PHY140Y

Spring Term – Week 2 Solutions

January 8, 1999

1. First let's work out the strength of the force of gravity acting on the packet of mail. That would be

$$F_r = \frac{GMm}{r^2}, \tag{1}$$

where G is the universal gravitational constant, M is the mass of the Earth that is under the packet, m is the mass of the packet, and r is the distance of the packet from the centre of the Earth. Remember that this force acts in the direction from the centre of the Earth to the packet. The mass of the earth under the packet is

$$M = \frac{4\pi r^3}{3}\rho_E = \left(\frac{r}{R}\right)^3 M_E, \tag{2}$$

where ρ_E and M_E are the density and mass of the Earth, respectively.The component of gravity that is along the tube is given by

$$F_x = -F_r \sin\theta, \tag{3}$$

where θ is the angle of the packet from the centre of the tube. Combining things gives,

$$F_x = -\frac{GM_E m}{R^3}x. \tag{4}$$

This is the equation for a simple harmonic oscillator with a spring constant

$$k = \frac{GM_E m}{R^3}. \tag{5}$$

We thus know that the period of oscillation of such a system is

$$T = 2\pi\sqrt{\frac{m}{k}}, \tag{6}$$

and the time to get from Vancouver to St. John's is

$$t = T/2 = \pi\sqrt{\frac{R^3}{GM_E}}, \tag{7}$$

or about 42 minutes.

Note that the time of delivery does not depend on the mass of the packet, or the distance travelled. It is also half the period of a satellite in low-earth orbit (can you see why?). end up there.

2. (a) There are relatively few options. To overtake the telescope, the shuttle must travel faster. However, a faster orbit at that altitude is no longer circular; it would become elleptic and would cause their orbital altitude to increase.

(b) There is a really easy way of working this out. Just calculate the radius of the circular orbit for which the speed is exactly 10 km/hr higher than the one they are at. The radius of the original orbit is

$$r_o = R_E + 2.5 \times 10^5 = 6.62 \times 10^6 \text{ m}, \tag{8}$$

where R_E is the radius of the earth. The speed of that orbit is

$$v_o = \sqrt{\frac{GM_E}{r_o}} = 7.7557 \times 10^3 \text{ m/s}. \tag{9}$$

The lower orbit must have a speed, v_l, that is 10 km/hr = 2.8 m/s faster. Thus, the radius of the lower orbit is

$$r_l = \frac{GM_E}{v_l^2} = 6.6153 \times 10^6 \text{ m}, \tag{10}$$

which corresponds to a drop in orbit of 4.7 km.

(c) This is not the most efficient intercept course. A more efficient one would be to put the shuttle on an elleptical orbit that intercepts the HST and doesn't require four separate orbital transitions (Note: to do it our way, the shuttle has to fire boosters to descend out of the original orbit, then fire boosters to speed up to establish the lower circular orbit, and repeat that in reverse an hour later. A direct intercept orbit only requires two such rocket firings (but more calculation).

3. (a) The force of the moon's gravity on the particle is

$$F_n = \frac{GMm}{(r-R)^2}. \tag{11}$$

(b) The corresponding force at the earth's centre is

$$F_c = \frac{GMm}{r^2}, \tag{12}$$

The difference in these forces is

$$\begin{aligned} \Delta F &= F_n - F_c & (13)\\ &= GMm\left[\frac{1}{(r-R)^2} - \frac{1}{r^2}\right] & (14)\\ &= \frac{GMm}{r^2}\left[\frac{1}{(1-R/r)^2} - 1\right] & (15)\\ &= \frac{GMm}{r^2}\left[1 + 2(R/r) + O([R/r]^2) - 1\right] & (16)\\ &\simeq \frac{2GMmR}{r^3}, & (17) \end{aligned}$$

where we have used a Taylor series expansion

$$(1+\delta)^{-2} = 1 - 2\delta + \text{terms of order } \delta^2 \tag{18}$$

to approximate the first term and have used the fact that $R \ll r$. This force is positive, and by the way I have defined things implies that there is a greater force on the particle of water towards the moon than the force on the earth itself.

(c) We can repeat the same calculation as in the previous section, except that the force difference will now be in the opposite direction. The force difference would cause the packet of water to experience a force that pushes it away from the centre of the earth.

(d) The fact that the two relative forces are in opposite directions explains the "bulges" of water that we see. This is in fact the tidal forces we are all aware of. From this, it is still a challenge to predict the height of the tides as they depend on the size of the body of water and local geography.