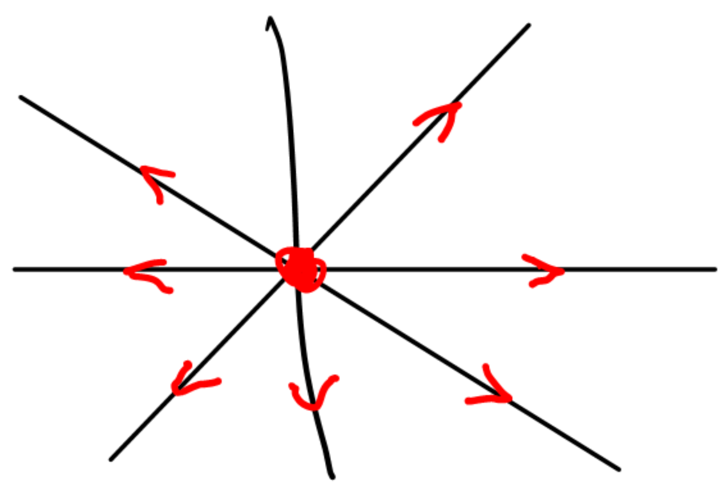
FIELD LINES, FLUX, GAUSS'S LAW:

WE HAVE COVERED ALL PHYSICS CONTENT OF ELECTROSTATICS — BUT $\int$s IN SOME PROBLEMS DIFFICULT OR IMPOSSIBLE

LET'S LOOK AT FIELD LINES

$$E(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

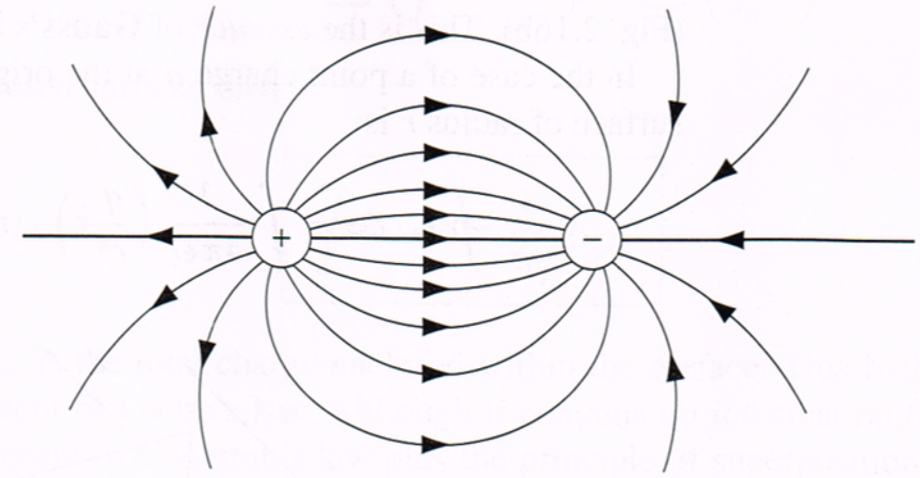
• LOOK @ DISCUSSION IN TEXT.



HOW TO REPRESENT FIELD STRENGTH?

FIELD LINES

- MAGNITUDE OF FIELD REPRESENTED BY DENSITY OF LINES



Opposite charges

- 2-d A BIT DECEPTIVE

  ↳ 3d

$$\text{DENSITY} = \frac{\text{NUMBER OF FIELD LINES}}{4\pi r^2} \sim \frac{1}{r^2}$$

← COULOMBS LAW

- FIELD LINES START ON +ve CHARGE END ON NEGATIVE $\vec{\nabla}\cdot\vec{E} \neq$ FREE SPACE

