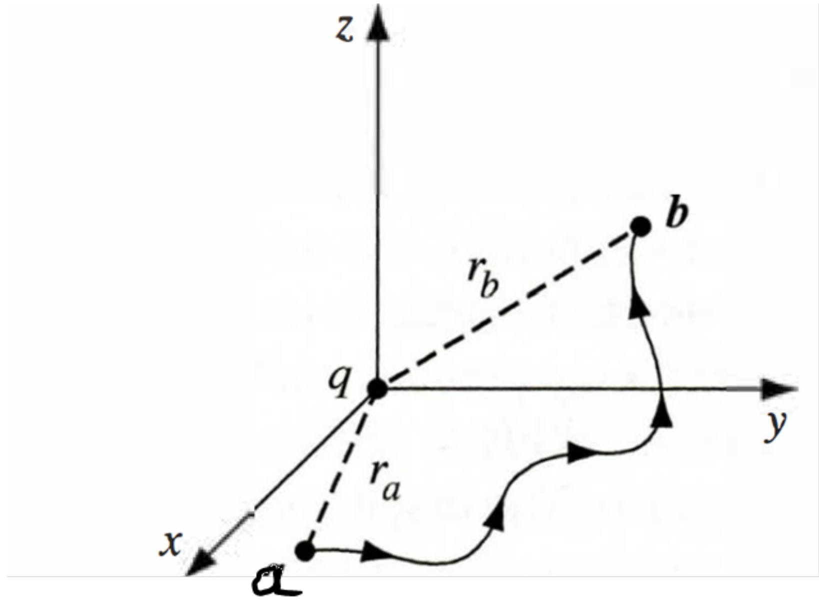
THE CURL OF $\bar{E}$ $\quad \bar{\nabla} \times \bar{E}$

POINT CHARGE AT ORIGIN

$$\bar{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$



$$\int_{\bar{a}}^{\bar{b}} \bar{E} \cdot d\bar{\ell}$$

$$d\bar{\ell} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

ONLY $\hat{r}\cdot\hat{r} = 1$

$$\bar{E} \cdot d\bar{\ell} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}\,dr$$

$$\int_{\bar{a}}^{\bar{b}} \bar{E} \cdot d\bar{\ell} = \frac{1}{4\pi\varepsilon_0} \int_{\bar{a}}^{\bar{b}} q/r^2 \cdot dr = \frac{-1}{4\pi\varepsilon_0} \frac{q}{r} \Bigg|_{r_a}^{r_b}$$

$$= \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{r_a} - \frac{q}{r_b} \right)$$

$$\int_{\bar{a}}^{\bar{b}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

IF $r_a = r_b \rightarrow$ CLOSED LOOP $\oint \bar{E}\cdot d\bar{\ell} = 0$

STOKES $\int_S \tilde{\nabla}\times\bar{v} = \oint_P \tilde{v}\cdot d\bar{\ell}$

SO $\vec{\nabla}\times\bar{E} = 0$ $\leftarrow$ MAKES NO REFERENCE TO ORIGIN $\therefore$ TRUE ANYWHERE

SUPERPOSTION $\Rightarrow$

$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \cdots\cdots$$

$$\bar{\nabla}\times\bar{E} = \bar{\nabla}\times(\bar{E}_1 + \bar{E}_2 + \cdots) = \bar{\nabla}\times\bar{E}_1 + \bar{\nabla}\times E_2 + \cdots$$

$\oint \bar{E}\cdot d\bar{\ell} = 0$, $\vec{\nabla}\times\bar{E} = 0$ TRUE FOR ANY <u>STATIC</u> CHARGE DISTRIBUTION

ELECTRIC POTENTIAL

$\bar{E} \rightarrow$ SPECIAL VECTOR

$\bar{\nabla} \times \bar{E} = 0$

$\oint \bar{E} \cdot d\bar{l} = 0$

USE TO REDUCE VECTOR PROBLEMS $\rightarrow$ SCALAR

SAW THAT IF $\bar{\nabla} \times \bar{V} = 0$ $\quad \bar{V} =$ GRADIENT (SCALAR)

$\oint \bar{E} \cdot d\bar{l} = 0 \Rightarrow$ LINE $\int$ IS SAME FOR ALL PATHS

ELSE $\bar{a} \rightarrow \bar{b} \neq \bar{b} \rightarrow \bar{a}$ $\quad \oint \bar{E} \cdot d\bar{l} \neq 0$

