1)

gram bisect each other. Consider the parallelogram $PQRS$ (see Fig. 19.5). Let the midpoint of PR be M, and the mid-point of QS

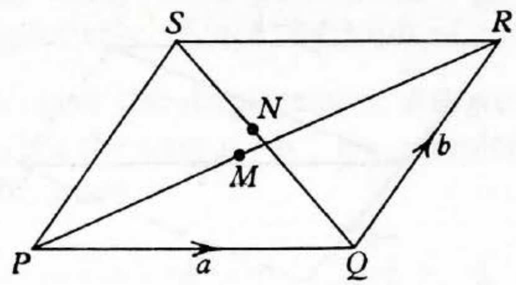


Fig. 19.5

be N. Then writing $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{QR} = \mathbf{b}$ we have

$$\overrightarrow{PN} = \overrightarrow{PQ} + \overrightarrow{QN} = \overrightarrow{PQ} + \tfrac{1}{2}\overrightarrow{QS} \tag{10}$$

$$= \mathbf{a} + \tfrac{1}{2}(\mathbf{b}-\mathbf{a}) = \tfrac{1}{2}(\mathbf{a}+\mathbf{b}). \tag{11}$$

Similarly

$$\overrightarrow{PM} = \tfrac{1}{2}\overrightarrow{PR} = \tfrac{1}{2}(\mathbf{a}+\mathbf{b}). \tag{12}$$

Consequently $\overrightarrow{PM} = \overrightarrow{PN}$; M and N are therefore the same point and the diagonals PR, QS bisect each other.

2)

Example 2. To find the angle between the vectors

$$\mathbf{r}_1 = \mathbf{i}+\mathbf{j}+\mathbf{k}, \quad \mathbf{r}_2 = -\mathbf{i}+2\mathbf{j}+3\mathbf{k}. \tag{44}$$

Here $|\mathbf{r}_1| = \sqrt{3}$, $|\mathbf{r}_2| = \sqrt{14}$. Consequently

$$\hat{\mathbf{r}}_1 = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}) \quad \text{and} \quad \hat{\mathbf{r}}_2 = \frac{1}{\sqrt{14}}(-\mathbf{i}+2\mathbf{j}+3\mathbf{k}) \tag{45}$$

and

$$\cos\theta = \hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2 = \frac{1}{\sqrt{42}}(-1+2+3) = \frac{4}{\sqrt{42}}. \tag{46}$$

3)

Here

$$\hat{\mathbf{r}}_1 = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}),$$

$$\hat{\mathbf{r}}_2 = \frac{1}{\sqrt{14}}(-\mathbf{i}+2\mathbf{j}+3\mathbf{k}), \tag{59}$$

whence

$$\hat{\mathbf{r}}_1 \wedge \hat{\mathbf{r}}_2 = \frac{1}{\sqrt{14}}(\mathbf{i}+\mathbf{j}+\mathbf{k}) \wedge \frac{1}{\sqrt{3}}(-\mathbf{i}+2\mathbf{j}+3\mathbf{k}) \tag{60}$$

$$= \frac{1}{\sqrt{42}}(\mathbf{i}-4\mathbf{j}+3\mathbf{k}). \tag{61}$$

Now since by definition $\hat{\mathbf{r}}_1 \wedge \hat{\mathbf{r}}_2 = \sin\theta\, \hat{\mathbf{n}}$, we have

$$|\, \hat{\mathbf{r}}_1 \wedge \hat{\mathbf{r}}_2 \,| = |\, \sin\theta \,|. \tag{62}$$

Using (61)

$$|\, \hat{\mathbf{r}}_1 \wedge \hat{\mathbf{r}}_2 \,|^2 = \frac{1}{42}(\mathbf{i}-4\mathbf{j}+3\mathbf{k}) \cdot (\mathbf{i}-4\mathbf{j}+3\mathbf{k}) \tag{63}$$

$$= \frac{1}{42}(1+16+9) = \frac{26}{42}. \tag{64}$$

Hence

$$|\, \sin\theta \,| = \sqrt{\frac{26}{42}} \quad \text{and} \quad \cos\theta = \frac{4}{\sqrt{42}}, \tag{65}$$

as before.

