ORBITAL STABILITY

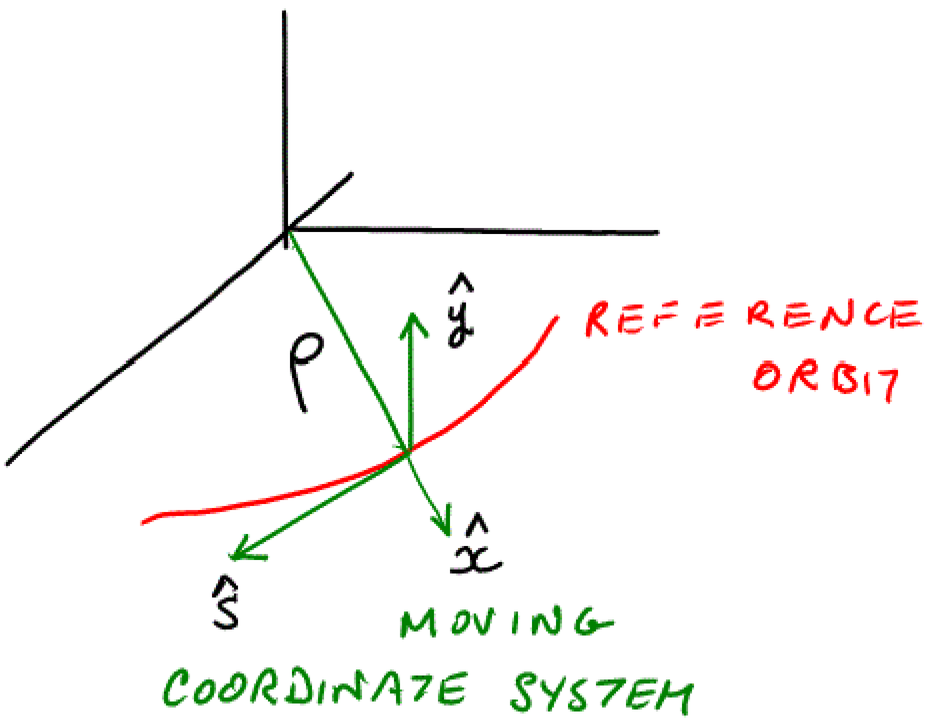
- IN DISCUSSION OF BETATRON OSCILLATIONS IN
- A WEAKLY FOCUSED MACHINE DERIVED CONDITIONS FOR STABILITY AROUND EQUILIBRIUM ORBIT
- FOR STRONG FOCUSING THE FIELD INDEX CAN BE $\gg 1$. IN FACT, WE CAN IMAGINE THAT IT WILL VARY AROUND THE RING

- WANT TO STUDY
  - SPATIAL STABILITY OF BEAM ENVELOPE $\rightarrow$ BETATRON OSCILLATIONS
  - PHASE STABILITY W.R.T THE PHASE OF ACCELERATING VOLTAGE

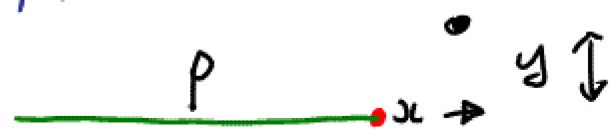
    $\rightarrow$ SYNCHROTRON OSCILLATIONS

WORK IN MOVING COORDINATE SYSTEM

- EQUILIBRIUM ORBIT IN A PARTICULAR MACHINE IS DEFINED BY CHOSEN MAGNET CONFIGURATION
- OUR MAIN INTEREST IS IN UNDERSTANDING HOW PARTICLES MOVE RELATIVE TO EQUILIBRIUM ORBIT
- MOTION EASIER TO TREAT IN COORDINATE SYSTEM THAT MOVES ALONG EQUILIBRIUM ORBIT.



POSITION OF PARTICLE



$$\bar{R} = r\hat{x} + y\hat{y}\,;\quad r = \rho + x$$

EQUATION OF MOTION

$$\frac{d\bar{p}}{dt} = e\,\bar{v} \times \bar{B}$$

MAGNETIC FIELD → RADIAL + VERTICAL COMPONENTS

→ NO COMPONENT IN $\hat{s}$-DIRECTION

$$\bar{v} \times \bar{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{s} \\ v_x & v_y & v_s \\ B_x & B_y & 0 \end{vmatrix} = -v_s B_y \hat{x} + v_s B_x \hat{y} + (v_x B_y - v_y B_x)\hat{s}$$

ASSUME NO ENERGY CHANGE → NO ACCELRATION IN $\hat{S}$

→ NO SYNCHROTRON RADIATION

$$e\,\bar{v} \times \bar{B} = \frac{d\bar{p}}{dt} = \frac{d}{dt}\left(\gamma m \dot{\bar{R}}\right) = \gamma m \ddot{\bar{R}}$$

$$\ddot{\bar{R}} = \frac{e\,\bar{v} \times \bar{B}}{\gamma m}$$

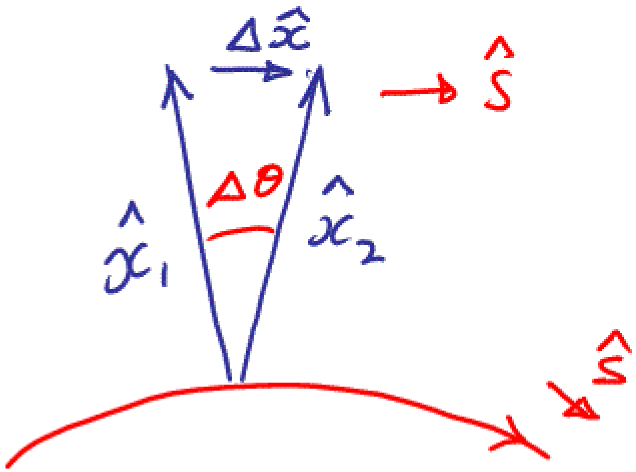
EVALUATE THIS IN LOCAL MOVING COORD SYSTEM

$$\bar{R} = r\hat{x} + y\hat{y} \rightarrow \dot{\bar{R}} = \dot{r}\hat{x} + r\dot{\hat{x}} + \dot{y}\hat{y}$$

