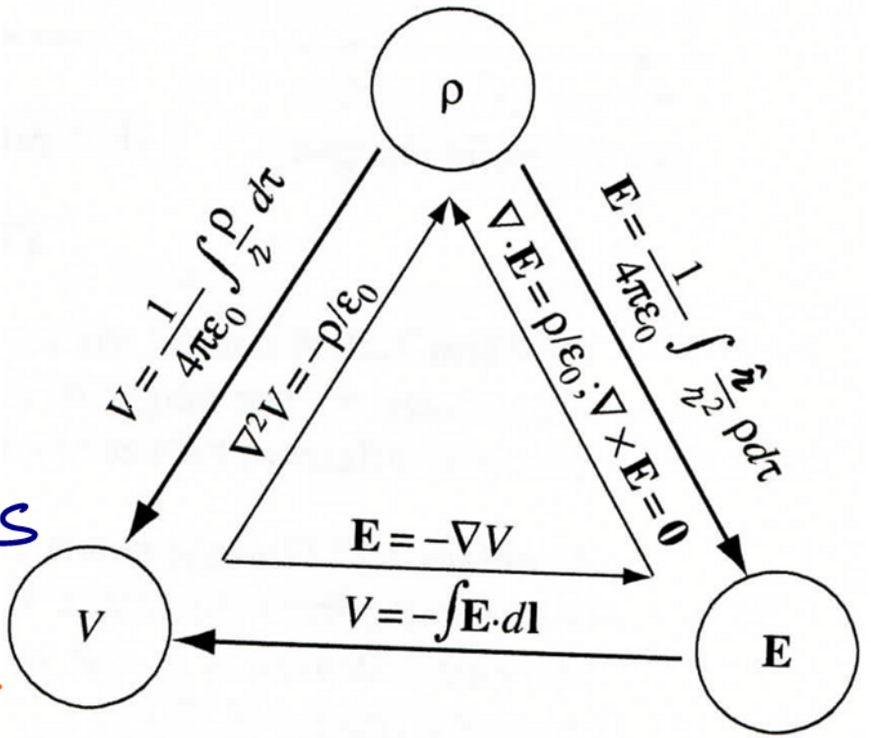
BOUNDARY CONDITIONS

TYPICALLY — KNOW ρ
WANT $\vec{E}$

IF NO SYMMETRY
↳ BEST FIND POTENTIAL

3 FUNDAMENTAL QUANTITIES
IN ELECTROSTATICS

ρ, $\vec{E}$, V



SIX FORMULAS RELATING THEM

THESE FORMULAE COME FROM 2 EXPERIMENTAL FACTS

1) SUPERPOSITION

2) COULOMBS LAW

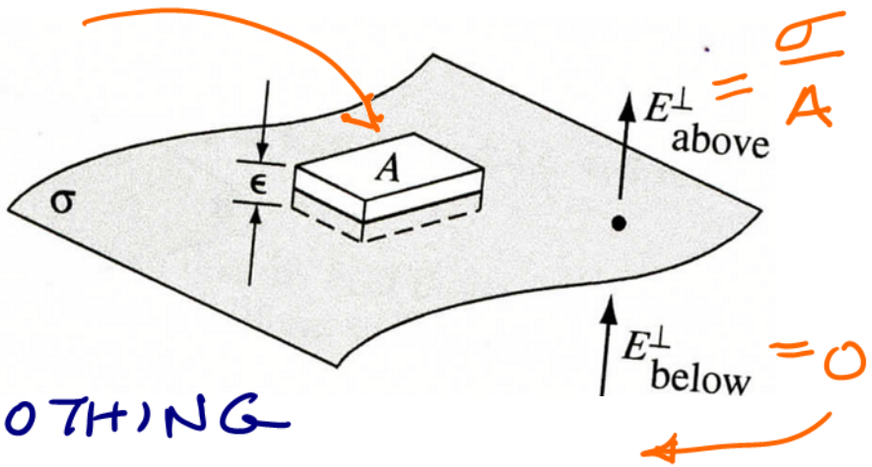
ELECTRIC FIELD ALWAYS UNDERGOES DISCONTINUITY WHEN YOU CROSS A SURFACE CHARGE

CAN FIND AMOUNT OF CHANGE IN E AT BOUNDARY

VERY THIN GAUSSIAN PILL BOX

GAUSS'S LAW

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$



SIDES OF BOX CONTRIBUTE NOTHING

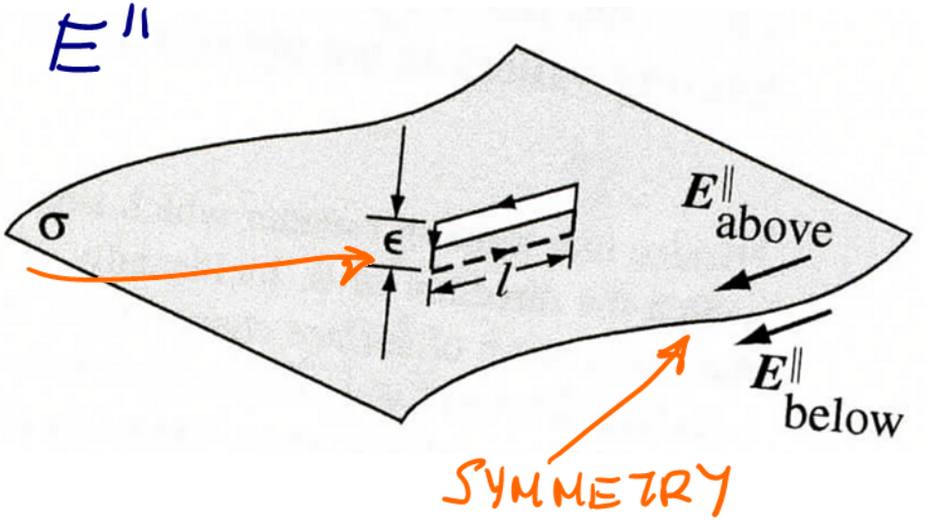
AS THICKNESS OF BOX $\longrightarrow$ 0

$$E^{\perp}_{ABOVE} - E^{\perp}_{BELOW} = \frac{\sigma}{\varepsilon_0}$$

NORMAL COMPONENT OF $\vec{E}$ IS DISCONTINUOUS BY AN AMOUNT σ/ε_0 AT ANY BOUNDARY

$\longrightarrow$ NO SURFACE CHARGE $E^{\perp}$ CONTINUOUS

TANGENTIAL COMPONENT $E^{\parallel}$ IS ALWAYS CONTINUOUS



THIN RECTANGULAR LOOP

$$\oint \bar{E} \cdot d\bar{\ell} = 0$$

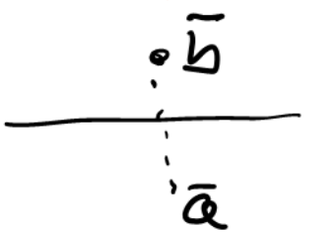
ENDS GIVE NOTHING, SIDES $E^{\parallel}_{ABOVE} = E^{\parallel}_{BELOW}$

BOUNDARY CONDITION COMBINED