4)

Example 4. To show that if three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ satisfy the relation $\mathbf{a}+\mathbf{b}+\mathbf{c} = 0$, then $\mathbf{a} \wedge \mathbf{b} = \mathbf{b} \wedge \mathbf{c} = \mathbf{c} \wedge \mathbf{a}$.

This is easily proved by forming the vector product

$$(\mathbf{a}+\mathbf{b}+\mathbf{c}) \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{b}+\mathbf{c} \wedge \mathbf{b} = 0. \tag{66}$$

Hence

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{c} \wedge \mathbf{b} = \mathbf{b} \wedge \mathbf{c}. \tag{67}$$

The other result follows in a similar way by taking the vector product of $\mathbf{a}+\mathbf{b}+\mathbf{c}$ and $\mathbf{c}$.

5)

and **c**, and $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ is perpendicular to $\mathbf{b} \wedge \mathbf{c}$ (see Fig. 19.11). To evaluate $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ we now write $\mathbf{b} \wedge \mathbf{c} = \mathbf{f}$. Then by (55) the x, y z components of **f** are

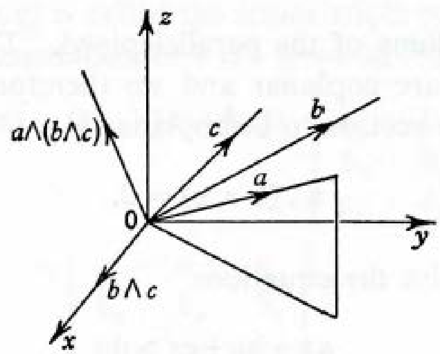


Fig. 19.11

$$f_x = \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix}, \quad f_y = \begin{vmatrix} b_z & b_x \\ c_z & c_x \end{vmatrix}, \quad f_z = \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix}. \tag{80}$$

Consequently

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \mathbf{a} \wedge \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ f_x & f_y & f_z \end{vmatrix} \tag{81}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} & \begin{vmatrix} b_z & b_x \\ c_z & c_x \end{vmatrix} & \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} \end{vmatrix} \tag{82}$$

$$\begin{aligned} &= \mathbf{i}\{a_y(b_xc_y - b_yc_x) - a_z(b_zc_x - b_xc_z)\} \\ &\quad + \mathbf{j}\{a_z(b_yc_z - b_zc_y) - a_x(b_xc_y - b_yc_x)\} \\ &\quad + \mathbf{k}\{a_x(b_zc_x - b_xc_z) - a_y(b_yc_z - b_zc_y)\} \end{aligned} \tag{83}$$

$$\begin{aligned} &= \mathbf{i}\{b_x(a_xc_x + a_yc_y + a_zc_z) - c_x(a_xb_x + a_yb_y + a_zb_z)\} \\ &\quad + \mathbf{j}\{b_y(a_xc_x + a_yc_y + a_zc_z) - c_y(a_xb_x + a_yb_y + a_zb_z)\} \\ &\quad + \mathbf{k}\{b_z(a_xc_x + a_yc_y + a_zc_z) - c_z(a_xb_x + a_yb_y + a_zb_z)\} \end{aligned} \tag{84}$$

$$= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \tag{85}$$

Similarly we find

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a}. \tag{86}$$

6)

(a) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c} \neq (\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}$.

(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \neq \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}\mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, a standard result.

(d) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d} = (\mathbf{a} \times \mathbf{b}) \cdot (\lambda \mathbf{a} + \mu \mathbf{b}) = \lambda (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} + \mu (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b}$
$= \lambda 0 + \mu 0 = 0$.

(e) $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c} \Rightarrow (\mathbf{a} - \mathbf{b}) \times \mathbf{c} = \mathbf{0} \Rightarrow \mathbf{a} - \mathbf{b} \parallel \mathbf{c}$
$\Rightarrow (\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = c\,|\mathbf{a} - \mathbf{b}| \Rightarrow \mathbf{c} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} = c\,|\mathbf{a} - \mathbf{b}|$.