VARIES IN TIME

CONSTANT

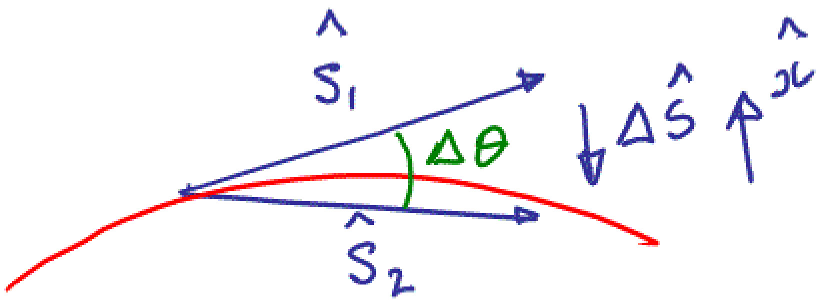
VARIATION IN TIME OF UNIT VECTORS



$$\Delta\hat{x} = \underbrace{\overbrace{|\hat{x}|}^{=1}\Delta\theta}_{\text{MAG}} \cdot \underbrace{\hat{s}}_{\text{DIRECTION}}$$

ASSUME $\hat{s}$ CONSTANT

$$\dot{\hat{x}} = \dot{\theta} \cdot \hat{s}$$



$$\Delta\hat{s} = |\hat{s}|\,\Delta\theta\,(-\hat{x})$$

$$\dot{\hat{s}} = -\dot{\theta}\,\hat{x}$$

$$\dot{\theta} = \frac{v_s}{\rho}$$

HAD

$$\dot{\vec{R}} = \dot{r}\,\hat{x} + r\,\dot{\hat{x}} + \dot{y}\,\hat{y}$$

$$\dot{R} = \dot{r}\,\hat{x} + r\underbrace{\dot{\theta}\,\hat{s}}_{\text{VARIES IN TIME}} + \dot{y}\,\hat{y}$$

$$\ddot{\vec{R}} = \ddot{r}\,\hat{x} + \dot{r}\,\dot{\hat{x}} + r\frac{d}{dt}(\dot{\theta}\,\hat{s}) + \dot{\theta}\,\hat{s}\,\dot{r} + \ddot{y}\,\hat{y}$$

$$= \ddot{r}\,\hat{x} + \dot{r}\,\underbrace{\dot{\hat{x}}}_{\dot{\theta}\,\hat{s}} + r\dot{\theta}\,\dot{\hat{s}} + r\ddot{\theta}\,\hat{s} + \dot{\theta}\,\hat{s}\,\dot{r} + \ddot{y}\,\hat{y}$$

$$= \ddot{r}\,\hat{x} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{s} + r\dot{\theta}\,\underbrace{\dot{\hat{s}}}_{-\dot{\theta}\,\hat{x}} + \ddot{y}\,\hat{y}$$

$$\ddot{\vec{R}} = \ddot{r}\,\hat{x} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{s} - r\dot{\theta}^2\hat{x} + \ddot{y}\,\hat{y}$$

$$\ddot{\vec{R}} = \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{\text{RADIAL MOTION}}\hat{x} + \underbrace{(2\dot{r}\dot{\theta} + r\ddot{\theta})}_{\text{ALONG ORBIT}}\hat{s} + \underbrace{\ddot{y}\,\hat{y}}_{\text{VERTICAL}}$$

X OR RADIAL MOTION FROM

$$\ddot{\bar{R}} = (\ddot{r} - r\dot{\theta}^2)\hat{x} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s} + \ddot{y}\hat{y}$$

$$F = \gamma m \ddot{x}$$

$$e(\bar{v}\times\bar{B})_x = -e v_s B_y \hat{x} = \frac{d\bar{p}}{dt} = \gamma m(\ddot{r} - r\dot{\theta}^2)\hat{x}$$

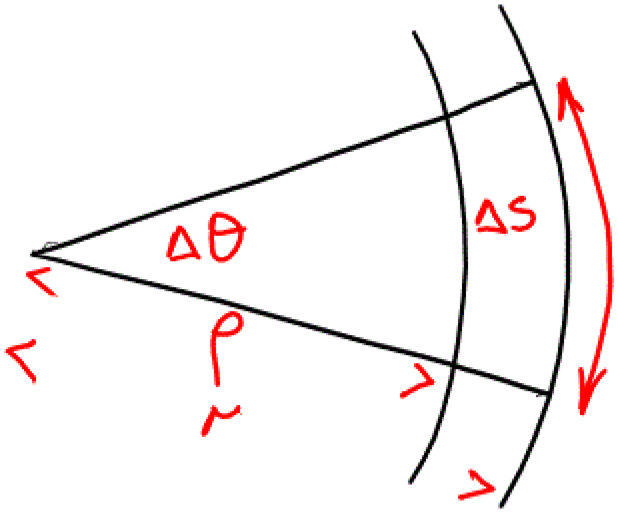
EQUATION OF MOTION $(\ddot{r} - r\dot{\theta}^2) = -\dfrac{e v_s B_y}{\gamma m}$

FOR $v_x \ll v_s$; $v_y \ll v_s$ $\quad p \to \gamma m v_s$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{e v_s^2 B_y}{p}$$

IN AN ACCELERATOR MORE INTERESTED IN HOW MOTION VARIES ALONG PERIODIC ORBIT.

