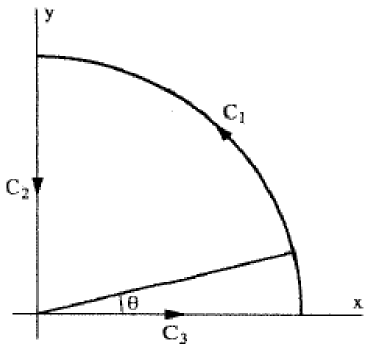
PHY250 Solutions to Prob Set#2

1)



b: $\mathbf{F} = \mathbf{i}y + \mathbf{j}z + \mathbf{k}x$. Hence $\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \int_{C_1} y dx$ (because $z = 0$ on C_1). Letting $x = \cos\theta$, $y = \sin\theta$, this integral becomes $-\int_0^{\pi/2} \sin^2\theta d\theta = -\pi/4$. The integrals over C_2 and C_3 are treated in exactly the same way and both yield $-\pi/4$. Hence $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -3\pi/4$.

A straightforward calculation gives $\nabla \times \mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$, while the equation of the surface is $z = \sqrt{1 - x^2 - y^2}$. Hence $\partial f/\partial x = -x/z$ and $\partial f/\partial y = -y/z$. We therefore have

$$\iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\; dS = \iint_R \left(-\frac{x}{z} - \frac{y}{z} - 1\right) dxdy$$

where R is the quarter circle of radius 1 lying in the xy-plane and centered at the origin. The integral can be written

$$-\iint_R \frac{x}{\sqrt{1 - x^2 - y^2}} dxdy - \iint_R \frac{y}{\sqrt{1 - x^2 - y^2}} dxdy - \iint_R dxdy.$$

But

$$\iint_R \frac{x}{\sqrt{1 - x^2 - y^2}} dxdy = \int_0^{\pi/2}\int_0^1 \frac{r^2\cos\theta\; drd\theta}{\sqrt{1 - r^2}} = \int_0^1 \frac{r^2\; dr}{\sqrt{1 - r^2}} = \frac{\pi}{4}.$$

The second integral above can be treated in exactly the same way and also yields $\pi/4$. The third integral is just the area of the quarter-circle and thus also equals $\pi/4$. It follows then that

$$\iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\; dS = -\frac{3\pi}{4},$$

in agreement with the line integral calculated above.

(2) $\vec{V} = (2r\cos\theta)\hat{r} + (3r\sin\theta)\hat{\theta} + (5r\sin\theta\cos\phi)\hat{\phi}$

DIV in spherical coords. — $\int$ volume.

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, V_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(V_\phi\right)$$

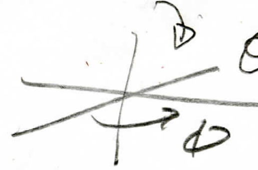
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(2r^3\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(3r\sin^2\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(5r\sin\theta\cos\phi\right)$$

$$= \frac{6r^2\cos\theta}{r^2} + \frac{1}{r\sin\theta}\cdot 3r\,2\sin\theta\cos\theta + \frac{1}{r\sin\theta}5r\sin\theta(-\sin\phi)$$

$$= 6\cos\theta + 6\cos\theta - 5\sin\phi$$

$$= 12\cos\theta - 5\sin\phi$$

(2)



$\theta \quad 0 \to \frac{\pi}{2}$

$\phi \quad 0 - 2\pi$

$$\int \vec{\nabla} \cdot \vec{V} \, d\tau = \int (12\cos\theta - 5\sin\phi)\, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^R r^2 dr \int_0^{\pi/2} \left[\int_0^{2\pi} (12\cos\theta - 5\sin\phi)\, d\phi \right] \sin\theta \, d\theta$$

$$= \frac{R^3}{3} \int_0^{\pi/2} \Big[\phi \cdot 12\cos\theta + \underbrace{5\cos\theta}_{\cos 2\pi - \cos 0 = 0} \Big]_0^{2\pi} \sin\theta \, d\theta$$

$$= \frac{R^3}{3} \int_0^{\pi/2} 24\pi \cos\theta \sin\theta \, d\theta$$

$$= \frac{24\pi R^3}{3} \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

$$= \frac{24\pi R^3}{3} \underbrace{\left[\frac{1}{2} \sin^2\theta \right]_0^{\pi/2}}_{1/2}$$

$$= \frac{12\pi R^3}{3} = 4\pi R^3$$

VOLUME

$\int$ over surface of hemisphere.