CAN DEFINE $V(\bar{r}) = \int_{O}^{\bar{r}} \bar{E} \cdot d\bar{l}$

SOME REFERENCE POINT

V DEPENDS ONLY ON $\bar{r} \rightarrow$ ELECTRIC POTENTIAL

POTENTIAL DIFFENCE BETWEEN TWO POINTS

$$V(\bar{b}) - V(\bar{a}) = -\int_{\mathcal{O}}^{\bar{b}} \bar{E}\cdot d\bar{\ell} + \int_{\mathcal{O}}^{\bar{a}} \bar{E}\cdot d\bar{\ell}$$

← LETTER $\mathcal{O}$ ORIGIN

$$= \int_{\mathcal{O}}^{\bar{b}} \bar{E}\cdot d\bar{\ell} - \int_{\bar{a}}^{\mathcal{O}} \bar{E}\cdot d\bar{\ell} = -\int_{\bar{a}}^{\bar{b}} \bar{E}\cdot d\bar{\ell}$$

FUNDEMENTAL THEOREM FOR GRADIENTS

$$V(\bar{b}) - V(\bar{a}) = \int_{\bar{a}}^{\bar{b}} (\bar{\nabla}\cdot V)\cdot d\bar{\ell}$$

$$\text{SO} \int_{\bar{a}}^{\bar{b}} (\bar{\nabla} V)\cdot d\bar{\ell} = -\int_{\bar{a}}^{\bar{b}} \bar{E}\cdot d\bar{\ell}$$

$$\bar{E} = -\bar{\nabla} V$$

$$E = -\bar{\nabla} V$$

PATH INDEPENDENDCE $\longrightarrow$ CRUCIAL!

IF $\int \bar{E} \cdot d\bar{l}$ DEPENDED ON PATH

$$V(\bar{r}) = -\int_{O}^{r} \bar{E} \cdot d\bar{l}$$ WOULD NOT DEFINE A UNIQE FUNCTION

ADVANTAGE OF POTENTIAL

KNOW $V \longrightarrow \bar{E}$

SCALAR

VECTOR

COMPONENTS OF $\bar{E}$ $\longrightarrow$ $\bar{\nabla} \times \bar{E} = 0$ $\longrightarrow$ RELATES COMPONENTS of $\bar{E}$

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \; ; \quad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z} \; ; \quad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

POTENTIAL FORMULATION EXPLOITS RELATION BETWEEN COMPONENTS OF $\bar{E}$ TO TURN A VECTOR PROBLEM → SCALAR → FORGET ABOUT COMPONENTS

REFERENCE POINT IS ARBITRARY

↳ CHANGING REFERENCE POINT IS EQUIVALENT TO ADDING CONSTANT TO V

$$V \rightarrow V + k$$

$$V'(\bar{r}) = -\int_{O'}^{\bar{r}} \bar{E} \cdot d\bar{l} = -\int_{O'}^{O} \bar{E} \cdot d\bar{l} - \int_{O}^{\bar{r}} \bar{E} \cdot d\bar{l} = k + V(\bar{r})$$

ADDING A CONSTANT DOES NOT EFFECT POTENTIAL DIFFERENCE

$$V'(\bar{b}) - V'(\bar{a}) = V(\bar{b}) - V(\bar{a})$$

FROM $V(\bar{b}) - V(\bar{a}) = -\int_{0}^{\bar{b}} \bar{E}\cdot d\bar{l} - \int_{\bar{a}}^{0} = -\int_{\bar{a}}^{\bar{b}} \bar{E}\cdot d\bar{l}$

CAN SEE THAT POTENTIAL DIFF INDEP OF PATH

GRADIENT IS ALSO NOT AFFECTED BY ADDING A CONSTANT TO POTENTIAL

$$\bar{\nabla} V' = \bar{\nabla} V \quad \text{SINCE} \quad \bar{\nabla} k = 0$$

POTENTIAL HAS NO PHYSICAL SIGNIFICANCE

cf DENVER $\leftrightarrow$ TORONTO $\rightarrow$ USE SEA LEVEL AS REFERENCE

NATURAL POINT TO REFER POTENTIAL TO IS POINT ∞LY FAR FROM CHARGE

EXCEPTION IS WHEN CHARGE EXTENDS TO ∞

POTENTIAL BLOWS UP

UNIFORMLY CHARGE CHARGED PLANE

$$E = \left(\frac{\sigma}{2\varepsilon_0}\right)\hat{n}$$ ← WE SHOWED THIS

THEN AT z ABOVE PLANE

$$V(z) = -\int_{\infty}^{z} \frac{1}{2\varepsilon_0}\sigma\, dz = \frac{-1}{2\varepsilon_0}\sigma(z-\infty)$$

JUST CHOOSE DIFFERENT REF. POINT.