CHANGE VARIABLE $t \to s$



$$\Delta s = \rho\, \Delta\theta$$

$$v_s\, \Delta t = r\, \Delta\theta$$

$$\Delta t = \frac{r}{v_s} \cdot \Delta\theta$$

$$\Delta t = \frac{r}{v_s} \cdot \frac{\Delta s}{\rho}$$

$$\frac{1}{\Delta t^2} = v_s^2 \left(\frac{\rho}{r}\right)^2 \frac{1}{\Delta s^2}$$

$$\frac{d}{dt} = \frac{ds}{dt} \cdot \frac{d}{ds}$$

HAD $$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{e v_s^2}{p} B_y$$

$$\hookrightarrow r = \rho + x \rightarrow \rho \text{ IS CONSTANT}$$

$$\frac{d\theta}{dt} = \frac{v_s}{r}$$

$$\frac{d^2 x}{dt^2} - r\left(\frac{v_s}{r}\right)^2 = -e\frac{v_s^2}{p} \cdot B_y$$

$$\frac{1}{\Delta t^2} = v_s^2\left(\frac{\rho}{r}\right)^2 \frac{1}{\Delta s^2} \longrightarrow \cancel{v_s^2}\frac{\rho^2}{r^2} \cdot \frac{d^2 x}{ds^2} - r\frac{\cancel{v_s^2}}{r^2} = -\frac{e\cancel{v_s^2} B_y}{p}$$

$$\frac{e}{p} = \frac{1}{B\rho} \rightarrow \frac{d^2 x}{ds^2} - \frac{r}{\rho^2} = -\frac{r^2}{\rho^2} \cdot \frac{B_y}{B\rho}$$

$$r = \rho + x \longrightarrow \frac{d^2 x}{ds^2} - \left(\frac{\rho + x}{\rho^2}\right) = -\left(1 + \frac{x}{\rho}\right)^2 \frac{B_y}{B}$$

EQUATION OF MOTION IN x - DIRECTION

Y - DIRECTION, HAD!

$$\ddot{\bar{R}} = (\ddot{r} - r\dot{\theta}^2)\hat{x} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s} + \ddot{y}\,\hat{y}$$

$$F = \gamma m \ddot{y}$$

$$e(\bar{v} \times \bar{B})_y = +e\,v_s B_x \hat{y} = \gamma m \ddot{y}\,\hat{y}$$

$$v_x, v_y << v_s \qquad p \to \gamma m v_s$$

$$\frac{d^2 y}{dt^2} = -\frac{e\,v_s^2}{p} \cdot v_s$$

CHANGE $dt^2 \to ds^2$

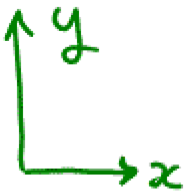
$$\frac{d^2 y}{ds^2} = +\left(1 + \frac{x}{\rho^2}\right)\frac{B_x}{B\rho}$$

EQUATION OF MOTION IN Y-DIRECTION

IN ENGINEERING TERMS, THE MAGNETIC FIELDS CAN BE QUITE COMPLEX – NONLINEAR BUT ASSUME LINEAR FOR SIMPLICITY

$$B_x = B_x(0,0) + \frac{\partial B_x}{\partial y}\cdot y + \frac{\partial B_x}{\partial x}\cdot x$$

$$B_y = B_y(0,0) + \frac{\partial B_y}{\partial x}\cdot x + \frac{\partial B_y}{\partial y}\cdot x$$



$B_x(0,0) = 0$ $\downarrow B_y$ $\rightarrow B_x$; $B_y(0,0) \rightarrow B$

DON'T WANT x & y MOTION COUPLED

$\therefore$ ASSUME $\dfrac{\partial B_x}{\partial x} = \dfrac{\partial B_y}{\partial y} = 0$

$$B_x = \frac{\partial B_x}{\partial y} \cdot y$$

$$B_y = B + \frac{\partial B_y}{\partial x} \cdot x$$

$$\bar{\nabla} \times \bar{B} = 0 \rightarrow \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) = 0$$

$$B_x = \frac{\partial B_y}{\partial x} \cdot y$$

$$B_y = B + \frac{\partial B_y}{\partial x} \cdot x$$

MOTION IN x - DIRECTION

$$\frac{d^2x}{ds^2} + \frac{\rho + x}{\rho^2} + \left(1 + \frac{x}{\rho}\right)^2 \cdot \frac{1}{B\rho} \frac{\partial B_y(s)}{\partial x} \cdot x = 0$$

FOR SMALL x $\left(1 + \frac{x}{\rho}\right)^2 \sim \left(1 + \frac{2x}{\rho}\right)$

$$\frac{\partial B_y(s)}{\partial x} = B + x \frac{\partial B_y}{\partial x}$$

$$\left(B + x \frac{\partial B_y}{\partial x}\right)\left(1 + \frac{2x}{\rho}\right) \frac{1}{B\rho} = \left(B + \frac{2xB}{\rho} + \cancel{2x^2} \frac{\partial B_y}{\partial x} + x \frac{\partial B_y}{\partial x}\right) \frac{1}{B\rho}$$

$$\frac{d^2x}{ds^2} - \frac{1}{\rho} - \frac{x}{\rho^2} + \frac{1}{\rho} + \frac{2x}{\rho^2} + \frac{x}{B\rho} \frac{\partial B_y}{\partial x}$$

FOR SMALL OSCILLATIONS, x & y EQUATIONS OF MOTION

x $\leftrightarrow$

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2} + \frac{1}{B\rho}\frac{\partial B_y(s)}{\partial x}\right]\cdot x = 0 \qquad (1)$$

$\updownarrow$ y

$$\frac{d^2y}{ds^2} - \frac{1}{B\rho}\cdot\frac{\partial B_y}{\partial x}\cdot y = 0 \qquad (2)$$

IN (1) GO BACK TO FORM

$$\frac{d^2x}{ds^2} - \frac{\rho + x}{\rho^2} = \frac{-B_y}{B\rho}\left(1 + \frac{x}{\rho}\right)^2$$

ON EQUILIBRIUM ORBIT $\frac{d^2x}{ds^2} = 0$; $x = 0$

$$\frac{1}{\rho} = \frac{B_y}{B\rho} \longrightarrow \frac{1}{\rho} = \frac{e}{p}\cdot B_y \longrightarrow$$

