PHY250 Solutions to Prob Set#3

1)

(a) In the configuration of four point charges q at the four corners of a square of side L, each of the four charges is a distance L from two other charges, and a distance $\sqrt{2}L$ from a third charge. Consequently, the potential energy is given by

$$U_0 = \frac{1}{2}\sum_{i,j\neq i}\left[\frac{q_i q_j}{|\mathbf{r_i}-\mathbf{r_j}|}\right] = \frac{4}{2}\left[\frac{2q^2}{L}+\frac{q^2}{\sqrt{2}L}\right] = \frac{\sqrt{2}q^2}{L}(1+2\sqrt{2}). \qquad (1.7)$$

(b) The kinetic energy of the four charges (each of mass m) at a long time after their release (so that $L\to\infty$) is given by

$$4\left(\frac{1}{2}mv^2\right) = U_0 = \frac{\sqrt{2}q^2}{L}(1+2\sqrt{2}), \qquad (1.8)$$

$$\text{so} \quad v = \left[\frac{q^2(1+2\sqrt{2})}{\sqrt{2}mL}\right]^{\frac{1}{2}}, \text{ as in Prob. 1.} \qquad (1.9)$$

2)

The simplest way to solve this is to build up the sphere by adding a shell of radius dr to a sphere of radius r, and integrate the work done as the sphere is built up from $r = 0$ to $r = R$.

Since the charge is uniformly distributed, the charge on a sphere of radius r is

$$Q_1 = \frac{Qr^3}{R^3}.$$

The charge contained in a spherical shell of radius r and thickness dr is

$$Q_2 = Q\frac{4\pi r^2 dr}{\frac{4}{3}\pi R^3} = \frac{3Qr^2 dr}{R^3}.$$

The work done in bringing charges Q_1 and Q_2 from infinite separation to separation r is

$$\frac{Q_1 Q_2}{4\pi\varepsilon_0 r},$$

so we can write the increase dE in the potential energy resulting from adding a thickness dr to the sphere as

$$dE = \frac{Qr^3}{R^3}\frac{3Qr^2 dr}{R^3}\frac{1}{4\pi\varepsilon_0 r} = \frac{3Q^2 r^4}{4\pi\varepsilon_0 R^6}dr.$$

Integrating this from $r = 0$ to $r = R$ gives

$$E = \frac{3Q^2}{4\pi\varepsilon_0 R^6}\int_0^R r^4\, dr = \frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 R}$$

3) No marks for this part. Just mark (a) and (b)

Use Eq. 2.43, in the version appropriate to surface charges:

$$W = \frac{1}{2}\int \sigma V\, da.$$

Now, the potential at the surface of this sphere is $(1/4\pi\epsilon_0)q/R$ (a constant—Ex. 2.7), so

$$W = \frac{1}{8\pi\epsilon_0}\frac{q}{R}\int \sigma\, da = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}.$$

Solution 2

Use Eq. 2.45. Inside the sphere, $\mathbf{E} = \mathbf{0}$; outside,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}, \quad \text{so} \quad E^2 = \frac{q^2}{(4\pi\epsilon_0)^2 r^4}.$$

Therefore,

$$W_{\text{tot}} = \frac{\epsilon_0}{2(4\pi\epsilon_0)^2}\int_{\text{outside}} \left(\frac{q^2}{r^4}\right)(r^2\sin\theta\, dr\, d\theta\, d\phi)$$

$$= \frac{1}{32\pi^2\epsilon_0}q^2 4\pi\int_R^\infty \frac{1}{r^2}\, dr = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}.$$

3)

(a) $W = \frac{\epsilon_0}{2}\int E^2\, d\tau$. $\quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ $(a < r < b)$, zero elsewhere.

$$W = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2\int_a^b \left(\frac{1}{r^2}\right)^2 4\pi r^2 dr = \frac{q^2}{8\pi\epsilon_0}\int_a^b \frac{1}{r^2} = \boxed{\frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)}.$$

(b) $W_1 = \frac{1}{8\pi\epsilon_0}\frac{q^2}{a}$, $W_2 = \frac{1}{8\pi\epsilon_0}\frac{q^2}{b}$, $\quad \mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$ $(r > a)$, $\mathbf{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{-q}{r^2}\hat{\mathbf{r}}$ $(r > b)$. So $\mathbf{E}_1 \cdot \mathbf{E}_2 = \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{-q^2}{r^4}$, $(r > b)$, and hence $\int \mathbf{E}_1 \cdot \mathbf{E}_2\, d\tau = -\left(\frac{1}{4\pi\epsilon_0}\right)^2 q^2\int_b^\infty \frac{1}{r^4}4\pi r^2 dr = -\frac{q^2}{4\pi\epsilon_0 b}$.

$W_{\text{tot}} = W_1 + W_2 + \epsilon_0\int \mathbf{E}_1 \cdot \mathbf{E}_2\, d\tau = \frac{1}{8\pi\epsilon_0}q^2\left(\frac{1}{a}+\frac{1}{b}-\frac{2}{b}\right) = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$. ✓

4) a) $\sigma_R = \dfrac{q}{4\pi R^2}$; $\sigma_a = \dfrac{-q}{4\pi a^2}$; $\sigma_b = \dfrac{q}{4\pi b^2}$.