(f) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{b}) = \mathbf{b}\,[\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})] - \mathbf{a}\,[\mathbf{b} \cdot (\mathbf{c} \times \mathbf{b})]$
$= \mathbf{b}\,[\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})] - 0 = \mathbf{b}\,[\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})] = -\mathbf{b}\,[\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})]$
$\neq \mathbf{b}\,[\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})]$.

Thus only (c), (d) and (e) are true.

7)

Direct substitutions and the expansion formula for a triple vector product (proved in 7.9) enable the verifications to be made as follows. We make repeated use of the general result $(\mathbf{p} \times \mathbf{q}) \cdot \mathbf{p} = 0 = (\mathbf{p} \times \mathbf{q}) \cdot \mathbf{q}$.

(a) $\quad \mathbf{a}' \cdot \mathbf{a} = \dfrac{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = 1.$ Similarly for $\mathbf{b}' \cdot \mathbf{b}$ and $\mathbf{c}' \cdot \mathbf{c}$.

(b) $\quad \mathbf{a}' \cdot \mathbf{b} = \dfrac{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = 0.$ Similarly for $\mathbf{a}' \cdot \mathbf{c}$, $\mathbf{b}' \cdot \mathbf{a}$ etc.

(c)
$$[\mathbf{a}', \mathbf{b}', \mathbf{c}'] = \frac{\mathbf{a}' \cdot \{(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})\}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2}$$
$$= \frac{\mathbf{a}' \cdot \{[\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})]\,\mathbf{a} - [\mathbf{a} \cdot (\mathbf{c} \times \mathbf{a})]\,\mathbf{b}\}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2}$$
$$= \frac{1\,[\mathbf{b}, \mathbf{c}, \mathbf{a}] - 0\,(\mathbf{a}' \cdot \mathbf{b})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2}, \text{ using results (a) and (b),}$$
$$= \frac{1}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}.$$

(d)
$$\frac{\mathbf{b}' \times \mathbf{c}'}{[\mathbf{a}', \mathbf{b}', \mathbf{c}']} = \frac{[\mathbf{b}, \mathbf{c}, \mathbf{a}]\,\mathbf{a} - 0\,\mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2\,[\mathbf{a}', \mathbf{b}', \mathbf{c}']}, \qquad \text{as in part (c),}$$
$$= \mathbf{a}, \qquad \text{from result (c).}$$

8)

1. a. (i). $\mathbf{F} = \mathbf{i}\frac{\partial}{\partial x}(xyz) + \mathbf{j}\frac{\partial}{\partial y}(xyz) + \mathbf{k}\frac{\partial}{\partial z}(xyz) = \mathbf{i}yz + \mathbf{j}xz + \mathbf{k}xy.$

(ii). $\mathbf{F} = \mathbf{i}\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + \mathbf{j}\frac{\partial}{\partial y}(x^2 + y^2 + z^2) + \mathbf{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2) =$

$= 2(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z).$

(iii). $\mathbf{F} = \mathbf{i}\frac{\partial}{\partial x}(xy + yz + xz) + \mathbf{j}\frac{\partial}{\partial y}(xy + yz + xz) + \mathbf{k}\frac{\partial}{\partial z}(xy + yz + xz)$

$= \mathbf{i}(y + z) + \mathbf{j}(x + z) + \mathbf{k}(y + x).$

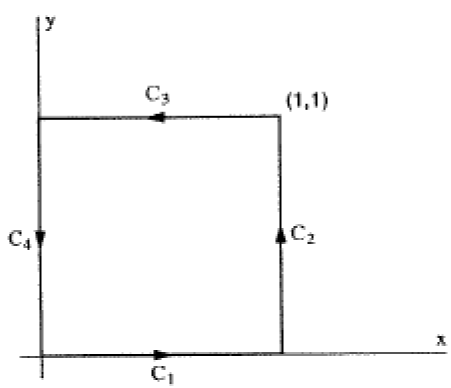
(iv). $\mathbf{F} = \mathbf{i}\frac{\partial}{\partial x}(3x^2 - 4z^2) + \mathbf{j}\frac{\partial}{\partial y}(3x^2 - 4z^2) + \mathbf{k}\frac{\partial}{\partial z}(3x^2 - 4z^2)$