$$\bar{E}_{ABOVE} - \bar{E}_{BELOW} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

POTENTIAL IS CONTINUOUS ACROSS BOUNDARY

$$V_{ABOVE} - V_{BELOW} = -\int_a^b \bar{E} \cdot d\bar{\ell}$$



AS $d\ell \to 0$ $\quad V_{ABOVE} \longrightarrow V_{BELOW}$

$$V_{ABOVE} = V_{BELOW}$$

GRADIENT OF V DISCONTINUOUS $\longrightarrow \vec{E}$

$$\vec{E} = -\vec{\nabla} V$$

$$\vec{\nabla} V_{ABOVE} - \vec{\nabla} V_{BELOW} = -\frac{1}{\varepsilon_0} \sigma \hat{n}$$

WRITE $\frac{\partial V}{\partial n} = \vec{\nabla} V \cdot \hat{n}$

NORMAL DERIVATIVE
↳ RATE OF CHANGE NORMAL TO SURFACE

$$\frac{\partial V_{ABOVE}}{\partial n} - \frac{\partial V_{BELOW}}{\partial n} = -\frac{1}{\varepsilon_0} \sigma$$

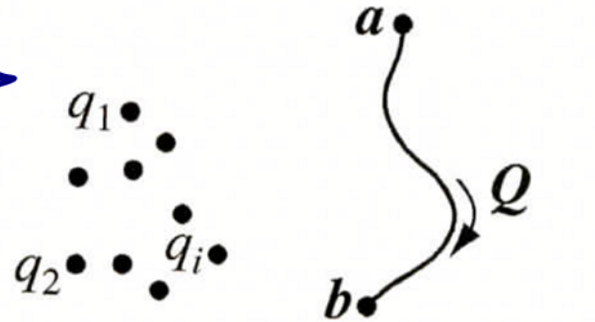
DERIVITIVES ARE LIMITING VALUES

↳ INFINITESIMALLY CLOSE TO THE BOUNDARY SURFACE

WORK & ENERGY IN ELECTROSTATICS

WORK TO MOVE A CHARGE

HOW MUCH WORK DOES IT TAKE TO MOVE FROM $a \to b$



FORCE ON CHARGE $\bar{F} = Q\bar{E}$

WORKING AGAINST FORCE $-Q\bar{E}$

$$\text{WORK} = \int_{\bar{a}}^{\bar{b}} \bar{F} \cdot d\bar{\ell} = -Q \int_{\bar{a}}^{\bar{b}} \bar{E} \cdot d\bar{\ell} = Q\left[V(\bar{b}) - V(\bar{a})\right]$$