(b) $V(0) = -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_b^a (0)dr - \int_a^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_R^0 (0)dr = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{q}{b} + \frac{q}{R} - \frac{q}{a}\right).}$

(c) $\boxed{\sigma_b \to 0}$ (the charge "drains off"); $V(0) = -\int_\infty^a (0)dr - \int_a^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_R^0 (0)dr = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{q}{R} - \frac{q}{a}\right).}$

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5)

A reasonable guess for the dimensions of a thundercloud might be a 1-km cube. If all the charge is separated to opposite faces of the cloud, it behaves like a parallel-plate capacitor with an area of 10^6 m^2 and a separation of 10^3 m. The formula $C = \varepsilon_0 A/d$ gives the capacitance as 8.85×10^{-9} F, but this precision is clearly not justified so we could estimate the capacitance as $\approx 10^{-8}$ F.

The charge stored by a capacitor is given by CV, where V is the potential difference across the plates. If the field is 3×10^6 V m^{-1} and the separation of the 'plates' is 10^3 m, the potential difference is 3×10^9 V, giving a stored charge of about 30 C.

[A lightning discharge takes typically 1 ms, so the current is about 30 kA.] $W = \frac{1}{2}CV^2 = \frac{1}{2} \times 10^{-8} \times (3 \times 10^9)^2 = 4.5 \times 10^{10}$ Joule

6)

The capacitance C of the air-spaced capacitor is $\varepsilon_0 l^2/t$.

The capacitance C' of the capacitor when the gap is entirely filled with dielectric of relative permittivity ε_r is $\varepsilon_0\varepsilon_r l^2/t$.

(4) (i) The energy stored by a capacitor of capacitance C carrying a charge Q is $Q^2/2C$, so the energy change on inserting the dielectric when Q is held constant must be

$$\Delta U = \frac{Q^2}{2}\left(\frac{1}{C'} - \frac{1}{C}\right) = \frac{Q^2 t}{2\varepsilon_0 l^2}\left(\frac{1}{\varepsilon_r} - 1\right).$$

(6) (ii) The energy stored by a capacitor of capacitance C charged to a potential difference V is $V^2C/2$, so the change in this energy on inserting the dielectric when V is held constant must be

$$\frac{V^2}{2}(C' - C) = \frac{V^2\varepsilon_0 l^2}{2t}(\varepsilon_r - 1).$$

However, the charge stored on the capacitor increases by $(C' - C)V$ so the energy stored by the battery must decrease by $(C' - C)V^2$. The change in the total energy stored by the system is thus $-(C' - C)V^2/2$, so we can write

$$\Delta U = -\frac{V^2\varepsilon_0 l^2}{2t}(\varepsilon_r - 1).$$

(7) i) To find the force on the dielectric slab when it is partially inserted, it is simplest to find an expression for the total energy of the system and then to differentiate it. The system consists, in effect, of two capacitors connected in parallel across a battery of potential V. The first of these capacitors is air-spaced and has an area of $l(l - x)$ and a separation t, so its capacitance is

$$C_1 = \varepsilon_0 l(l - x)/t.$$

The energy stored in its electric field is thus

$$C_1V^2/2 = \varepsilon_0 l(l - x)V^2/2t.$$

The second capacitor is filled with the dielectric medium and has an area of lx and a separation t, so its capacitance is

$$C_2 = \varepsilon_0\varepsilon_r lx/t.$$

The energy stored in its electric field is

$$C_2V^2/2 = \varepsilon_0\varepsilon_r lxV^2/2t,$$

so the total energy stored in the electric fields is

$$\frac{\varepsilon_0 lV^2}{2t}(l - x + \varepsilon_r x).$$

As before, however, we must also consider the energy stored in the battery. The total charge withdrawn from the battery is $C_1V + C_2V$, so the energy stored in it has *decreased* by $(C_1 + C_2)V^2$, which is numerically twice the value of the energy stored in the electric fields. The total energy of the system can thus be written as

$$U = -\frac{\varepsilon_0 l V^2}{2t}(l - x + \varepsilon_r x).$$

Differentiating this with respect to x gives

$$\frac{dU}{dx} = -\frac{\varepsilon_0 l V^2}{2t}(\varepsilon_r - 1),$$

so the magnitude of the force is $\varepsilon_0(\varepsilon_r - 1)lV^2/2t$. Since U decreases as x increases, the force must act in the direction tending to increase x. The force is thus inwards, i.e. tending to move the dielectric further between the plates. (7) ii

8) LHC

$$P = QBR$$

$$R = \frac{P}{QB}$$

$$= \frac{7000\left[\frac{GeV}{c}\right] \times 5.34\times10^{-19}\ kg.m.s^{-1}}{\underbrace{1.6\times10^{-19}}_{Q} \times 8\,[T]}$$

$$= 2.92\times10^{3}\ m$$

$$\text{circum} = 2\pi \times (\ \downarrow\ ) = 1.83\times10^{4}$$

If you google circumference of LHC, you get 27km, we get 18 because we assume a circle. the real LHC has straight sections