$= 6\mathbf{i}x - 8\mathbf{k}z.$

(v). $\mathbf{F} = \mathbf{i}\frac{\partial}{\partial x}(e^{-x}\sin y) + \mathbf{j}\frac{\partial}{\partial y}(e^{-x}\sin y) + \mathbf{k}\frac{\partial}{\partial z}(e^{-x}\sin y)$

$= -\mathbf{i}e^{-x}\sin y + \mathbf{j}e^{-x}\cos y$

b. Using $f = x^2 + y^2 + z^2$, we have $\mathbf{F} = 2(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)$. For the square path shown in the figure, $z = 0$ so $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\oint_C (xdx + ydy)$. On C_1 $y = 0$ so $\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\int_0^1 xdx = 1$. On C_2 $dx = 0$ so $\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\int_0^1 ydy = 1$. On C_3 $dy = 0$ so $\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\int_1^0 xdx = -1$. On C_4 $\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}ds = 2\int_1^0 ydy = -1$. Hence $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}ds = 1 + 1 - 1 - 1 = 0.$



9)a)

14. (a). $\frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}z^2 = 2(x + y + z).$

(b). $\frac{\partial}{\partial x}yz + \frac{\partial}{\partial y}xz + \frac{\partial}{\partial z}xy = 0.$

(c). $\frac{\partial}{\partial x}e^{-x} + \frac{\partial}{\partial y}e^{-y} + \frac{\partial}{\partial z}e^{-z} = -(e^{-x} + e^{-y} + e^{-z}).$

(d). $\frac{\partial}{\partial x}1 + \frac{\partial}{\partial y}(-3) + \frac{\partial}{\partial z}z^2 = 2z.$

(e). $\frac{\partial}{\partial x}\left[-\frac{xy}{x^2 + y^2}\right] + \frac{\partial}{\partial y}\left[\frac{xy}{x^2 + y^2}\right] = -\frac{y}{x^2 + y^2}.$

(f). $\frac{\partial}{\partial z}\sqrt{x^2 + y^2} = 0.$

(g). $\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3.$

(h). $\frac{\partial}{\partial x}\left[-\frac{y}{\sqrt{x^2 + y^2}}\right] + \frac{\partial}{\partial y}\left[\frac{x}{\sqrt{x^2 + y^2}}\right] =$

$$xy(x^2 + y^2)^{-3/2} - xy(x^2 + y^2)^{-3/2} = 0.$$

9)b)

15. (a). In the following we evaluate the function at the center of the relevant face of the cube.

On S_1 $\mathbf{F}\cdot\hat{\mathbf{n}} = \mathbf{F}\cdot\mathbf{i} = (x_0 + s/2)^2$

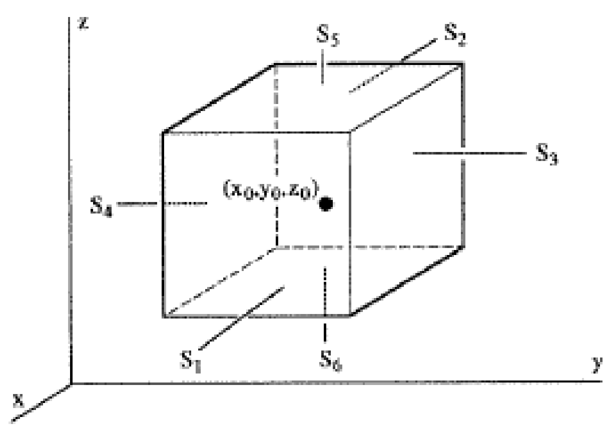
On S_2 $\mathbf{F}\cdot\hat{\mathbf{n}} = -\mathbf{F}\cdot\mathbf{i} = -(x_0 - s/2)^2$

On S_3 $\mathbf{F}\cdot\hat{\mathbf{n}} = \mathbf{F}\cdot\mathbf{j} = (y_0 + s/2)^2$

On S_4 $\mathbf{F}\cdot\hat{\mathbf{n}} = -\mathbf{F}\cdot\mathbf{j} = -(y_0 - s/2)^2$