POTENTIAL OBEYS SUPERPOSITION

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots$$

DIVIDE BY Q

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots$$

$\int$ COMMON REF

$$V = \underbrace{V_1 + V_2 + \cdots}_{\text{SCALAR SUM}}$$

SCALAR SUM, $\vec{E}$ IS VECTOR SUM

UNITS

FORCE → NEWTONS

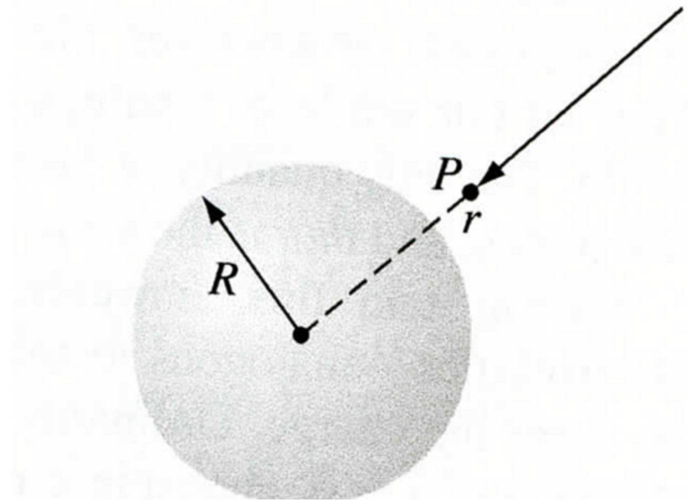
CHARGE → COULOMBS

ELECTRIC FIELD → NEWTONS PER COULOMB

POTENTIAL → NEWTON - METERS PER COULOMB

→ $\underbrace{\text{JOULES PER COULOMB}}_{\text{VOLT}}$

EXAMPLE: POTENTIAL INSIDE AND OUTSIDE A SPHERICAL SHELL OF RADIUS R UNIFORM SURFACE CHARGE



FIELD INSIDE $= 0 \leftarrow Q_{ENC} = 0$

OUTSIDE $r > R$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \rightarrow V(r) = \int_{\mathcal{O}}^{r} \vec{E}\cdot d\vec{l}$$

$$= \frac{-1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}\Big|_{\infty}^{r}$$

POTENTIAL $\sim \frac{1}{r}$

E-FIELD $\sim \frac{1}{r^2}$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

POTENTIAL INSIDE BREAK $\int \bar{E} \cdot d\bar{l}$ INTO $\underbrace{\text{TWO PIECES}}$

$r < R$

USE FIELD IN EACH REGION

$$V(r) = \frac{-1}{4\pi\varepsilon_0} \int_{\infty}^{R} \frac{q}{r'^2}\, dr' - \int_{R}^{r} (0)\, dr'$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q}{r'} \Big|_{\infty}^{R} + 0 = \frac{1}{4\pi\varepsilon_0} \frac{q}{R}$$

r OUTSIDE

POTENTIAL INSIDE SPHERE NOT ZERO

V IS CONSTANT INSIDE $\bar{\nabla} V = 0$

- CANNOT DERIVE V INSIDE FROM FIELD INSIDE
  V INSIDE IS SENSITIVE TO WHAT IS OUTSIDE
- PLACE SECOND UNIFORMLY CHARGED SPHERE
  OUTSIDE ORIGINAL $R' > R$

→ POTENTIAL INSIDE R CHANGES

FIELD INSIDE STILL ZERO

FIELD $\Rightarrow$ GAUSS'S LAW SAYS THAT CHARGE EXTERIOR TO A GIVEN POINT PRODUCES NO $\vec{E}$-FIELD AT THAT POINT

POTENTIAL $\Rightarrow$ CHANGES IF CHARGE IS ADDED BEYOND POINT

FOR $\sigma \rightarrow \infty$

CYLINDRICAL OR SPHERICAL SYMMETRY

POISSON'S & LAPLACE'S EQUATION

ELECTRIC FIELD CAN BE WRITTEN

GRADIENT (SCALAR POTENTIAL)

$$\bar{E} = -\bar{\nabla} V$$

IN TERMS OF V → WHAT DO DIV & CURL $\bar{E}$ LOOK LIKE?