CIRCULAR MOTION IN DIPOLE FIELD

EQUATIONS OF MOTION

$\leftrightarrow x$
$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2} + \frac{1}{B\rho}\frac{\partial B_y(s)}{\partial x}\right] \cdot x = 0 \qquad (1)$$

OSCILLATIONS $\longrightarrow$ $\frac{1}{\rho^2} << \frac{1}{B\rho}$

$\updownarrow y$
$$\frac{d^2y}{ds^2} - \frac{1}{B\rho} \cdot \frac{\partial B_y}{\partial x} \cdot y = 0 \qquad (2)$$

BOTH ARE OF FORM

$$x'' + K(s)x = 0$$

HILL'S EQUATION
SIMPLE HARMONIC MOTION WITH VARIABLE SPRING CONSTANT

TRANSVERSE OSCILLATIONS VARY IN AMPLITUD & FREQUENCY ALONG ORBIT

PIECEWISE SOLUTION FOR REAL ACCELERATOR

$$x'' + K(s)\,x = 0$$

$K(s)$ CONSTANT IN EACH ELEMENT OF LATTICE → DIPOLE, 4-POLE, DRIFT etc

- CAN USE SHM SOLUTION IN EACH ELEMENT & MATCH BOUNDARIES

$K = 0$ DRIFT OR CONSTANT B_y

$K > 0$ SIMPLE HARMONIC OSCILLATOR

$K < 0$ HYPERBOLIC

$K=0$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{OUT} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{IN}$$

$K>0$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{OUT} = \begin{pmatrix} \cos(\sqrt{K}\,l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}\,L) \\ -\sqrt{K} \sin(\sqrt{K}\,l) & \cos(\sqrt{K}\,L) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{IN}$$

$K<0$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{OUT} = \begin{pmatrix} \cosh(\sqrt{|K|}\,l) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}\,l) \\ \sqrt{|K|} \sinh(\sqrt{|K|}\,L) & \cosh(\sqrt{|K|}\,l) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{IN}$$

$l \to 0 \quad \to$ THIN LENS

ANALYTICAL SOLUTION OF HILL'S EQUATION.

WANT SOME FORMALISM WHICH DOES NOT JUST DESCRIBE ONE PARTICLE

- WANT TO DESCRIBE BEAM ENVELOPE → PHASE AMPLITUDE

$$x'' + K(s)\,x = 0$$

FOR A PERIODIC STRUCTURE $K(s) = K(s+c)$

SOLUTION $\quad x = A\,\omega(s)\cos\left(\phi(s) + \delta\right)$

$A, \delta \rightarrow$ INITIAL CONDITIONS

K CONSTANT → SHM $\quad x = A\cos\left(\phi(s) + \delta\right)$

- A CONSTANT
- $\phi(s) = s\sqrt{K}$ INCREASES LINEARLY

HERE A VARIES WITH S, $\phi(s)$ DOES NOT INCREASE LINEARLY WITH S

TRY SOLUTION $x(s) = A\omega(s)\cos(\phi(s) + \delta)$

IN $x'' + K(s)\cdot x = 0$

$$x'' + Kx = A(2\omega'\phi' + \omega\phi'')\sin(\phi+\delta) +$$

$$A(\omega'' - \omega\phi'^2 + K\omega)\cos(\phi+\delta) = 0$$

IF δ ARBITRARY, COEFFICIENTS OF SIN, COS $= 0$

$$\omega(2\omega'\phi' + \omega\phi'') = 0 \rightarrow (\omega^2\phi')' = 0$$

$\phi' = k/\omega^2$ k CONSTANT OF INTEGRATION

ABSORB INTO A, $k \rightarrow 1$

$$\Phi' = \frac{1}{\omega^2}$$

COEFFICIET OF COSINE TERM

$$\omega'' - \omega\phi'^2 + K(s)\omega \rightarrow \omega'' - \omega \cdot \frac{1}{\omega^4} + K(s)\omega = 0$$

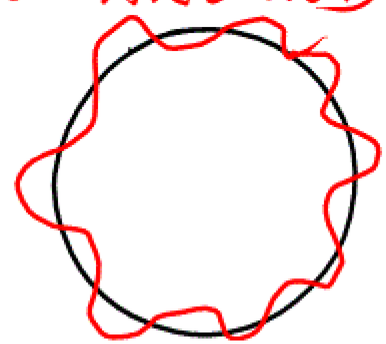
$$\omega^3(\omega'' + K(s)\omega) = 1$$

NOW HAVE AN EQUATION FOR $\omega(s)$ AS A FUNCTION OF POSITION AROUND ACCELERATOR RING

$$x(s) = A\,\omega(s)\cos(\phi(s) + \delta)$$

$$\omega'' + K(s)\omega = \frac{1}{\omega^3} \quad ; \quad \omega^2\phi' = 1$$

AMPLITUDE OF TRANSVERSE OSCILLATIONS AS A FUNCTION OF POSITION AROUND ACCELERATOR RING



$$\omega'' + K(s)\,\omega = \frac{1}{\omega^3} \quad ; \quad \omega^2 \phi' = 1$$

DEFINE COURANT - SNYDER PARAMETERS

$\beta(s) = \omega^2(s)$ AMPLITUDE FUNCTION

$$\alpha(s) = -\frac{1}{2}\beta'(s)$$

$$\gamma(s) = 1 + \alpha^2(s) / \beta(s)$$

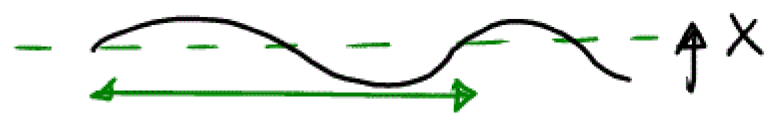
SO $\phi'(s) = \dfrac{1}{\beta(s)} \rightarrow \Delta\phi(s) = \displaystyle\int_0^s \frac{ds}{\beta(s)}$

$$x = A\,\omega(s) \cos\left(\phi(s) + \delta\right)$$

PHASE ADVANCE ALONG PATH S.