surface element $d\bar{a} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

↑ constant

ϕ $0 \to 2\pi$, θ $0 \to \frac{\pi}{2}$

$\int \bar{v} \cdot d\bar{a}$ → just $\hat{r}$ component ($2r\cos\theta$)

$$\int \bar{v} \cdot d\bar{a} = \int \underbrace{(2r\cos\theta)}_{v_r} R^2 \sin\theta \, d\theta \, d\phi$$

$$= 2R^3 \underbrace{\int_0^{\pi/2} \sin\theta\cos\theta \, d\theta}_{\text{again } \frac{1}{2}} \int_0^{2\pi} d\phi$$

(constant → R^3)

$$= 2R^3 \left(\frac{1}{2}\right) 2\pi = 2\pi R^3 \quad (*)$$

$\int$ over flat bottom, θ constant $= \frac{\pi}{2}$

$$d\bar{a} = (dr \, r \underbrace{\sin\theta}_{=1} \, d\phi) \, \hat{\theta}$$

$$= r \, dr \, d\phi$$

$$\int_{\text{flat surface}} \bar{v} \cdot d\bar{a} = \int (3 r \sin\theta) \, r \, dr \, d\phi$$

$\uparrow$ $\hat{\theta}$ comp

$$= 3 \int_0^R r^2 dr \int_0^{2\pi} d\phi = R^3 \cdot 2\pi$$

Total surface = $2\pi R^3 + 2\pi R^3$

$= 4\pi R^3$ Sigh!

SURFACE

3) a) $\int_3^7 (5x^2 - 3x - 1)\,\delta(x-4)\,dx$

4 is in range 3 →

$= (5 \cdot 16 - 3 \cdot 4 - 1) = 67$

b) $\int_{-4}^{+4} (5x+2)\,\delta(2x)\,dx$

$= \int_{-4}^{+4} (5x+2)\,\frac{1}{2}\,\delta(x)\,dx$

traps $x = 0$

$= (0+2)/2 = 1$

c) $\int_3^5 (5x^2 - 3x - 1)\,\delta(x-6)\,dx$

traps $x = 6$, outside range

$= 0$

c1) = d) $\int_0^4 (x^4 + 3x^2 + 2x)\,\delta(1-x)\,dx$

$\delta(1-x) = \delta(x-1)$

$= 1 + 3 + 2 = 6$

d)/e) $$\int_{-2}^{+2} 16x^2 \delta(4x+1)\,dx$$

$$= \int_{-2}^{+2} 16x^2 \frac{1}{4}\left(x+\frac{1}{4}\right) dx$$

$$= 16 \times \left(\frac{1}{4}\right)^2 \times \frac{1}{4} = \frac{1}{4}$$

e)/f) $$\int_{-\infty}^{d} \delta(x-c)\,dx$$

$= 1$ if $d > c$

$= 0$ if $d < c$.

NB. In some of the following solutions Gaussian units are used. You did them using SI. To go from Gaussian to SI you just put $\frac{1}{4\pi\varepsilon_0}$ in front of the fields, and change the 4π in Gauss's law into $\frac{1}{\varepsilon_0}$

4)

The four point charges q are located at the corners of a square with sides of length L. The distance from each charge to a point z above the square, on the perpendicular axis of the square, is $\sqrt{z^2 + L^2/2}$. The horizontal fields cancel, and the magnitude of the vertical field is given by

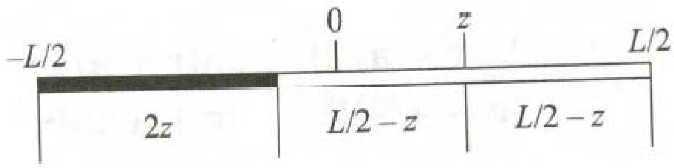
$$E_z = E\cos\theta = \frac{4qz}{(z^2 + L^2/2)^{\frac{3}{2}}}.$$

5)

(a) The field on the axis of the uniformly charged wire, a distance z from the center of the wire, is given for $z > L/2$ by

$$\begin{aligned} E_z &= \frac{Q}{L}\int_{\frac{L}{2}}^{+\frac{L}{2}} \frac{dz'}{(z - z')^2} \\ &= \frac{Q}{L}\left[\frac{1}{z - L/2} - \frac{1}{z + L/2}\right] \\ &= \frac{Q}{z^2 - L^2/4}. \end{aligned} \qquad (1.16)$$

(b) For $-L/2 < z < L/2$, the point z is a distance $(L/2 - z)$ from the end of the wire. The wire can be thought of as two parts.