$$\bar{\nabla} \cdot \bar{E} = \rho/\varepsilon_0 \qquad \bar{\nabla} \times \bar{E} = \bar{0}$$

$$\bar{\nabla} \cdot \bar{E} = \bar{\nabla}(-\bar{\nabla} V) = -\nabla^2 V$$

$$\nabla^2 V = -\rho/\varepsilon_0 \qquad \text{POISSON}$$

$\rho = 0$

$$\nabla^2 V = 0 \qquad \text{LAPLACE}$$

$$\bar{\nabla} \times \bar{E} = \bar{\nabla} \times (-\bar{\nabla} V) = \bar{0}$$

NOT A CONDITION ON V $\rightarrow$ $\bar{\nabla} \times (\bar{\nabla})$ ALWAYS ZERO

$\bar{\nabla} \times \bar{E} = \bar{0}$ LEADS TO $\bar{E} = -\bar{\nabla} V$

$\bar{E} = -\bar{\nabla} V$ GUARANTEES $\bar{\nabla} \times \bar{E} = 0$

POISSON $\rightarrow$ ONLY ONE DIFFERENTIAL EQUATION TO DETERMINE V

SCALAR

$\rightarrow$ NEED TWO FOR $\bar{E}$

VECTOR

$$\bar{\nabla} \cdot \bar{E} = \rho / \varepsilon_0$$

$$\bar{\nabla} \times \bar{E} = \bar{0}$$

POTENTIAL of LOCALIZED CHARGE DISTRIBUTION

V DEFINED IN TERMS of $\bar{E}$

USUALLY DON'T KNOW $\bar{E}$ → WANT TO FIND IT

EASIER TO FIND V (SCALAR) → GET $\bar{E}$ FROM $-\bar{\nabla} V$

TYPICALLY → KNOW CHARGE

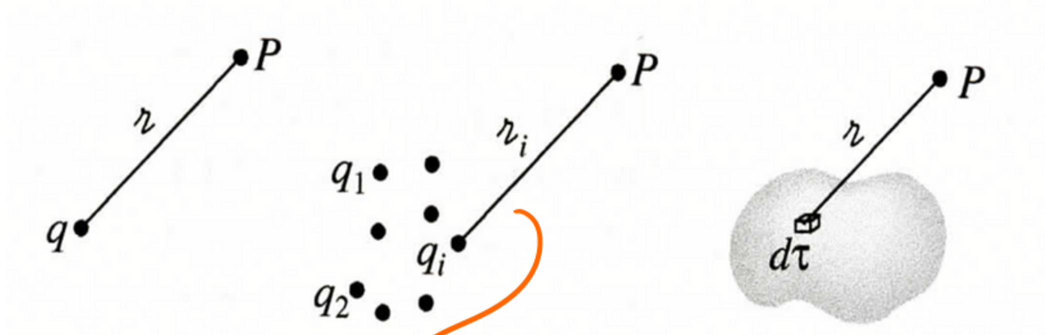
USE POISSON $\nabla^2 V = -\rho/\varepsilon_0$ ← $\underbrace{\text{GIVE } \rho \text{ IF KNOW } V}_{\text{IN VERT}}$

START WITH POINT CHARGE AT ORIGIN

$$\bar{E} = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \hat{r}$$

$$d\bar{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\bar{E} \cdot d\bar{l} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$



DISTANCE FROM CHARGE TO "TEST" POINT

$$\bar{E}\cdot d\bar{\ell} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$ → PUT REF. POINT → ∞

$$V(r) = -\int_{\mathcal{O}}^{r}\bar{E}\cdot d\bar{\ell} = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}\Big|_{\infty}^{r}$$

LOWER LIMIT → 0

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$ → +VE CHARGE

+VE POTENTIAL

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}}$$

SUPERPOSITION → POTENTIAL OF A COLLECTION OF CHARGES

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i} \xrightarrow{\text{CONTINUOUS}} V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{\mathscr{r}}\,dq$$

VOLUME

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\bar{r}')}{\mathscr{r}}\,d\tau'$$

KNOW ρ → GET $V(r)$

SOLUTION OF POISSON'S EQUATION

cf $\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\bar{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau'$ $\quad V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\bar{r}')}{\mathscr{r}} d\tau'$