$\omega^2 \rightarrow$ OSC FREQUENCY

$2\pi\beta \rightarrow$ OSCILLATION WAVE LENGTH



$\lambda = 2\pi\beta(s)$

HAD $\qquad \omega'' - K(s)\,\omega - \frac{1}{\omega^3} = 0$

$$\omega = \sqrt{\beta} \longrightarrow \omega' = \frac{1}{2\sqrt{\beta}}\,\beta'$$

$$\omega'' = -\frac{1}{4}\,\beta^{-3/2}(\beta')^2 + \frac{1}{2}\,\beta^{-\frac{1}{2}}\beta''$$

$$\rightarrow \quad \frac{1}{2}\,\beta^{-\frac{1}{2}}\beta'' - \frac{1}{4}(\beta')^2 + K\beta^2 - 1 = 0$$

$$\left(2\beta\beta'' - \beta'^2 + 4K\beta^2\right) = 4$$

$$K\beta = \gamma + \alpha'$$

FOR A PERIODIC ACCELERATOR PARTICLE ORBITS ARE SOLUTIONS OF

$$\left.\begin{aligned} x'' + K(s)\,x &= 0 \\ y'' + K(s)\,y &= 0 \end{aligned}\right\} \quad (1)$$

AND

$$2\beta\beta'' - \beta'^2 + 4\beta^2 K(s) = 0 \quad (2)$$

(1) DESCRIBES INDIVIDUAL PARTICLE

(2) DESCRIBES BEAM ENVELOPE

CALL x or y y

$$y'' + K(s)\,y = 0$$ SOLN $$y(s) = A\sqrt{\beta(s)}\cos(\phi(s)+\delta) \quad ①$$

DIFFERENTIATE SOLUTION

$$y'(s) = A\frac{d}{ds}\sqrt{\beta}\cos(\phi+\delta) + A\sqrt{\beta}\frac{d}{ds}\left\{\cos(\phi+\delta)\right\}$$

$$\alpha(s) = -\frac{1}{2}\frac{d\beta}{ds}, \quad \frac{d\sqrt{\beta}}{ds} = -\frac{1}{\sqrt{\beta}}\cdot\alpha$$

DROP δ FOR NOW

$$y' = -\frac{A\alpha}{\sqrt{\beta}}\cos\phi - A\sqrt{\beta}\frac{d\phi}{ds}\sin\phi$$

$$\longleftarrow \frac{1}{\beta}$$

$$y' = -\frac{A\alpha}{\sqrt{\beta}}\cos\phi - \frac{A}{\sqrt{\beta}}\sin\phi$$

$$\beta y' + \alpha y = -A\sqrt{\beta}\,\alpha\cos\phi - A\sqrt{\beta}\sin\phi + A\sqrt{\beta}\,\alpha\cos\phi$$

$$= -A\sqrt{\beta}\sin\phi \quad (2)$$

TAKE $(1)^2 + (2)^2$

$$(\beta y' + \alpha y)^2 + y^2 = A^2\beta\sin^2\phi + A^2\beta\cos^2\phi$$

$$\beta^2 y'^2 + \alpha^2 y^2 + y^2 + 2\alpha\beta y' y = A^2\beta$$

$$\underbrace{\frac{1+\alpha^2}{\beta}}_{\gamma} y^2 + \beta y'^2 + 2\alpha y' y = A^2$$

$$\beta y'^2 + 2\alpha y y' + \gamma y^2 = A^2$$

COURANT – SNYDER INVARIANT – BEAM EMITTANCE

$$\text{(1)} \quad \underbrace{\beta y'^2 + 2\alpha y y' + \gamma y^2}_{\text{THIS IS AN ELLIPSE IN } y\,y' \text{ SPACE}} = A^2 \leftarrow$$

INITIAL CONDITIONS
INVARIANT OF MOTION

THIS IS AN ELLIPSE IN $y\ y'$ SPACE

ELLIPSE $\quad a x^2 + 2bxy + c y^2 = d$

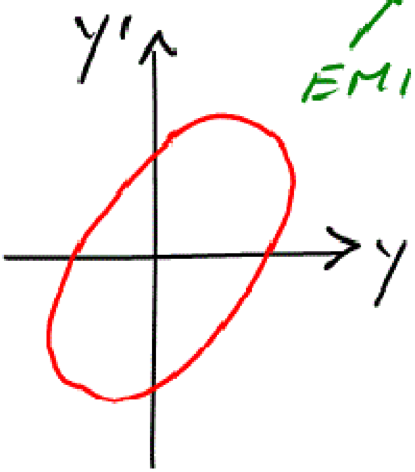
$$\text{AREA} = \pi d / \sqrt{ac - b^2}$$

$$\text{IN (1) AREA} = \pi A^2 / \underbrace{\sqrt{\beta\gamma - \alpha^2}}_{=1} = \pi A^2$$