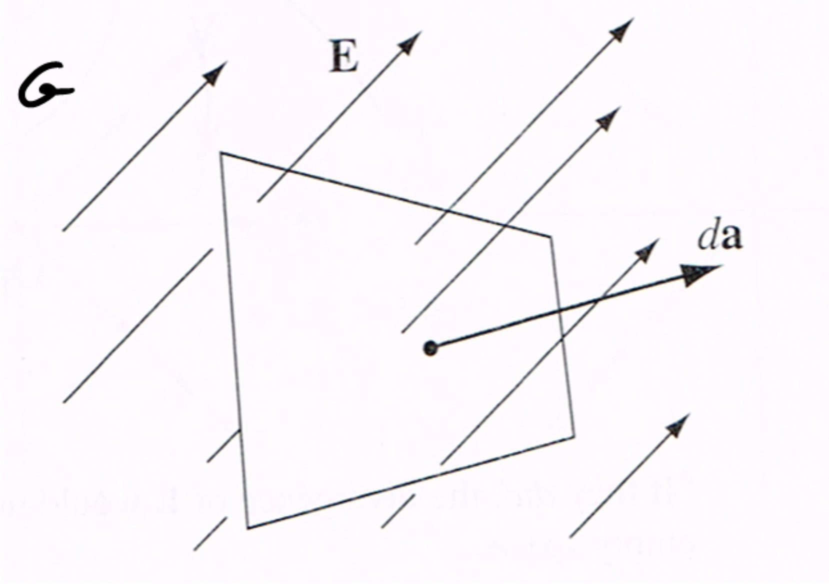
- ONLY TERMINATE ON CHARGES

FLUX OF $\bar{E}$ THROUGH SURFACE S

$$\Phi_E \equiv \int_S \bar{E} \cdot d\bar{a}$$

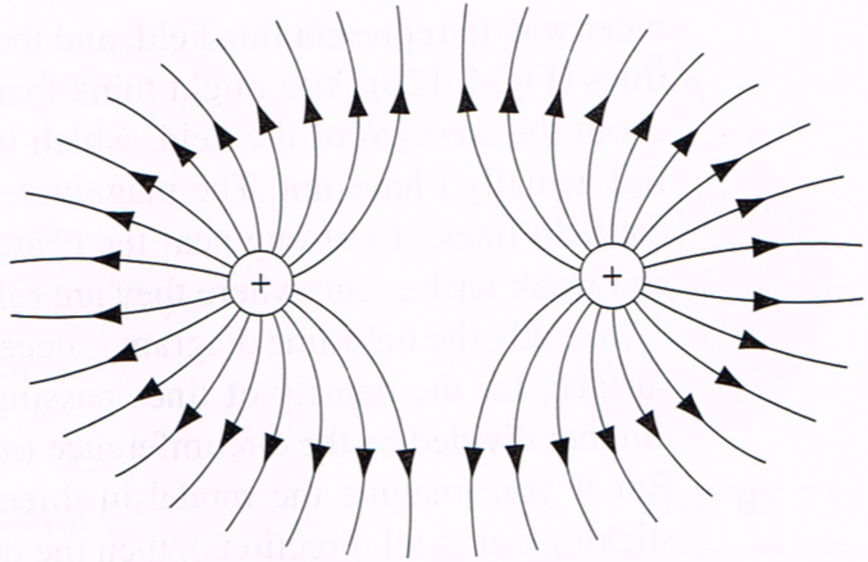
IS A MEASURE OF "NUMBER OF FIELD LINES PASSING THRU S"