VANISHES

FORGET COMPONENTS OF $\bar{E}$

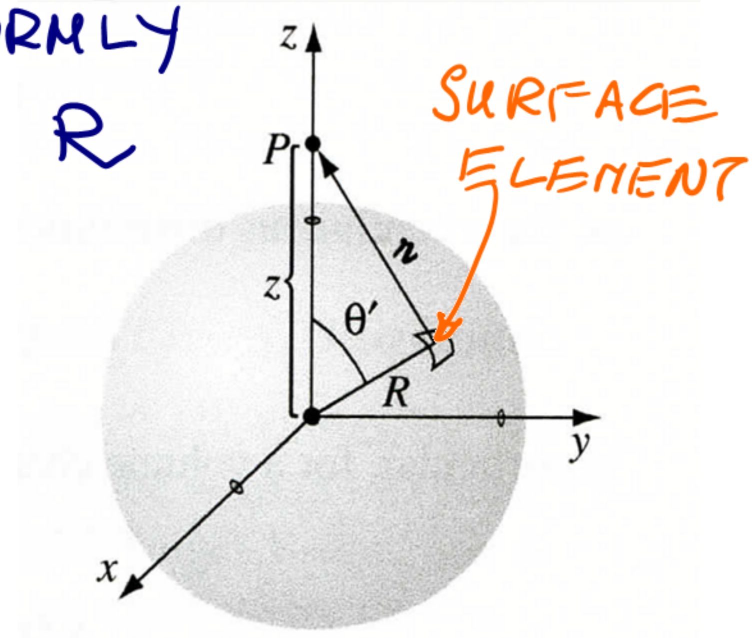
$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\bar{r}')}{\mathscr{r}} dl'$$ LINE CHARGE

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\bar{r}')}{\mathscr{r}} da'$$ SURFACE CHARGE

ALL THESE IMPLICITLY USE $\mathcal{O} \to \infty$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$ CAME FROM $\int_{\infty}^{r}$

EXAMPLE: FIND POTENTIAL of UNIFORMLY CHARGED SPHERICAL SHELL RAIDIUS R



$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma}{\mathscr{r}}\, da'$$

$$|\bar{C}|^2 = |\bar{A}|^2 + |\bar{B}|^2 - 2|\bar{A}||\bar{B}|\cos\theta$$

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'$$

ELEMENT OF SURFACE $\rightarrow R^2 \sin\theta'\, d\theta'\, d\phi'$

$$4\pi\varepsilon_0 V(z) = \sigma \int \frac{R^2 \sin\theta'\, d\theta'\, d\phi'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

$$= \sigma \int_0^{2\pi} d\phi' \int_0^{\pi} \frac{R^2 \sin\theta'\, d\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

$$= 2\pi\sigma R^2 \int_0^{\pi} \frac{\sin\theta'\, d\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

THIS IS FORM $\displaystyle\int \frac{\sin\theta\, d\theta}{\sqrt{a-b\cos\theta}}$

PUT $u = a - b\cos\theta$, $du = b\sin\theta\, d\theta$

$$\int \frac{\sin\theta}{\sqrt{u}} \cdot \frac{du}{b\sin\theta} = \frac{1}{b}\int u^{-1/2}\, du$$

$$= \frac{2}{b} u^{1/2} = \frac{2}{b}\sqrt{a-b\cos\theta}$$

$$\int \rightarrow \frac{2}{2Rz}\sqrt{R^2+z^2-2Rz\cos\theta}$$

$$\rightarrow 2\pi R^2\sigma \left(\frac{1}{Rz}\sqrt{R^2+z^2-2Rz\cos\theta} \right)\Bigg|_0^{\pi}$$

$$= \frac{2\pi R}{z}\sigma\left(\sqrt{R^2+z^2+2Rz} - \sqrt{R^2+z^2-2Rz}\right)$$

$$= \frac{2\pi R}{z}\sigma\left(\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right.$$

TAKE +VE $\sqrt{}$ FOR POINT OUTSIDE SPHERE

$\sqrt{(R-z)^2} = z - R$ $\quad z > R$

FOR POINTS INSIDE SPHERE $z < R$

SO $\sqrt{(R-z)^2} = R - z$

OUTSIDE $V(z) = \frac{R\sigma}{2\varepsilon_0 z}\left[z + R - (z - R)\right] = \frac{R^2\sigma}{2\varepsilon_0 z}$

INSIDE $V(z) = \frac{R\sigma}{2\varepsilon_0 z}\left[z + R - (R - z)\right] = \frac{R\sigma}{\varepsilon_0}$.

TOTAL CHARGE ON SPHERE $= q = 4\pi R^2 \sigma$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \quad r \geq R$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \quad r \leq R$$