The part of the wire from $z' = z - (L/2 - z) = 2z - L/2$ to $z' = L/2$ is symmetric about the point z. This means that the field due to that portion of the wire will cancel. The remaining part of the wire has a length $L' = L - 2(L/2 - z) = 2z$ and a charge $Q' = 2zQ/L$. The midpoint of this part of the wire is at $z_0 = (2z - L/2 - L/2)/2 = z - L/2$. Thus the electric field from this part of the wire is

$$E_z = \frac{Q'}{[(z - z_0)^2 - L'^2/4]}$$

$$= \frac{2Qz}{L[(z - z + L/2)^2 - z^2]}$$

$$= \frac{2Qz}{L[L^2/4 - z^2]}. \qquad (1.17)$$

6)a)

By the differential form of Gauss's law,

$$\bar{\nabla} \circ \bar{E} = \rho / \varepsilon_0$$

In spherical coordinates,

$$\bar{\nabla} \circ \bar{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(E_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial \phi}$$

In this case,

$$a r^2, E_\theta = \frac{b\cos(\theta)}{r}, \text{ and } E_\phi = c, \text{ so}$$

$$\bar{\nabla} \circ \bar{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\frac{b\cos(\theta)\sin(\theta)}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(c)$$
$$= 4ar + \frac{b}{r^2 \sin\theta}(-\sin^2\theta + \cos^2\theta) + 0$$

Using

$$\cos^2\theta = 1 - \sin^2\theta,$$

This means

$$4ar + \frac{b}{r^2}\left(\frac{1 - 2\sin^2\theta}{\sin\theta}\right) = \rho / \varepsilon_0$$

Or

$$\rho = 4ar\varepsilon_0 + \frac{b\varepsilon_0}{r^2}\left(\frac{1}{\sin\theta} - 2\sin\theta\right) \quad \text{(answer)}$$

6)b)

The differential form of Gauss's law relates the divergence of $\bar{E}$ to the charge density:

$$\bar{\nabla} \circ \bar{E} = \rho / \varepsilon_0$$

In cylindrical coordinates, the divergence is given by Eq. 1.21 as

$$\bar{\nabla} \circ \bar{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}$$

In this case, $E_r = az/r$, $E_\phi = br$, and $E_z = cr^2z^2$, so

$$\bar{\nabla} \circ \bar{E} = \frac{1}{r}\frac{\partial}{\partial r}\left[r(\frac{az}{r})\right] + \frac{1}{r}\frac{\partial}{\partial \phi}(br) + \frac{\partial}{\partial z}(cr^2z^2) = \rho / \varepsilon_0$$

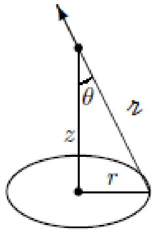
$$\frac{1}{r}(0) + \frac{1}{r}(0) + 2zcr^2 = \rho / \varepsilon_0$$

or

$$\rho = 2zcr^2\varepsilon_0$$

7)

Problem 2.5



"Horizontal" components cancel, leaving: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{\int \frac{\lambda dl}{\mathscr{r}^2}\cos\theta\right\}\hat{\mathbf{z}}$.
Here, $\mathscr{r}^2 = r^2 + z^2$, $\cos\theta = \frac{z}{\mathscr{r}}$ (both constants), while $\int dl = 2\pi r$. So

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda(2\pi r)z}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}}.}$$

Problem 2.6

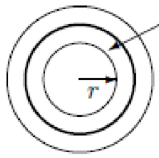
Break it into rings of radius r, and thickness dr, and use Prob. 2.5 to express the field of each ring. Total charge of a ring is $\sigma \cdot 2\pi r \cdot dr = \lambda \cdot 2\pi r$, so $\lambda = \sigma dr$ is the "line charge" of each ring.