$\bar{E} \cdot d\bar{a} \propto$ # OF LINES PASSING THRU $d\bar{a}$

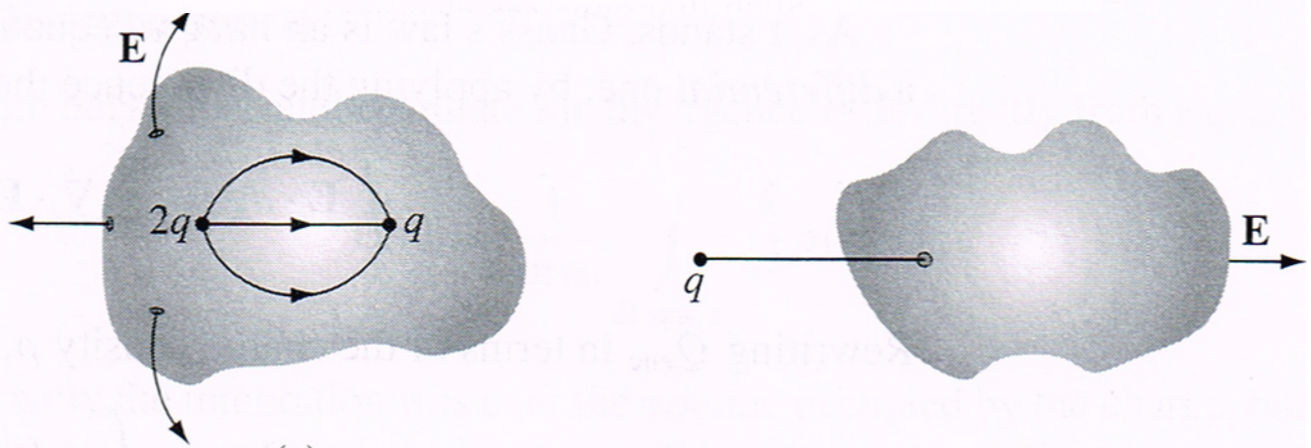


- DOT PRODUCT ENSURES COMPONENT OF $d\bar{a}$ IN DIRECTION OF $\bar{E}$

- FLUX THRU CLOSED SURFACE IS A MEASURE OF CHARGE ENCLOSED → GAUSS'S LAW



Equal charges



(a) (b)

- FIELD LINES FROM +VE CHARGE MUST PASS THRU SURFACE, OR END ON −VE CHARGE

- LINES FROM EXTERNAL PASS IN ONE SIDE OUT OTHER SIDE → CANCEL OUT.

POINT CHARGE AT ORIGIN, FLUX OF $\bar{E}$ THRU SPHERICAL SURFACE

↳ USE SPHERICAL COORDINATES

$$\oint \bar{E}\cdot d\bar{a} = \int \underbrace{\frac{1}{4\pi\varepsilon_0}\left(\frac{q\,\hat{r}}{r^2}\right)}\underbrace{\left(r^2\sin\theta\, d\theta\, d\phi\, \hat{r}\right)}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{q\,\hat{r}\cdot\hat{r}}{r^2}\cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\cdot q$$

$$\oint \bar{E}\cdot d\bar{a} = q/\varepsilon_0$$

ENCLOSED CHARGE.

RADIUS OF SPHERE CANCELS OUT

↳ SURFACE AREA $\propto r^2$

FIELD $\propto 1/r^2$

ANY CLOSED SURFACE WOULD DO.

SINGLE CHARGE $\longrightarrow$ SEVERAL CHARGES SCATTERD IN VOLUME

SUPERPOSITION $\longrightarrow$ $\bar{E} = \sum_{i=1}^{n} \bar{E}_i$

FLUX THRU SURFACE ENCLOSING THEM

$$\oint \bar{E} \cdot d\bar{a} = \sum_{i=1}^{n} \left(\oint \bar{E}_i \cdot da \right) = \sum_{i=1}^{n} \left(\frac{q_i}{\varepsilon_0} \right)$$

FOR ANY CLOSED SURFACE

$$\oint \bar{E} \cdot d\bar{a} = \frac{1}{\varepsilon_0} Q_{ENC} \quad \Big\} \text{GAUSS'S LAW}$$