DEFINE $\varepsilon = \pi A^2$

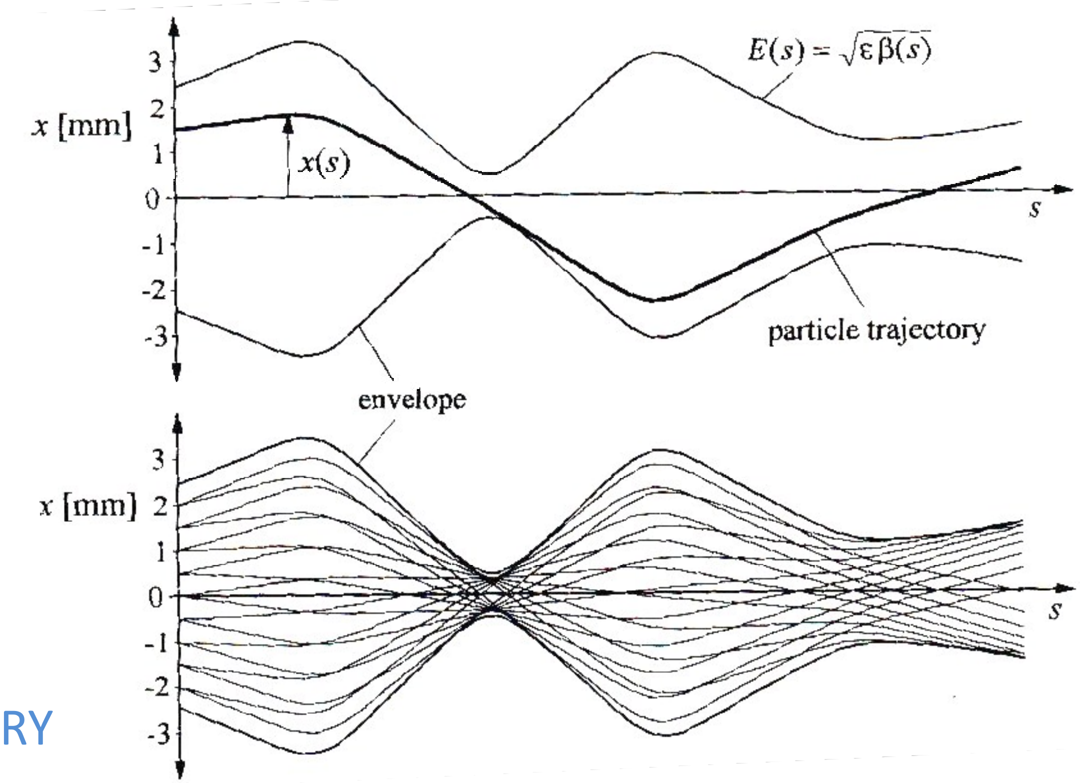
EMITTANCE

$$\beta y'^2 + 2\alpha y y' + \gamma y^2 = \frac{\varepsilon}{\pi}$$



$$y(s) = A\,w(s)\cos(\phi(s) + \delta)$$

$$y(s) = \sqrt{\frac{\varepsilon\beta(s)}{\pi}}\,\cos(\phi(s) + \delta)$$



TRAJECTORY

$$\frac{d^2Y}{ds^2}+K(s)Y=0$$

$$Y(s)=A\omega(s)\cos(\phi(s)+\delta)$$

$$Y(s)=\sqrt{\frac{\varepsilon}{\pi}\beta(s)}\cos(\phi(s)+\delta)$$

BEAM ENVELOPE

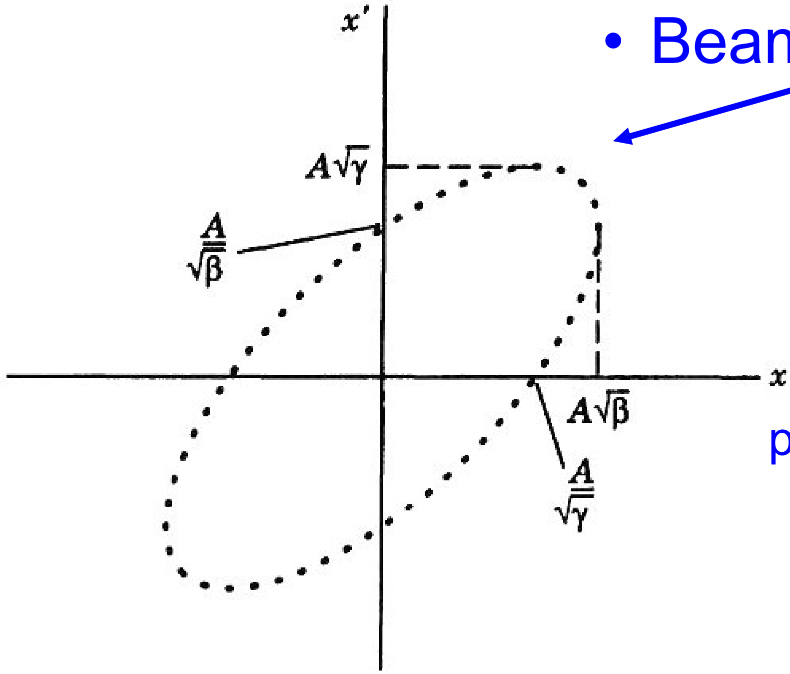
$$\frac{d^2\omega}{ds^2}+K(s)\omega=\frac{1}{\omega^3}$$

$$\beta(s)=\omega^2(s)$$

Amplitude of betatron oscillations

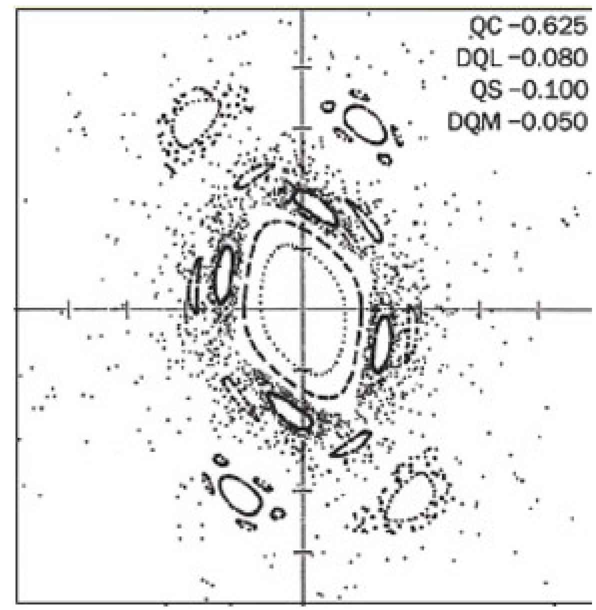
- Single Particle Phase Space
- Beam Envelope