$$E_{\text{ring}} = \frac{1}{4\pi\epsilon_0}\frac{(\sigma dr)2\pi r z}{(r^2+z^2)^{3/2}}; \quad E_{\text{disk}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma z\int_0^R \frac{r}{(r^2+z^2)^{3/2}}dr.$$

$$\boxed{\mathbf{E}_{\text{disk}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma z\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]\hat{\mathbf{z}}.}$$

8)

Problem 2.11

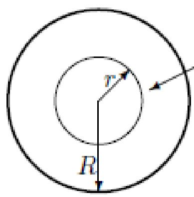


Gaussian surface: *Inside*: $\oint \mathbf{E}\cdot d\mathbf{a} = E(4\pi r^2) = \frac{1}{\epsilon_0}Q_{\text{enc}} = 0 \Rightarrow \boxed{\mathbf{E} = 0.}$

Gaussian surface: *Outside*: $E(4\pi r^2) = \frac{1}{\epsilon_0}(\sigma 4\pi R^2) \Rightarrow \boxed{\mathbf{E} = \frac{\sigma R^2}{\epsilon_0 r^2}\hat{\mathbf{r}}.}$

(As in Prob. 2.7.)

Problem 2.12



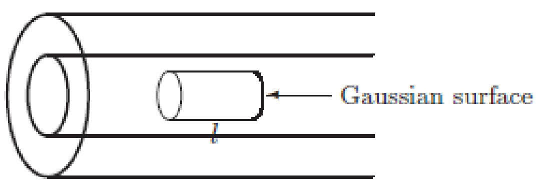
$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 4\pi r^2 = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3\rho.$ So

$$\boxed{\mathbf{E} = \frac{1}{3\epsilon_0}\rho r\hat{\mathbf{r}}.}$$

Since $Q_{\text{tot}} = \frac{4}{3}\pi R^3\rho$, $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}$ (as in Prob. 2.8).

Problem 2.16

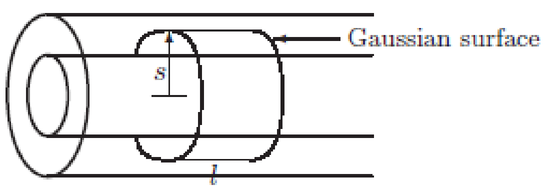
(i)



$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\rho\pi s^2 l;$

$$\boxed{\mathbf{E} = \frac{\rho s}{2\epsilon_0}\hat{\mathbf{s}}.}$$

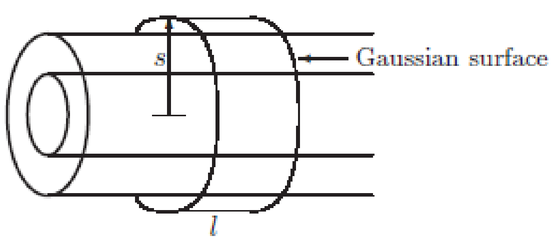
(ii)



$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\rho\pi a^2 l;$

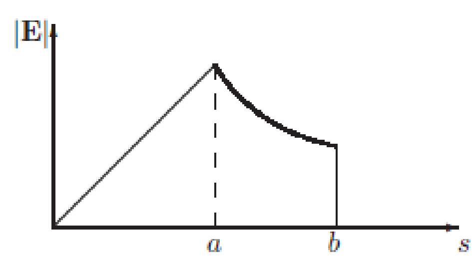
$$\boxed{\mathbf{E} = \frac{\rho a^2}{2\epsilon_0 s}\hat{\mathbf{s}}.}$$

(iii)



$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{\text{enc}} = 0;$

$$\boxed{\mathbf{E} = 0.}$$



9)

(a) The long straight wire has radius R and uniform charge density ρ. By symmetry, the electric field is perpendicular to the axis of the wire. We consider a Gaussian cylinder of length L and radius r, concentric with the wire. Then, for $r \leq R$, Gauss's law becomes

$$\begin{aligned}\oint \mathbf{E}\cdot\mathbf{dA} &= 4\pi Q \\ 2\pi r L E &= 4\pi^2 r^2 \rho L \\ E &= 2\pi\rho r, \quad r \leq R.\end{aligned} \tag{1.58}$$