TOTAL ENCLOSED CHARGE

- DEPENDS ON $\frac{1}{r^2}$

NATURE OF FORCE

$$\oint \bar{E} \cdot d\bar{a} = Q_{ENC} / \varepsilon_0$$

INTEGRAL EQTN $\longrightarrow$ DIFFERENTIAL EQTN.
USE DIVERGENCE THEOREM

$$\int_V (\bar{\nabla} \cdot \bar{v})\, d\tau = \oint_S \bar{v} \cdot d\bar{a}$$

SO

$$\oint \bar{E} \cdot d\bar{a} = \int_V (\bar{\nabla} \cdot \bar{E})\, d\tau$$

$$Q_{ENC} = \int \rho\, d\tau$$ CHARGE DENSITY

$$\int (\bar{\nabla} \cdot \bar{E})\, d\tau = \oint \bar{E} \cdot d\bar{a} = \frac{Q_{ENC}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_V \rho\, d\tau$$

$$\bar{\nabla} \cdot \bar{E} = \frac{1}{\varepsilon_0} \rho$$

DIFFENTIAL FORM OF GAUSS'S LAW

DIVERGENCE OF $\bar{E}$

$\bar{\nabla} \cdot \bar{E}$ DIRECTLY

$$\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\text{SPACE}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho(\bar{r}')\, d\tau'$$

$\bar{\mathscr{r}} = \bar{r} - \bar{r}'$

$$\bar{\nabla} \cdot \bar{E} = \frac{1}{4\pi\varepsilon_0} \int \bar{\nabla} \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) \rho(\bar{r}')\, d\tau'$$

$$\bar{\nabla} \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\, \delta^3(\bar{\mathscr{r}})$$ ← SHOWED EARLIER

$$\bar{\nabla} \cdot \bar{E} = \frac{1}{4\pi\varepsilon_0} \int 4\pi\, \delta^3(\bar{r} - \bar{r}') \rho(\bar{r}')\, d\tau' = \frac{1}{\varepsilon_0} \rho(\bar{r})$$

$$\bar{\nabla} \cdot \bar{E} = \frac{1}{\varepsilon_0} \rho(\bar{r})$$

GAUSS'S DIFFERENTIAL LAW OBTAINED DIRECTLY

CAN GO BACK TO INTEGRAL FORM

$\int$ OVER VOLUME $\rightarrow$ USE DIVERGENCE THEOREM

$$\int_V \bar{\nabla}\cdot\bar{E}\,d\tau = \oint_S \bar{E}\cdot d\bar{a} = \frac{1}{\varepsilon_0}\int \rho\,d\tau = \frac{1}{\varepsilon_0} Q_{ENC}$$

BACK TO: $\displaystyle\int_V \bar{\nabla}\cdot\bar{E}\,d\tau = \frac{1}{\varepsilon_0} Q_{ENC}$

CONSISTENT.

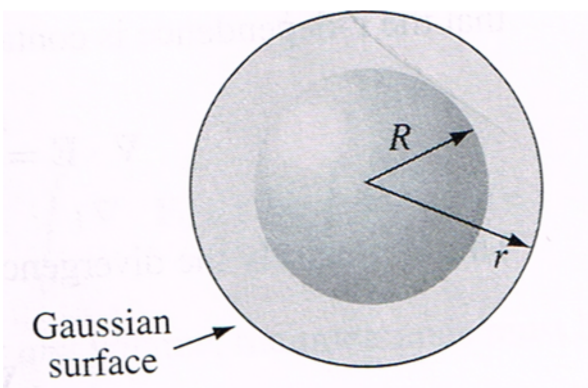
APPLICATIONS OF GAUSS'S LAW

IN SITUATIONS OF SYMMETRY

QUICK, EASY $\longrightarrow$ E

EG. 2.3 FIND FIELD OUTSIDE UNIFORMLY CHARGED SOLID SPHERE

GAUSSIAN SURFACE IS AN IMAGINARY SPHERICAL SURFACE



$$\int \bar{E} \cdot d\bar{a} = \frac{Q_{ENC}}{\varepsilon_0}$$

RADIAL DUE TO SPHERICAL SYMMETRY

$$\int_S \bar{E} \cdot d\bar{a} = \int |\bar{E}| \, da$$

$$\int \bar{E} \cdot d\bar{a} = \int |\bar{E}| \, da$$

SURFACE ELEMENT

CONSTANT OVER SURFACE

$$\int |\bar{E}| \, da = |\bar{E}| \int da = |\bar{E}| \, 4\pi r^2 = \frac{1}{\varepsilon_0} q$$