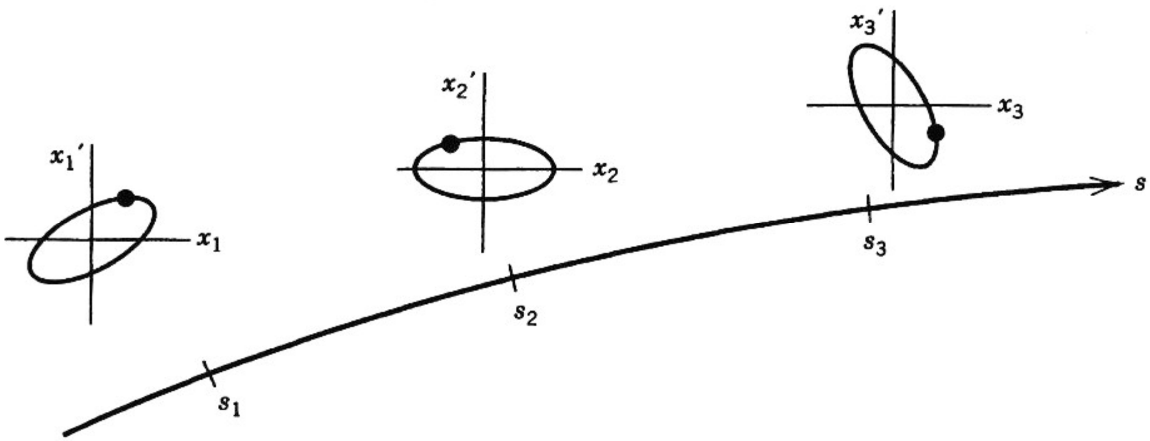
angle



position

- Real Accelerator
- Non-linear





Shape of phase space changes along accelerator lattice
Area constant -> Liouville

FROM ELLIPSE PLOT, AT ANY POINT IN ACCELERATOR MAXIMUM $x = A\sqrt{\beta}$

AT SOME POINT IN THE LATTICE WILL HAVE APERTURE $2a$ WHICH DEFINES MAXIMUM ELLIPSE WHICH WILL FIT THRU MACHINE ADMITTANCE

$$2a(s) = 2A\sqrt{\beta} \rightarrow A^2 = a^2(s)/\beta$$

$$\beta y'^2 + 2\alpha y y' + \gamma y^2 = \frac{\varepsilon}{\pi} = A^2$$

$$\text{ADMITTANCE} = \frac{\pi a^2}{\beta_{MAX}}$$

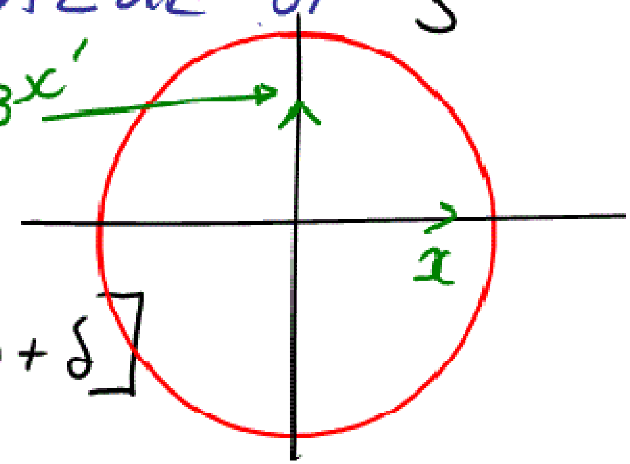
IN A REAL ACCELERATOR THERE ARE MANY PARTICLES

USUALLY ADMITTANCE / EMITTANCE DEFINED IN TERM OF PHASE SPACE BOUNDARY WHICH CONTAINS FRACTION F OF BEAM

NUMBER DISTRIBUTION ASSUMED GAUSSIAN

$$n(x)\,dx = \frac{1}{\sqrt{2\pi}\cdot\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx$$

ASSUME THIS DISTRIBUTION IS CONSTANT IN TIME AT A GIVEN VALUE OF S



$$x(s) = A\sqrt{\beta}\cos[\phi(s) + \delta]$$

$$\alpha(s)x(s) + \beta(s)x'(s) = -A\sqrt{\beta}\sin[\phi(s) + \delta]$$

IN THIS TRANSFORMED PHASE SPACE WHERE BOUNDARY IS A CIRCLE

$$n(x, \alpha x+\beta x')\,dx\,d(\alpha x+\beta x')$$
$$= \frac{1}{2\pi\sigma^2} \exp\left\{-\left[x^2+(\alpha x+\beta x')^2\right]/2\sigma^2\right\} dx\, d(\alpha x+\beta x')$$

CHANGE TO POLAR COORDINATES

$$r^2 = x^2 + (\alpha x+\beta x')^2$$
$$n(r, \theta)\, r\, dr\, d\theta = \frac{1}{2\pi\sigma^2}\, e^{-r^2/2\sigma^2}\, r\, dr\, d\theta$$

IF FRACTION F OF BEAM IS WITHIN RADIUS a

$$F = \int_0^{2\pi}\int_0^{a} n\, r\, dr\, d\theta = \int_0^{a} e^{-r^2/2\sigma^2}\, \frac{r\,dr}{\sigma^2}$$

$$F = \int_0^a e^{-r^2/2\sigma^2} \frac{r\,dr}{\sigma^2} \longrightarrow a^2 = -2\sigma^2 \ln(1-F)$$

AND $$\frac{\beta\varepsilon}{\pi} = \beta\left(\frac{1+\alpha^2}{\beta}\right)x^2 + 2\alpha\beta x x' + \beta^2 x'^2$$

$$= x^2 + (\alpha x + \beta x')^2$$

F CONTAINED IN $a^2 = \frac{\beta\varepsilon}{\pi} = -2\sigma^2 \ln(1-F)$

$$\varepsilon = -\frac{2\pi\sigma^2}{\beta} \ln(1-F)$$

ε	F %
σ^2/β	15
$\pi\sigma^2/\beta$	39
$4\pi\sigma^2/\beta$	87

IN DISCUSING HILL'S EQUATION USE THE SOLUTION

$$x(s) = A\,w(s)\cos(\phi(s)+\delta)\,;\quad \phi' = \frac{1}{w^2(s)}$$

CAN ALSO WRITE A!