POTENTIAL

WORK DONE → INDEP of PATH → CONSERVATIVE FORCE

$$V(\bar{b}) - V(\bar{a}) = W/Q$$

POTENTIAL DIFF BETWEE $\bar{a}$, $\bar{b}$ is WORK PER UNIT CHARGE $\bar{a} \to \bar{b}$. TO BRING CHARGE FROM FAR AWAY → N

$$W = Q\left[V(\bar{r}) - V(\infty)\right]$$

$$W = Q[V(r) - V(\infty)]$$

IF HAVE SET REFERENCE POINT AT ∞

$$V(\infty) = 0$$

$$W = QV(r)$$

POTENTIAL IS POTENTIAL ENERGY PER UNIT CHARGE

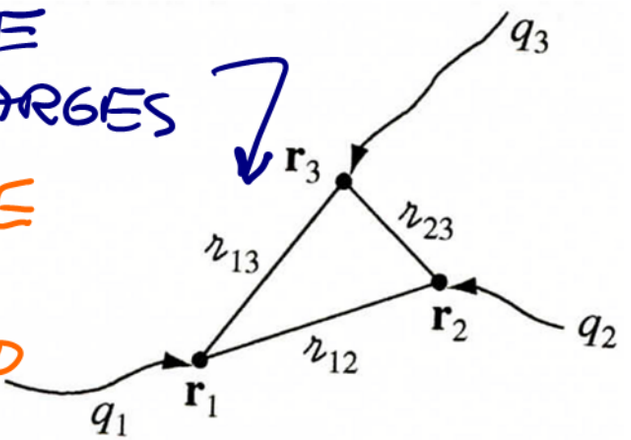
→ WORK IT TAKES TO ASSEMBLE SYSTEM

REMEMBER → FIELD IS FORCE / UNIT CHARGE

ENERGY of A POINT CHARGE DISTRIBUTION

HOW MUCH WORK DOES IT TAKE TO ASSEMBLE A GROUP OF CHARGES

BRING THEM FROM ∞ ONE BY ONE



$-q_1$ FIRST CHARGE – NO WORK NO FIELD

$-q_2$ $\quad W = QV(\vec{r}) \rightarrow W_2 = \frac{1}{4\pi\varepsilon_0} q_2 \left(\frac{q_1}{r_{12}}\right)$

$$W_3 = \frac{1}{4\pi\varepsilon_0} q_3 \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right)$$

$$W_4 = \frac{1}{4\pi\varepsilon_0} q_4 \left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}} \right)$$

$$\text{TOTAL WORK} = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}} \right)$$

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{r_{ij}}$$

DOESN'T COUNT EACH PAIR TWICE

DOES COUNT EACH PAIR TWICE

$$W = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \frac{q_i q_j}{r_{ij}}$$

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i \left(\sum_{j\neq i}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{r_{ij}} \right)$$

POTENTIAL AT q_i DUE TO ALL OTHER CHARGES

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)$$

YOU GET THIS WORK BACK IF YOU DISMANTLE ASSEMBLY

ENERGY STORED IN CONFIGURATION of CHARGES

ENERGY of CONTINUOUS CHARGE DISTRIBUTION

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\bar{r}_i) \longrightarrow \frac{1}{2}\int \rho V \, d\tau$$

← VOLUME CHARGE DENSITY

USE GAUSS LAW $\rho \rightarrow \bar{E}$

$$\rho = \varepsilon_0 \bar{\nabla}\cdot\bar{E} \longrightarrow W = \frac{\varepsilon_0}{2}\int (\bar{\nabla}\cdot\bar{E})\, V \, d\tau$$

$\int$ BY PARTS $\rightarrow$ $\int_V f(\bar{\nabla}\cdot\bar{A})\, d\tau = -\int_V \bar{A}\cdot(\bar{\nabla} f)\, d\tau + \oint f\bar{A}\cdot d\bar{a}$

$$W = \frac{\varepsilon_0}{2}\left[-\int \bar{E}\cdot(\bar{\nabla}V)\, d\tau + \oint V\bar{E}\cdot d\bar{a}\right]$$

BUT $\bar{\nabla}V = -\bar{E}$

$$W = \frac{\varepsilon_0}{2}\int_V E^2 \, d\tau + \oint_S V\bar{E}\cdot d\bar{a}$$

$$W = \frac{\varepsilon_0}{2}\int_V E^2\,d\tau + \oint_S V\vec{E}\cdot d\vec{a}$$