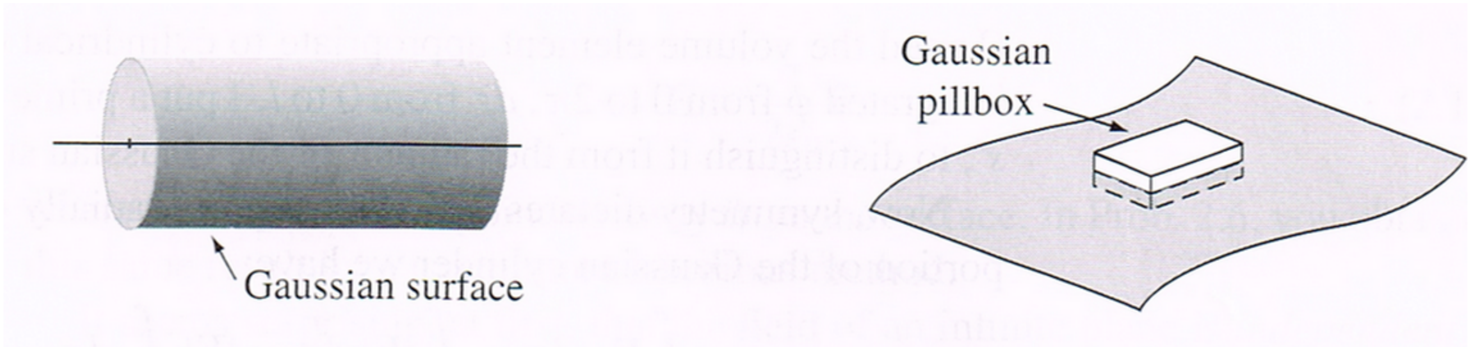
$$q = Q_{ENC}$$

$$\bar{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

MAKE IT A VECTOR

FIELD OUTSIDE SPHERE IS EXACTLY SAME AS IF ALL THE CHARGE WAS CONCENTRATED AT CENTRE OF SPHERE

$\Rightarrow$ SAME IS TRUE FOR GRAVITATIONAL FIELD OF A MASSIVE SPHERE



GAUSS'S LAW → ALWAYS TRUE

USEFUL IF HAVE SYMMETRY

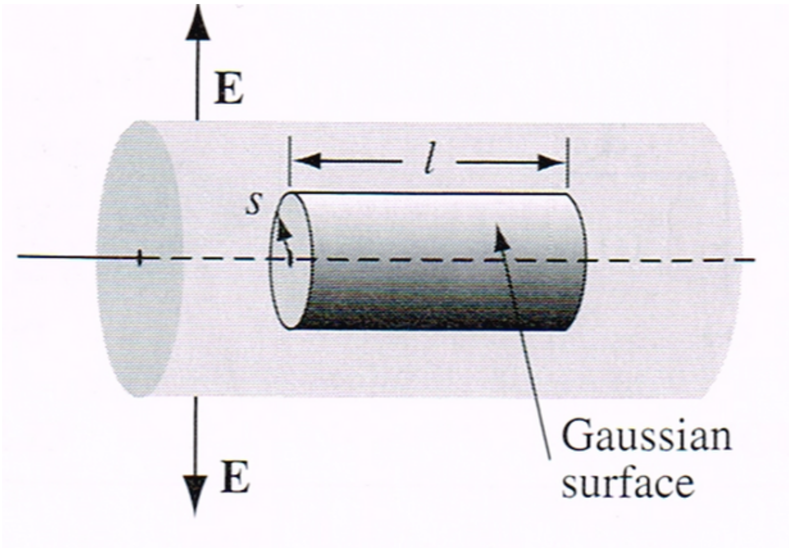
↳ $\int |E| da$

1) SPHERICAL SYMMETRY CONCENTRIC SPHERE

2) CYLINDRICAL SYMMETRY COAXIAL CYLINDER

3) PLANE SYMMETRY GAUSSIAN PILLBOX

LONG = ∞



LONG CYLINDER HAS CHARGE DENSITY $\rho = ks$

DISTANCE FROM AXIS

$$\oint_S \bar{E} \cdot d\bar{a} = \frac{1}{\varepsilon_0} Q_{ENC}$$

$$Q_{ENC} = \int \rho \, d\tau = \int \underbrace{(ks')}_{\rho} (s' ds' \, d\phi \, dz)$$