On S_5 $\mathbf{F}\cdot\hat{\mathbf{n}} = \mathbf{F}\cdot\mathbf{k} = (z_0 + s/2)^2$

On S_6 $\mathbf{F}\cdot\hat{\mathbf{n}} = -\mathbf{F}\cdot\mathbf{k} = -(z_0 - s/2)^2$



CONT NEXT PAGE

Hence $\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\, dS + \iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\, dS \cong s^2[(x_0 + s/2)^2 - (x_0 - s/2)^2] = 2x_0 s^3$,
with analogous results for the othe two pairs of faces. Hence

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS \cong 2s^3(x_0 + y_0 + z_0).$$

9)c)

(b). The volume of the cube is $V = s^3$ so

$$(1/V)\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \frac{2s^3(x_0 + y_0 + z_0)}{s^3} = 2(x_0 + y_0 + z_0).$$

By definition this is $\nabla\cdot\mathbf{F}$ at (x_0, y_0, z_0) and it agrees with Prob. II-14(a). [Note that there is no need to calculate the limit of this expression as $s \to 0$ since the result is independent of s.]

10)a)

a. $$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & x^2 & -y^2 \end{vmatrix} = 2(-\mathbf{i}y + \mathbf{j}z + \mathbf{k}x).$$

b. $$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3xz & 0 & -x^2 \end{vmatrix} = 5\mathbf{j}x.$$

c. $$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{-y} & e^{-z} & e^{-x} \end{vmatrix} = \mathbf{i}e^{-z} + \mathbf{j}e^{-x} + \mathbf{k}e^{-y}.$$

d. $$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = \mathbf{i}(x - x) + \mathbf{j}(y - y) + \mathbf{k}(z - z) = 0.$$

e. $$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -yz & xz & 0 \end{vmatrix} = -\mathbf{i}x - \mathbf{j}y + 2\mathbf{k}z.$$

f. $$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & x^2+y^2 \end{vmatrix} = 2(\mathbf{i}y - \mathbf{j}x).$$

g. $$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & y^2 & yz \end{vmatrix} = \mathbf{i}z - \mathbf{k}x.$$

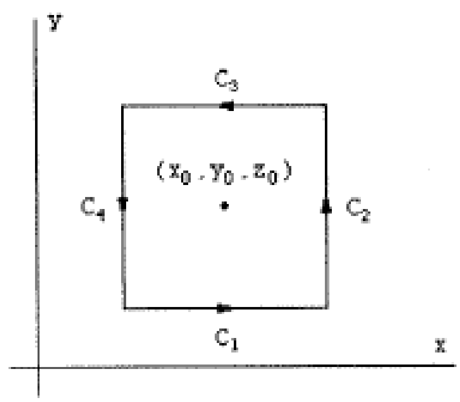
h. $$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/D & y/D & z/D \end{vmatrix} \quad \text{where } D = (x^2 + y^2 + z^2)^{3/2} \; .$$

The x component of this is

$$\frac{\partial}{\partial y}\frac{z}{D} - \frac{\partial}{\partial z}\frac{y}{D} = -\frac{3yz}{(x^2 + y^2 + z^2)^{5/2}} + \frac{3yz}{(x^2 + y^2 + z^2)^{5/2}} = 0.$$

The other two components yield 0 in the same way.

10)b)



4. a. $\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_1} z^2 ds = 0$ (because $z = 0$ on C_1),

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_2} x^2 ds = (x_0 + s/2)^2 s,$$

$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -\int_{C_3} z^2 ds = 0$ (because $z = 0$ on C_3), and

$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -\int_{C_4} x^2 ds = -(x_0 - s/2)^2 s.$$

Combining these results we find

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = [(x_0 + s/2)^2 - (x_0 - s/2)^2]s = 2x_0 s^2.$$

10)c)

b. Since the area of the square is s^2 we get

$$\frac{1}{s^2}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = 2x_0 = (\mathbf{k}\cdot\nabla\times\mathbf{F})_{(x_0,y_0,z_0)}.$$ (Note that there is no need to take the limit as $s \to 0$ because our result, $2x_0$, is independent of s.) This result agrees with Prob III3-(a).

Prob III3-(a) is 10)a)