THIS CAME FROM

$$\frac{1}{2}\int \rho V\,d\tau$$

$\int$ OVER LARGER SPACE W DOES NOT CHANGE $\rho = 0$ OUTSIDE VOLUME

AS INCREASE VOLUME $\int_V \vec{E}$ INCREASES

$\int_S$ DECREASES → W CONSTANT

W = $\int$ CORRECT FOR ANY VOLUME

$E \sim \frac{1}{r^2}$, $V \sim \frac{1}{r}$, SURFACE $\sim r^2$

$\int$ OVER ALL SPACE → SURFACE → 0

$$W = \frac{\varepsilon_0}{2}\int_{\text{ALL SPACE}} E^2\,d\tau$$

FIND ENERGY OF UNIFORMLY CHARGED SPHERICAL SHELL q R

① USE $W = \frac{1}{2}\int \sigma V da = \frac{1}{2}\int \sigma \cdot \frac{1}{4\pi\varepsilon_0}\frac{q}{R}\, da$

$= \frac{1}{8\pi\varepsilon_0}\frac{q}{R}\int \sigma\, da = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$ ($\int \sigma\, da \to q$)

OR ②

$W = \frac{\varepsilon_0}{2}\int_{\text{ALL SPACE}} E^2 d\tau$ INSIDE SPHERE $\vec{E} = \vec{0}$

OUTSIDE $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \to E^2 = \frac{q^2}{(4\pi\varepsilon_0)^2}\cdot\frac{1}{r^4}$

$W = \frac{\varepsilon_0}{2(4\pi\varepsilon_0)^2}\int_{\text{OUTSIDE}} \left(\frac{q^2}{r^4}\right)(r^2 \sin\theta\, dr\, d\theta\, d\phi)$ SPHERICAL COORDS

$= \frac{1}{32\pi^2\varepsilon_0}\cdot q^2 \cdot 4\pi \int_R^\infty \frac{1}{r^2}\, dr = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$

($\int \theta, \phi$)

SOME COMMENTS ON ES ENERGY

$$W = \frac{\varepsilon_0}{2}\int E^2 \, \tau$$ → IMPLIES W is ALWAYS +VE

BUT $W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\bar{r}_i)$ COULD BE -VE ($q \to \pm$)

↳ DOES NOT TAKE INTO ACCOUNT WORK TO CREATE CHARGE

↳ JUST WORK TO BRING THEM TOGETHER

$$W = \frac{\varepsilon_0}{2}\int E^2 d\tau$$ IMPLIES ENERGY OF POINT CHARGE IS ∞

$$W = \frac{\varepsilon_0}{2}\int \left(\frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\right)^2 r^2 \sin\theta \, dr\, d\theta\, d\phi = \frac{\varepsilon_0 q^2}{2(4\pi\varepsilon_0)^2}\int \frac{1}{r^2}\underbrace{\sin\theta \, d\theta\, d\phi}_{4\pi}\, dr$$

$$= \frac{\varepsilon_0 q^2 4\pi}{2(4\pi\varepsilon_0)^2}\int_0^\infty \frac{1}{r^2}\, dr \to \infty$$

∞ SELF ENERGY OF ELECTRON PUZZLED MANY GREAT 19TH CENTURY PHYSICISTS

↳ RENORMALIZATION IN QCD

FOR US → CHARGES GIVEN TO US

↳ WE JUST MOVE THEM AROUND

↳ ∞ SELF ENERGY IRRELVANT

ENERGY IS STORED IN FIELD

$$\left(\begin{array}{c}\text{ENERGY PER}\\ \text{UNIT VOLUME}\end{array}\right) = \frac{\varepsilon_0}{2} E^2$$

SUPERPOSITION & ENERGY

ELECTROSTATIC ENERGY QUADRATIC IN $\vec{E}$

→ DOES NOT OBEY SUPERPOSITION

ENERGY SYSTEM $\neq \sum$ OF PARTS

$$(E_1 + E_2)^2 = E_1^2 + E_2^2 + 2E_1E_2$$

$$W_{TOT} = \frac{\varepsilon_0}{2}\int E^2 d\tau = \frac{\varepsilon_0}{2}\int (\vec{E}_1+\vec{E}_2)^2 d\tau$$

$$= \frac{\varepsilon_0}{2}\int (E_1^2 + E_2^2 + 2\vec{E}_1\cdot\vec{E}_2)\, d\tau$$

$$= W_1 + W_2 + \varepsilon_0\int \vec{E}_1\cdot\vec{E}_2\, d\tau$$

DOUBLE CHARGE → QUADRUPLE ENERGY