$$x = w(s)\left(A_1\cos\phi(s) + A_2\sin\phi(s)\right)$$

AND

$$x' = \left(A_1 w' + A_2/w\right)\cos\phi + \left(A_2 w' - \frac{A_1}{w}\right)\sin\phi$$

INITIAL CONDITIONS $x_0,\ x_0',\ s=s_0,\ \phi=0$

$$A_1 = \frac{x_0}{w}\ ;\quad A_2 = x_0' w - x_0 w'$$

GOING FROM $S_0 \to S_0 + C$ PERIOD OF SOLUTION $w(S_0+C) = w(S_0)$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{S_0+C} = \begin{pmatrix} \cos\Delta\phi_c - ww'\sin\Delta\phi_c & w^2\sin\Delta\phi_c \\ -\frac{1+(ww')^2}{w^2}\sin\Delta\phi_c & \cos\Delta\phi_c + ww'\sin\Delta\phi_c \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_{S_0}$$

IN TERMS OF COURANT - SNYDER, LAST BECOMES

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+c} = \begin{pmatrix} \cos\Delta\phi_c + \alpha \sin\Delta\phi_c & \beta \sin\Delta\phi_c \\ -\gamma \sin\Delta\phi_c & \cos\Delta\phi_c - \alpha \sin\Delta\phi_c \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\phi(s_0 \to s_0+c) \equiv \Delta\phi_c = \int_{s_0}^{s_0+c} \frac{ds}{w^2(s)} \equiv \int_{s_0}^{s_0+c} \frac{ds}{\beta(s)}$$

THIS TRANSFER MATRIX CAN BE WRITTEN

$$M = I \cos\Delta\phi_c + J \sin\Delta\phi_c$$

$$J \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \; ; \quad J^2 = -I$$

OR

$$M = e^{J\Delta\phi_c}$$

SUPPOSE PRODUCT OF ALL TRANSFER MATRICES IN REPEAT PERIOD $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\Delta\phi_c + \alpha\sin\Delta\phi_c & \beta\sin\Delta\phi_c \\ -\gamma\sin\Delta\phi_c & \cos\Delta\phi_c - \alpha\sin\Delta\phi_c \end{pmatrix} \quad (1)$$

$$\cos\Delta\phi_c = \frac{1}{2}(a+d) = \frac{1}{2}\,\mathrm{Tr}\,M$$

IN DISCUSSION OF STABILITY CRITERION IN FODO

$$-1 \lesssim \frac{1}{2}\mathrm{Tr}\,M \lesssim 1 \rightarrow -1 \lesssim \cos\mu \lesssim 1$$

COMPARING μ IN $e^{i\mu}$ IS $\Delta\phi_c$

→ PHASE ADVANCE THROUGH REPEAT PERIOD.

FROM 1 ON LAST PAGE

$$\beta = \frac{b}{\sin \Delta\phi_c} \quad ; \quad \alpha = \frac{a-d}{2\sin \Delta\phi_c}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ IS TRANSFER MATRIX AT THIS POINT IN LATTICE

IF ONE KNOWS M CAN GET α, β WHICH ALLOW ONE TO CALCULATE THE PARTICLE MOTION

EQ M COULD BE

$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \text{ OR } \begin{pmatrix} \cos\sqrt{K}L & \frac{1}{\sqrt{K}}\sin(KL) \\ -\sqrt{K}\sin(KL) & \cos\sqrt{K}L \end{pmatrix} \text{ OR etc ---}$$

DRIFT FOCUSING QUAD

IF ONE DETERMINES β AT EVERY POINT ON LATTICE, MOTION FROM $1 \to 2$.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M(s_1 \to s_2) \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

GET EXPLICIT FORM OF M

START WITH

$$x = w(s)\left(A_1 \cos\phi + A_2 \sin\phi\right)$$

$$x' = \left(A_1 w'(s) + \frac{A_2}{w}\right)\cos\phi + \left(A_2 w'(s) - \frac{A_1}{w(s)}\right)\sin\phi$$

INITIAL CONDITIONS $x_1, x_1'; \ s = s_1$

$$\rightarrow \quad A_1 = \frac{x_1}{w_1}; \quad A_2 = x_1' w_1 - x_1 w_1'$$

$$w_1 = \sqrt{\beta_1} \quad \text{AND} \quad \alpha_1 = -\frac{\beta_1'}{2}$$

$$x_2 = \sqrt{\frac{\beta_2}{\beta_1}} \left(\cos \Delta\phi_c + \alpha_1 \sin \Delta\phi_c \right) \cdot x_1 \quad + \sqrt{\beta_1 \beta_2}\, \sin \Delta\phi_c \cdot x_1'$$

$$x_2' = - \left\{ \frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin \Delta\phi_c + \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1 \beta_2}} \cos \Delta\phi_0 \right\} \cdot x_1$$

$$+ \sqrt{\frac{\beta_1}{\beta_2}} \left(\cos \Delta\phi_c - \alpha_2 \sin \Delta\phi_c \right) \cdot x_1'$$

$$M = \begin{pmatrix} \left(\frac{\beta_2}{\beta_1}\right)^{1/2} \left(\cos \Delta\phi_c + \alpha_1 \sin \Delta\phi_c \right) & \left(\beta_1 \beta_2\right)^{\frac{1}{2}} \sin \Delta\phi_c \\ -\frac{1 + \alpha_1 \alpha_2}{\left(\beta_1 \beta_2\right)^{\frac{1}{2}}} \sin \Delta\phi_c + \frac{\alpha_1 - \alpha_2}{\left(\beta_1 \beta_2\right)^{\frac{1}{2}}} \cos \Delta\phi_c & \left(\frac{\beta_1}{\beta_2}\right)^{1/2} \left(\cos \Delta\phi_c - \alpha_2 \sin \Delta\phi_c \right) \end{pmatrix}$$

$$\Delta\phi_c = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}$$