For $r \geq R$, Gauss's law is

$$\begin{aligned}\oint \mathbf{E}\cdot\mathbf{dA} &= 4\pi Q \\ 2\pi r L E &= 4\pi^2 R^2 \rho L \\ E &= \frac{2\pi\rho R^2}{r}, \quad r \geq R.\end{aligned} \tag{1.59}$$

(b) We have to pick some radius for which $\phi = 0$. We choose $\phi(R) = 0$, so that the surface of the wire is at zero potential. Then, for $r \leq R$,

$$\phi(r) = -\int_R^r \mathbf{E}\cdot\mathbf{dr} = -\int_R^r 2\pi\rho r\,dr = \pi\rho(R^2 - r^2). \tag{1.60}$$

For $r \geq R$,

$$\phi(r) = -\int_R^r \mathbf{E}\cdot\mathbf{dr} = -\int_R^r \frac{2\pi R^2 \rho dr}{r} = 2\pi\rho R^2 \ln(R/r). \tag{1.61}$$

10)

(a) For $\phi = qe^{-\mu r}/r$, the electric field is

$$\mathbf{E} = -q\boldsymbol{\nabla}(e^{-\mu r}/r) = -qe^{-\mu r}\boldsymbol{\nabla}\left(\frac{1}{r}\right) - \frac{q}{r}\boldsymbol{\nabla}(e^{-\mu r})$$
$$= -qe^{-\mu r}\left(-\frac{\hat{\mathbf{r}}}{r^2} - \frac{\mu\hat{\mathbf{r}}}{r}\right) = \frac{q\hat{\mathbf{r}}e^{-\mu r}}{r^2}(1+\mu r). \quad (1.111)$$

(b) The charge density is given by

$$\rho = \frac{1}{4\pi}\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{q}{4\pi}\boldsymbol{\nabla}\cdot\left[e^{-\mu r}\left(\frac{\mathbf{r}}{r^3} + \frac{\mu\mathbf{r}}{r^2}\right)\right], \quad (1.112)$$

where we have written $\mathbf{E}$ in a more convenient form for taking its divergence. We differentiate each term in turn, leading to

$$\rho = \frac{q}{4\pi}\left\{e^{-\mu r}\left[\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{r}}{r^3}\right) + \boldsymbol{\nabla}\cdot\left(\frac{\mu\mathbf{r}}{r^2}\right)\right] + \left[\frac{\mathbf{r}}{r^3} + \frac{\mu\mathbf{r}}{r^2}\right]\cdot\boldsymbol{\nabla}(e^{-\mu r})\right\}$$
$$= \frac{qe^{-\mu r}}{4\pi}\left[4\pi\delta(\mathbf{r}) + \frac{3\mu}{r^2} - \frac{2\mu}{r^2} - \frac{\mu}{r^2} - \frac{\mu^2}{r}\right]$$
$$= q\delta(\mathbf{r}) - \frac{\mu^2 qe^{-\mu r}}{4\pi r}. \quad (1.113)$$

Note that above we isolated the term $\boldsymbol{\nabla}\cdot(\mathbf{r}/r^3)$, because we knew that it equals $4\pi\delta(\mathbf{r})$. This correctly accounted for the singular behavior at the origin. The term $q\delta(\mathbf{r})$ corresponds to a point charge q at the origin. The other part of Eq. (1.113) represents a negative charge distribution surrounding the point charge.

(c) Gauss's law is

$$\oint \mathbf{E}\cdot\mathbf{dA} = 4\pi Q. \quad (1.114)$$

We choose a sphere of radius R as our Gaussian surface. The integral of $\mathbf{E}\cdot\mathbf{dA}$ over the surface of the sphere is

$$\oint \mathbf{E}\cdot\mathbf{dA} = 4\pi R^2 E_r(R) = 4\pi qe^{-\mu R}(1+\mu R), \quad (1.115)$$

where we have taken E_r from Eq. (1.111). The charge within the Gaussian sphere is given (using Eq. (1.113)) by

$$Q = \int_{r\leq R} \rho d^3r = q - 4\pi \int_0^R \frac{\mu^2 q e^{-\mu r} r^2 dr}{4\pi r}$$
$$= 4\pi q e^{-\mu R} (1 + \mu R) , \qquad (1.116)$$

which agrees with Eq. (1.115), and Gauss's law is satisfied. (The last integral above was done using integration by parts.)

Note the importance of the proper treatment of the delta function at the origin.