CYLINDRICAL COORDS

$\int = 2\pi$ $\int = l$

$$= 2\pi k l \int_0^s s'^2 \, ds' = \frac{2}{3} \pi k l s^3 \leftarrow Q_{ENC}$$

SYMMETRY $\rightarrow$ $\bar{E}$ POINTS RADIALLY OUT

FOR CURVED PART OF GAUSSIAN SURFACE

$$\int \bar{E} \cdot d\bar{a} = \int |\bar{E}| \, da = |\bar{E}| \, 2\pi s \ell$$

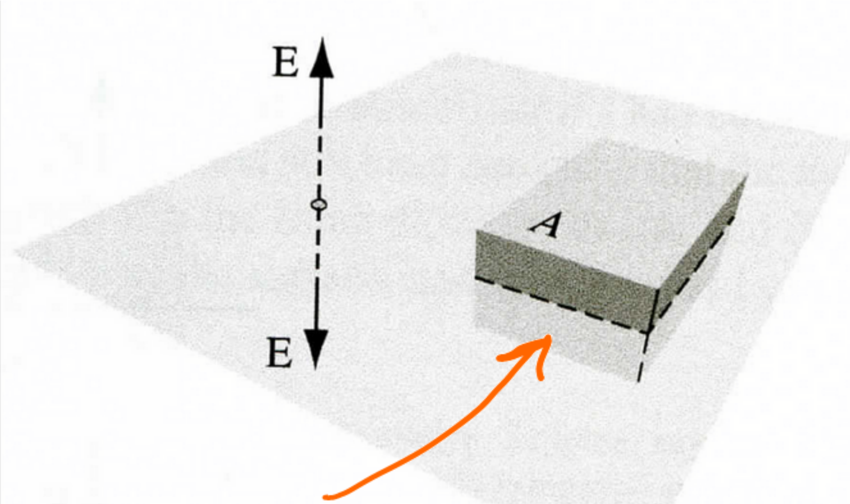
FIELD DOESN'T PASS THRU ENDS

↳ NO CONTRIBUTION

$$\int \bar{E} \cdot d\bar{a} = Q_{ENC} / \varepsilon_0$$

$$|\bar{E}| \, 2\pi s \ell = \frac{1}{\varepsilon_0} \frac{2}{3} \pi k \ell s^3$$

$$\bar{E} = \frac{1}{3\varepsilon_0} k s^2 \hat{s}$$



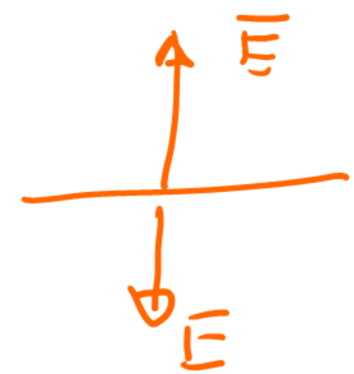
EQUAL DISTANCES
ABOVE & BELOW

AN INFINITE PLANE CARRIES A UNIFORM SURFACE CHARGE σ

$$\oint \bar{E} \cdot d\bar{a} = \frac{1}{\varepsilon_0} Q_{ENC}$$

$$Q_{ENC} = \sigma A$$

BY SYMMETRY ELECTRICAL FIELD



$$\int \bar{E} \cdot d\bar{a} = 2A|\bar{E}|$$

$$2A|\bar{E}| = \sigma A / \varepsilon_0 \rightarrow \bar{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

SIDES CONTRIBUTE NOTHING

? WHY IS FIELD INDEP OF DISTANCE $1/r^2$?